One Tile Suffices

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Abstract

We have found for all \( k \) larger than two a possibility to tile the plane completely with \( k \)-gons. We use infinite many copies of a single tile. The proofs are not by written words, but by pictures. Amongst others, we use the well-known tiling with hexagons. We show for \( k \) larger than 4 new ways to cover the plane.

We think that it is useful to repeat the definition of a simple polygon.

A simple polygon with \( k \) vertices consists of \( k \) different points of the plane \((x_1, y_1), (x_2, y_2), \ldots, (x_{k-1}, y_{k-1}), (x_k, y_k)\), called vertices, and the straight lines between \((x_i, y_i)\) and \((x_{i+1}, y_{i+1})\) for \(1 \leq i \leq k-1\), called edges. Also the straight line between \((x_k, y_k)\) and \((x_1, y_1)\) belongs to the polygon. We demand that it is homeomorphic to a circle, and that there are no three consecutive collinear points \((x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})\) for \(1 \leq i \leq k-2\). Also the three points \((x_k, y_k), (x_1, y_1), (x_2, y_2)\) and \((x_{k-1}, y_{k-1}), (x_k, y_k), (x_1, y_1)\) are not collinear.

We call this just described simple polygon a \( k \)-gon.

**Theorem 1.** Let \( k \) be a natural number larger than 2. Than there exists a tiling of the plane \( \mathbb{R}^2 \) by \( k \)-gons. We need infinite copies of only a single tile.

**Proof.** This theorem is well-known. Please see [1], p. 11.

There is another proof. For \( k = 3 \) and \( k = 4 \) and \( k = 6 \) the theorem is trivial. For \( k = 5 \) please see Figure 1. We take a regular 6-gon and cut it into identical halves. See also the tiling in the case \( k = 6 \).

Now let \( k \) be a natural number larger than 6.

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• Possibility 1: $k \equiv 0 \mod 4$.
The numbers $k$ are $8, 12, 16, \ldots$
See Figure 2. As an example, we show one tile for the case $k = 12$. The measures for the
big square are $4 \times 4$, while the two small squares have measures of $2 \times 2$.

• Possibility 2: $k \equiv 1 \mod 4$.
The sequence of the numbers of $k$ is $9, 13, 17, \ldots$
See Figure 3. There we show one tile for the case $k = 13$.
The big square has sidelengths of $4$, while the small square has sidelengths of $2$. The two
horizontal edges on the left have lengths $4$ and $2$, respectively. They have a distance of $1$.
The two sloped edges both have a length of $\sqrt{2}$.

• Possibility 3: $k \equiv 2 \mod 4$.
The sequence of the numbers of $k$ is $10, 14, 18, \ldots$
See Figure 4. We show a 14-gon. The square has a sidelength of $4$, the rectangle has mea-
sures of $2 \times 4$. The triangle on the left has sidelengths $4, 2, \sqrt{20}$. The triangle on the
right has sidelengths $4$ and $\sqrt{32}$.

• Possibility 4: $k \equiv 3 \mod 4$.
The sequence of the numbers of $k$ is $7, 11, 15, \ldots$
See Figure 5. Here we show a 15-gon.
The square also has a sidelength of $4$, both rectangles have measures of $2 \times 4$. □

Figure 1:
$k = 5$ and $k = 6$

Figure 2:
$k = 12$
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Figure 3:
k = 13

Figure 4:
k = 14

Figure 5:
k = 15

References


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