\[ \pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers} \]

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Abstract

It is proved that \( \pi - e, \pi + e, \pi e \) and \( \frac{\pi}{e} \) all are irrational numbers. The proof is essentially elementary, it is an argument by contradiction.

Notation and reminder

\( \pi \): known as Archimedes constant, is the ratio of a circle’s circumference to its diameter and \( 3 < \pi < 4 \).

\( e = \sum_{m=0}^{\infty} \frac{1}{m!} \): known as Euler’s number and \( 2 < e < 3 \).

\( \mathbb{N}^* := \{1, 2, 3, 4, \ldots\} \) the natural numbers.

\( \mathbb{Z} := \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, \ldots\} \) the integers and \( \mathbb{Z}^* := \mathbb{Z} \setminus \{0\} \).

\( \mathbb{Q} := \left\{ \frac{p}{q} : (p, q) \in \mathbb{Z} \times \mathbb{Z}^* \text{ and } p \wedge q = 1 \right\} \) the set of rational numbers.

\( \mathbb{R} \): the set of real numbers.

\( \mathbb{R} \setminus \mathbb{Q} := \{x \in \mathbb{R} : x \notin \mathbb{Q}\} \) the set of irrational numbers.

\( p \wedge q := \max\{d \in \mathbb{N}^* : d/p \text{ and } d/q\} \) the greatest common divisor of \( p \) and \( q \).

\( \forall \): the universal quantifier and \( \exists \): the existential quantifier.

Introduction

Irrational numbers are the type of real numbers that cannot be expressed in the rational form \( \frac{p}{q} \), where \( p, q \) are integers and \( q \neq 0 \). In simple words, all the real numbers that are not rational numbers are irrational. In this paper we show that \( \sqrt{3} - \sqrt{2} \) and \( \sqrt{3} + \sqrt{2}, e \) and \( \pi, \pi - e, \pi + e, \pi e \) and \( \frac{\pi}{e} \) all are irrational numbers. It is an argument by contradiction.
\[ \pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers} \]

**Theorem 1.** \( \sqrt{6} \in \mathbb{R} \setminus \mathbb{Q} \). In other words, \( \sqrt{6} \) is an irrational number.

**Proof.** An argument by contradiction. Suppose that \( \sqrt{6} \in \mathbb{Q} \), and as \( \sqrt{6} > 0 \) then \( \exists p, q \in \mathbb{N}^* \) such that \( \sqrt{6} = \frac{p}{q} \) and \( p \wedge q = 1 \), then \( (\sqrt{6})^2 = \left(\frac{p}{q}\right)^2 \), then \( 6 = \frac{p^2}{q^2} \) and \( 6q^2 = p^2 \Rightarrow p^2 \) is even and \( p \in \mathbb{N}^* \Rightarrow p \) is even or \( p = 2k \): \( k \in \mathbb{N}^* \Rightarrow 6q^2 = (2k)^2 = 4k^2 \Rightarrow 3q^2 = 2k^2 \) and \( 3 \wedge 2 = 1 \Rightarrow 2 \) divides \( q^2 \) and \( 2 \) is prime \( \Rightarrow 2 \) divides \( q \) and \( q \in \mathbb{N}^* \Rightarrow q \) is even or \( q = 2k' \): \( k' \in \mathbb{N}^* \), hence \( p \wedge q \geq 2 \), and we get a contradiction because \( p \wedge q = 1 \).

**Main Theorem 1.** \( \sqrt{3} - \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \) and \( \sqrt{3} + \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q} \).

In other words, \( \sqrt{3} - \sqrt{2} \) and \( \sqrt{3} + \sqrt{2} \) both are irrational numbers.

**Proof.** An argument by contradiction. First, suppose that \( \sqrt{3} - \sqrt{2} \in \mathbb{Q} \), then \( \exists r \in \mathbb{Q} \) such that \( \sqrt{3} - \sqrt{2} = r \) implies that \( (\sqrt{3} - \sqrt{2})^2 = r^2 \in \mathbb{Q} \Rightarrow 5 - 2\sqrt{6} = r^2 \in \mathbb{Q} \Rightarrow \sqrt{6} = \frac{5 - r^2}{2} \in \mathbb{Q} \), and we get a contradiction.

Second, suppose that \( \sqrt{3} + \sqrt{2} \in \mathbb{Q} \), then \( \exists r \in \mathbb{Q} \) such that \( \sqrt{3} + \sqrt{2} = r \) implies that \( (\sqrt{3} + \sqrt{2})^2 = r^2 \in \mathbb{Q} \Rightarrow 5 + 2\sqrt{6} = r^2 \in \mathbb{Q} \Rightarrow \sqrt{6} = \frac{r^2 - 5}{2} \in \mathbb{Q} \), and we get a contradiction.

**Theorem 2.** \( \forall n \in \mathbb{N}^* \) we have \( \sin(n) \neq 0 \). Several proofs are possible.

**Proof.** Indeed, \( \forall n \in \mathbb{N}^* \) we have \( \cos(n) \in \mathbb{R} \setminus \mathbb{Q} \) see [1, Theorem 2.5], then \( |\cos(n)| \neq 1 \) and \( \cos^2(n) + \sin^2(n) = 1 \Rightarrow \sin(n) \neq 0 \).

**Main Theorem 2.** \( e \in \mathbb{R} \setminus \mathbb{Q} \) and \( \pi \in \mathbb{R} \setminus \mathbb{Q} \).

In other words, \( e \) and \( \pi \) both are irrational numbers.

**Proof.** An argument by contradiction. First, Suppose that \( e \in \mathbb{Q} \), and as \( 2 < e < 3 \) then \( \exists p, q \in \mathbb{N}^* \) such that \( e = \frac{p}{q} \) and \( q > 1 \) and \( p \wedge q = 1 \),

then \( q! e = q! \frac{p}{q} = (q-1)! p \Rightarrow q! e \in \mathbb{N}^* \).

We also have \( q! \sum_{m=0}^{q} \frac{1}{m!} = \sum_{m=0}^{q} \frac{q!}{m!} = q! + q! \frac{1}{2!} + \cdots + 1 \Rightarrow q! \sum_{m=0}^{q} \frac{1}{m!} \in \mathbb{N}^* \), and \( e = \sum_{m=0}^{+\infty} \frac{1}{m!} > \sum_{m=0}^{q} \frac{1}{m!} \Rightarrow q! e > q! \sum_{m=0}^{q} \frac{1}{m!} \) and \( q! e > q! \left( e - \sum_{m=0}^{q} \frac{1}{m!} \right) \in \mathbb{N}^* \).

\[ |x| := \max\{-x, x : x \in \mathbb{R}\} \text{ the absolute value of } x. \]

\[ ]0,1[ := \{x \in \mathbb{R} : 0 < x < 1 \} \text{ the open interval with endpoints } 0 \text{ and } 1. \]
Now, \( q! \left( e - \sum_{m=0}^{q} \frac{1}{m!} \right) = q! \left( \sum_{m=0}^{\infty} \frac{1}{m!} - \sum_{m=0}^{q} \frac{1}{m!} \right) = q! \sum_{m=q+1}^{\infty} \frac{1}{m!} = \sum_{m=q+1}^{\infty} \frac{q!}{m!} \),
and \( 0 < \sum_{m=q+1}^{\infty} \frac{q!}{m!} = \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \)
< \frac{1}{(q+1)} + \frac{1}{(q+1)(q+1)} + \frac{1}{(q+1)(q+1)(q+1)} + \cdots = \sum_{i=1}^{\infty} \frac{1}{(q+1)^i} = \frac{1}{q} \), we get
a contradiction because we have found an integer on \( ]0,1[ \).

Second, suppose that \( \pi \in \mathbb{Q} \), and as \( 3 < \pi < 4 \) then \( \exists p, q \in \mathbb{N}^* \) such that \( \pi = \frac{p}{q} \) and \( p \land q = 1 \Rightarrow p = q \pi \) and \( \sin(p) = \sin(q \pi) = 0 \), we get a contradiction according to [Theorem 2].

**Properties.** The sine function satisfies the following properties:

The sine function (or \( \sin(\theta) \)) is defined, continuous, odd and \( 2\pi \)-periodic on \( \mathbb{R} \).

\[ \forall \theta \in \mathbb{R} \text{ we have } \sin(2k\pi + \theta) = \sin(\theta) \text{ and } \sin(2k\pi - \theta) = -\sin(\theta) : k \in \mathbb{Z}. \]

\[ \forall \theta \in \mathbb{R} \text{ we have } \sin(\theta) = 0 \iff \theta \in \{ k\pi : k \in \mathbb{Z} \}. \]

Let \( \{ \theta_n : n \in \mathbb{N}^* \} \subset \mathbb{R} \) we have \( \lim_{n \to +\infty} \sin(\theta_n) = 0 \iff \lim_{n \to +\infty} \theta_n \in \{ k\pi : k \in \mathbb{Z} \}. \)

**Lemma.** We have \( \lim_{n \to +\infty} \sum_{m=n+1}^{\infty} \frac{n!}{m!} = 0. \)

**Proof.** \( \forall n \in \mathbb{N}^*, \sum_{m=n+1}^{\infty} \frac{n!}{m!} = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots \)
< \frac{1}{n+1} + \frac{1}{(n+1)(n+1)} + \frac{1}{(n+1)(n+1)(n+1)} + \cdots
= \sum_{i=1}^{\infty} \frac{1}{(n+1)^i} = \frac{1}{n}, \)
then \( 0 < \sum_{m=n+1}^{\infty} \frac{n!}{m!} < \frac{1}{n} \) and \( \lim_{n \to +\infty} \frac{1}{n} = 0 \Rightarrow \lim_{n \to +\infty} \sum_{m=n+1}^{\infty} \frac{n!}{m!} = 0. \)

Two other proofs that \( e \) is an irrational number are available at [2, Théorème 15.2] by Dimitris Koukoulopoulos (This proof was found by Fourier in 1815) and at [3] by Jonathan Sondow, and two other proofs that \( \pi \) is an irrational number are available at [4] by Ivan Niven and at [5] by Miklós Laczkovich (This proof was found by Lambert in 1761).
\[ \pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers} \]

**Theorem 3.** We have

\[
\begin{align*}
\lim_{n \to +\infty} \sin \left( n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} \right) &= 0 \\
\lim_{n \to +\infty} \sin \left( n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} \right) &= 0 \\
\lim_{n \to +\infty} \sin \left( n! \pi e - \pi \sum_{m=0}^{n} \frac{n!}{m!} \right) &= 0 \\
\lim_{n \to +\infty} \sin \left( n! pe - p \sum_{m=0}^{n} \frac{n!}{m!} \right) &= 0.
\end{align*}
\]

**Proof.** First,

\[
\begin{align*}
\lim_{n \to +\infty} \sin \left( n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} \right) &= \lim_{n \to +\infty} \sin \left( n! \pi - n! e + \sum_{m=0}^{n} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} \sin \left( n! \pi - \sum_{m=0}^{+\infty} \frac{n!}{m!} + \sum_{m=0}^{n} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} \sin \left( n! \pi - \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} -\sin \left( \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \right) = -\sin(0) = 0.
\end{align*}
\]

Second,

\[
\begin{align*}
\lim_{n \to +\infty} \sin \left( n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} \right) &= \lim_{n \to +\infty} \sin \left( n! \pi + n! e - \sum_{m=0}^{n} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} \sin \left( n! \pi + \sum_{m=0}^{+\infty} \frac{n!}{m!} - \sum_{m=0}^{n} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} \sin \left( n! \pi + \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} \sin \left( \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \right) = \sin(0) = 0.
\end{align*}
\]

Third,

\[
\begin{align*}
\lim_{n \to +\infty} \sin \left( n! \pi e - \pi \sum_{m=0}^{n} \frac{n!}{m!} \right) &= \lim_{n \to +\infty} \sin \left( \pi \sum_{m=0}^{+\infty} \frac{n!}{m!} - \pi \sum_{m=0}^{n} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} \sin \left( \pi \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \right) = \sin(0) = 0.
\end{align*}
\]

Fourth, let \( p \in \mathbb{N}^* \) we have

\[
\begin{align*}
\lim_{n \to +\infty} \sin \left( n! pe - p \sum_{m=0}^{n} \frac{n!}{m!} \right) &= \lim_{n \to +\infty} \sin \left( p \sum_{m=0}^{+\infty} \frac{n!}{m!} - p \sum_{m=0}^{n} \frac{n!}{m!} \right) \\
&= \lim_{n \to +\infty} \sin \left( p \sum_{m=n+1}^{+\infty} \frac{n!}{m!} \right) = \sin(0) = 0.
\end{align*}
\]
Main Theorem 3. \( \pi - e \in \mathbb{R} \setminus \mathbb{Q} \) and \( \pi + e \in \mathbb{R} \setminus \mathbb{Q} \) and \( \pi e \in \mathbb{R} \setminus \mathbb{Q} \) and \( \frac{\pi}{e} \in \mathbb{R} \setminus \mathbb{Q} \). In other words, \( \pi - e, \pi + e, \pi e \) and \( \frac{\pi}{e} \) all are irrational numbers.

Before starting the proof, we recall that \( \forall n \in \mathbb{N}^* \) we have \( n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} > 0 \) and \( n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} > 0 \), and according to [Main Theorem 2] we have \( \{k \pi : k \in \mathbb{Z}\} \subset \mathbb{R} \setminus \mathbb{Q} \cup \{0\} \).

Proof. An argument by contradiction. First, suppose that \( \pi - e \in \mathbb{Q} \), and as \( \pi - e > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \pi - e = \frac{p}{q} \) and \( p \wedge q = 1 \),

then \( \lim_{n \to +\infty} \sin \left( n! (\pi - e) + \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin \left( n! \frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!} \right) \).

We put \( a_n = n! \frac{p}{q} + \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \geq q\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k \pi : k \in \mathbb{Z}\} \),

this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].

Second, suppose that \( \pi + e \in \mathbb{Q} \), and as \( \pi + e > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \pi + e = \frac{p}{q} \) and \( p \wedge q = 1 \),

then \( \lim_{n \to +\infty} \sin \left( n! (\pi + e) - \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin \left( n! \frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!} \right) \).

We put \( a_n = n! \frac{p}{q} - \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \geq q\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k \pi : k \in \mathbb{Z}\} \),

this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].

Third, suppose that \( \pi e \in \mathbb{Q} \), and as \( \pi e > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \pi e = \frac{p}{q} \) and \( p \wedge q = 1 \),

then \( \lim_{n \to +\infty} \sin \left( n! \pi e - \pi \cdot \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin \left( n! \frac{p}{q} - \pi \cdot \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} (-1)^{n+1} \sin \left( n! \frac{p}{q} \right) \).

We put \( a_n = n! \frac{p}{q} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \geq q\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k \pi : k \in \mathbb{Z}\} \),

this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \) and \( \lim_{n \to +\infty} (-1)^{n+1} \sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].
\[
\pi - e, \pi + e, \pi e \text{ and } \frac{\pi}{e} \text{ all are irrational numbers}
\]

Fourth, suppose that \( \frac{\pi}{e} \in \mathbb{Q} \), and as \( \frac{\pi}{e} > 0 \), then \( \exists p, q \in \mathbb{N}^* \) such that \( \frac{\pi}{e} = \frac{p}{q} \) and \( pq = 1 \) implies that \( pe = q\pi \).

Then \( \lim_{n \to +\infty} \sin \left( n!pe - p \cdot \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} \sin \left( n!q\pi - p \cdot \sum_{m=0}^{n} \frac{n!}{m!} \right) = \lim_{n \to +\infty} -\sin \left( p \cdot \sum_{m=0}^{n} \frac{n!}{m!} \right) \).

We put \( a_n = p \cdot \sum_{m=0}^{n} \frac{n!}{m!} : n \in \mathbb{N}^* \), and it is clear that \( a_n \) is strictly increasing and \( \{a_n : n \in \mathbb{N}^*\} \subset \mathbb{N}^* \), then \( \lim_{n \to +\infty} a_n \notin \{k\pi : k \in \mathbb{Z}\} \),

this implies that \( \lim_{n \to +\infty} \sin(a_n) \neq 0 \) and \( \lim_{n \to +\infty} -\sin(a_n) \neq 0 \), and we get a contradiction according to [Theorem 3].

Finally, we conclude that \( \pi - e, \pi + e, \pi e \) and \( \frac{\pi}{e} \) all are irrational numbers.

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References


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