Topological property of Newton's theory of gravitation

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By assuming that the Ricci curvature tensor consists of a set of subset fields or a set of curvature components, complex scalar fields, in the case of a weak field and the Newtonian limit, we derive the equation of Newton's theory of gravitation in (2+1)-dimensional space-time expressed using the Clebsch variables. These variables obey the topological quantum condition. The Chern-Simons action is interpreted as a gravitational knot.

Keywords: *general theory of relativity, weak field, Chern-Simons action, gravitational knot.*

I. INTRODUCTION

It is commonly believed there exists no topological object in the linear theory, such as Newton's theory of gravitation. It is because a topological theory must be a non-linear theory[1]. How could topological object (a gravitational knot) exist in Newton's linear theory of gravitation as the weak-field limit of Einstein's non-linear theory of gravitation?

We consider that in analogy to the existence of a topological structure in Maxwell's linear theory of vacuum space[1,2], the curvature (the set of the solutions of Einstein field equations) of an empty space-time has a set of subset fields or a set of curvature components with a topological structure. An empty space-time here means that there is no matter present and there is no physical fields exist except the weak gravitational field. The weak gravitational field does not disturb the emptiness of space-time. But other fields disturb[3].

A set of curvature components is locally equal to curvature i.e. curvature can be obtained by patching together a set of curvature components (except in a zero-measure set) but globally different. The difference between a set of curvature components and the curvature in an empty space-time is global instead of local since a set of curvature components obey the topological quantum condition but the curvature does not.

Curvature satisfies a linear field equation, Newton's theory of gravitation, but a set of curvature components satisfies a non-linear field equation. Both, curvature and a set of curvature components, satisfy a linear field equation in the case of a weak field of gravitation. It means that, in the case of a weak field, a non-linear field theory reduces to Newton's linear theory of gravitation.

In this article, we propose there exists a gravitational knot in Newton's theory of gravitation of (2+1)-dimensional empty space-time. This gravitational knot could exist in Newton's theory of gravitation of an empty space-time because Newton's theory of gravitation of an empty space-time is the weak-field limit[2] of a non-linear field theory. To the best of our knowledge[1,4–6], the formulation of a weak field gravitational knot in Newton's theory of gravitation has not been done yet.

II. WEAK-FIELD LIMIT OF GRAVITATION

In the limit of weak gravitational field, low velocities (of sources), and small pressure, the general theory of relativity reduces to Newton's theory of gravitation[7]. In the case of a weak field, linearization (we assume that we ignore the non-linear terms of connection[8]) of the Ricci curvature tensor yields[7]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} \tag{1}$$

This equation is identical to Abelian field strength tensor equation in electromagnetic theory where the curvature (the Ricci curvature tensor), $R_{\mu\nu}$, is identical to the field strength tensor, $F_{\mu\nu}$, and the connection (Christoffel symbol), $\Gamma^\alpha{}_{\mu\alpha}$, is identical to the gauge potential, A_μ.

The time-time component of Ricci curvature tensor (1) can be written as[7]

$$R_{tt} = \partial_\alpha \Gamma^\alpha_{tt} - \partial_t \Gamma^\alpha_{t\alpha} \tag{2}$$

where the second term in the right-hand side of (2) is assumed zero, $\partial_t \Gamma^\alpha{}_{t\alpha} = 0$ (due to the test body of the gravitational source moving very slowly or static). We consider that the choice of the time-time component of the Ricci curvature tensor is the simplest choice which relates to the Ricci curvature tensor in the case of the weak field and the Newtonian limit. Eq.(2) becomes

$$R_{tt} = \partial_\alpha \Gamma^\alpha_{tt} \tag{3}$$

where α denotes the components of (3-dimensional) space.

In the case of a weak-field limit where the source of gravitation is static[8] or moving very slow compared to the speed of light, we could write Newton's theory of gravitation[7,9,10] as a linear equation below

$$R_{tt} = \vec{\nabla} \cdot \vec{g} = \vec{\nabla} \cdot \vec{\nabla}\phi = \nabla^2\phi \tag{4}$$

where R_{tt} is the time-time component of the Ricci curvature tensor, $\vec{g}$ is the gravitational field, ϕ is the (scalar) potential of gravitation, ∇^2 (div of grad) is the Euclidean Laplacian operator with respect to space, and

$$\nabla^2\phi = 4\pi\rho \tag{5}$$

is Poisson's equation[7], ρ is the mass density.

By substituting eq.(5) into eq.(4) we obtain Newton's theory of gravitation expressed as Newtonian field equation[7]

$$R_{tt} = 4\pi\rho \tag{6}$$

We see from eqs.(4),(6), the time-time component of Ricci curvature tensor is dominant. It is because, as we had mentioned, we ignore the non-linear terms of connection i.e. the higher order of the space and the mixed components of the Ricci curvature tensor. Physically, it means that in the case of infinite distance from the gravitational source, the gravitational field (the perturbation) is so weak.

III. SET OF SUBSET FIELDS PROPERTY AND MAPS $S^3 \to S^2$

Let us consider maps of a set of subset fields (consisting of complex scalar fields) from a finite radius r to an infinite r implies from the stronger field to the weak field. A scalar field has properties that, by definition, its value for a finite r depends on the magnitude and the direction of the position vector, $\vec{r}$, but for an infinite r it is well-defined[2] (it depends on the magnitude only). In other words, for an infinite r, a scalar field is isotropic. Throughout this article, we will work with the classical scalar field.

The property of a set of subset fields can be interpreted as maps

$$\text{a set of subset fields} : S^3 \to S^2 \tag{7}$$

where S^3 and S^2 are 3-dimensional and 2-dimensional spheres respectively i.e. after identifying via stereographic projection, 3-dimensional physical space, $R^3 \cup \{\infty\}$, with the sphere S^3 and the complete complex plane, $C \cup \{\infty\}$, with the sphere S^2[1]. These maps can be classified in homotopy classes labeled by the value of the corresponding Hopf indexes, integer numbers, and the topological invariants[1,2]. The other names of the topological invariants are the topological charge, and the winding number (the degree of a continuous mapping)[11]. The topological charge is independent of the metric tensor, it can be interpreted as energy[12].

We see there exists (one) dimensional reduction in such maps (7). We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of (complex) scalar fields for an infinite r. The property of scalar fields as a function of space seems likely in harmony with the property of space-time itself. Space-time could be locally anisotropic but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

IV. HOPF INVARIANT AND ABELIAN CHERN-SIMONS

Let us discuss the maps above more formally. As we mentioned we have complex scalar fields as a function of the position vector, $e^a(\vec{r})$, $e^{a^*}(\vec{r})$, with a property that can be interpreted using the non-trivial Hopf maps written below[1,2]

$$e^a(\vec{r}),\ e^{a^*}(\vec{r}) : S^3 \to S^2 \tag{8}$$

These non-trivial Hopf maps are related to the Hopf invariant[13], $\mathcal{H}$, expressed as an integral[13–15]

$$\mathcal{H} = \int_{S^3} \omega \wedge d\omega \tag{9}$$

where ω is a 1-form on S^3[13] and $d\omega$ is a 2-form. We see eq.(9) is identical to the formulation of circulation in hydrodynamics[21] where circulation is identical to Hopf invariant, ω and $d\omega$ are identical to velocity field and vorticity, respectively.

The relation between the Hopf invariant and the Hopf index, h, can be written explicitly as[1]

$$\mathcal{H} = h\ \gamma^2 \tag{10}$$

where γ is the total strength of the field which is the sum of the strengths of all the tubes formed by the integral lines of electric and magnetic fields[1]. Related to gauge theory and magnetohydrodynamics (self-helicity), it can be interpreted naturally that the Hopf invariant has a deep relationship with the Chern-Simons action (the Chern-Simons integral)[13]. We will see that the Hopf invariant is identical to the Chern-Simons action itself.

The Hopf invariant is just the winding number of Gauss mapping[13] (so probably, there exists a relationship between Gauss mapping and non-trivial Hopf maps). Hopf invariant or the Chern-Simons integral is an important topological invariant to describe the topological characteristics of the knot family[13,16]. In a more precise expression, the Hopf invariant or the Chern-Simons integral is the total sum of all the self-linking and all the linking numbers of the knot family[13,16]. The self-linking and linking numbers by themselves have a topological structure.

V. NON-LINEAR FIELD AND LINEARIZED RICCI THEORIES

We assume that the property of a set of curvature components consisting of complex scalar fields, e^a, e^{a^*}, could be described by the maps of (3+1) to (2+1)-dimensional space-time gravitational theory written below

$$e^a(\vec{r},t),\ e^{a^*}(\vec{r},t) : M^{3+1} \to M^{2+1} \tag{11}$$

where M denotes manifold. We see from eq.(8) that complex scalar fields in non-trivial Hopf maps are written as $e^a(\vec{r})$, $e^{a^*}(\vec{r})$, i.e. time-independent complex scalar fields.

It differs from time-dependent complex scalar fields written as $e^a(\vec{r},t)$, $e^{a^*}(\vec{r},t)$, in eq.(11). This problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[17].

These maps (11) have a consequence (by considering that the field strength is identical to the curvature) that we could write the Ricci curvature tensor as

$$R^a_{\mu\nu} = \frac{\sqrt{c}}{2\pi i}\left(\frac{\partial_\mu e^{a^*}\partial_\nu e^a - \partial_\nu e^{a^*}\partial_\mu e^a}{(1+e^{a^*}e^a)^2}\right) \tag{12}$$

where e^a is the set of components of the Ricci curvature tensor, and e^{a^*} is the complex conjugate of e^a. In analogy to non-linear field theory in electromagnetism[1], we consider c as an action constant, introduced so that the Ricci curvature tensor will have suitable dimensions for the curvature. Eq.(12) is the non-linear equation where the nonlinearity is shown by the $e^{a^*}e^a$ term in the denominator. The superscript index a in e^a represents a set of indices that label the components of curvature.

In the case of a weak field of gravitation, the complex scalar fields are very small, $|e^{a^*}e^a| << 1$, so eq.(12) reduces to a linear equation as written below

$$R^a_{\mu\nu} = \frac{\sqrt{c}}{2\pi i}\left(\partial_\mu e^{a^*}\partial_\nu e^a - \partial_\nu e^{a^*}\partial_\mu e^a\right) \tag{13}$$

This linear equation (13) is equivalent to eq.(1). It means that the linearized Ricci theory (1) could be interpreted as the same as the Ricci theory in the case of a weak field (13).

VI. SCALAR AND VECTOR POTENTIALS

We consider a set of curvature components, complex scalar fields, $e^a(\vec{r},t)$, $e^{a^*}(\vec{r},t)$, as scalar potentials and it could be interpreted similarly to linearized metric perturbations. Linearized metric perturbations take a role as "potentials" in linearized gravitation identical to electric (scalar) and magnetic (vector) potentials in electromagnetism[18]. Linearized (small) metric perturbations

$$h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta} \tag{14}$$

can be written as[18]

$$h_{\alpha\beta} = \rho_{\alpha\beta}\ e^{i\vec{k}\cdot\vec{r}} \tag{15}$$

where $g_{\alpha\beta}$ is the metric tensor, $\eta_{\alpha\beta}$ is the metric of flat space-time, $\rho_{\alpha\beta}$ is amplitude and $\vec{k}$ is wave vector. Small metric perturbations mean that $|h_{\alpha\beta}| << 1$ for all α and β. The subscript index α, β, represent space-time coordinates. In an empty space-time, a weak field, the amplitude is constant. Eq.(15) shows us that the linearized metric perturbations can be understood in terms of the wave.

In analogy to eq.(15), we propose that the scalar field or the scalar potential and the related vector potential could be written in terms of the wave, respectively as[19]

$$e^a = \rho^a e^{iq} \tag{16}$$

and

$$e^a_\rho = f^a\ \partial_\rho q \tag{17}$$

where ρ^a is the amplitude, q is the phase, the notation e in e^{iq} is exponential ($e^{iq} = \exp(iq)$), $f^a = -1/\left\{2\pi[1+(\rho^a)^2]\right\}$, f^a and q are the Clebsch variables[17]. The subscript index ρ in e^a_ρ represents space-time coordinates.

We consider that the Ricci curvature tensor (13) is identical to the field strength tensor of electromagnetic, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By using eq.(17), the Ricci curvature tensor (13) could be written as[17]

$$R^a_{\mu\nu} = \frac{\sqrt{c}}{2\pi i}\ \{\partial_\mu(f^a\ \partial_\nu q) - \partial_\nu(f^a\ \partial_\mu q)\} \tag{18}$$

This is the Ricci curvature tensor written in term of the Clebsch variables.

The time-time component of the linear Ricci curvature tensor (18) could be written as

$$R^a_{tt} = \frac{\sqrt{c}}{2\pi i}\{\partial_\alpha(f^a\ \partial_\alpha q) - \partial_t(f^a\ \partial_t q)\} \tag{19}$$

where the index α denotes the space component (space coordinate). The second term on the right-hand side of (19) is equal to zero. It is because, in the Newtonian limit, it is considered that the speed of the body as the gravitational source is very slow compared to the speed of light. So eq.(19) becomes

$$R^a_{tt} = \frac{\sqrt{c}}{2\pi i}\ \partial_\alpha(f^a\ \partial_\alpha q) \tag{20}$$

Eq.(20) is the equation of Newton's theory of gravitation expressed using the Clebsch variables. The first ∂_α means divergence and the second is gradient. Roughly speaking, eq.(20) says that the source of the potential of gravitation is the curvature. Eq.(20) is equivalent to (3).

VII. A GRAVITATIONAL KNOT

Roughly speaking, in three or (2+1)-dimensional space-time of the general theory of relativity, the dynamics is topology[22]. The (2+1)-dimensional general theory of relativity could be interpreted as a Chern-Simons three form[23] where Chern-Simons theory is topological gauge theory in three dimensions[22]. Chern-Simons theory was discovered in the context of anomalies and used as a rather exotic toy model for gauge systems in (2+1) dimensions ever since[25]. The Chern-Simons action precisely coincides with the (2+1)-dimensional space-time of the Einstein-Hilbert action[23,24].

The (2+1)-dimensional space-time of Abelian Chern-Simons action could be written as[23,24]

$$S_{CS} = \int_M \varepsilon^{\mu\nu\rho}\ e^a_\rho\ R^a_{\mu\nu}\ d^3r \tag{21}$$

where $\varepsilon^{\mu\nu\rho}$ is the Levi-Civita symbol. By substituting eqs.(17), (18), into eq.(21) we obtain

$$S_{CS} = \frac{\sqrt{c}}{2\pi i}\int_M \varepsilon^{\mu\nu\rho}\ f^a\ \partial_\rho q\left\{\partial_\mu(f^a\ \partial_\nu q) - \partial_\nu(f^a\ \partial_\mu q)\right\} d^3r \tag{22}$$

The action, S_{CS}, (22) is related to a topological object i.e. a knot[23], a gravitational knot (a gravitational helicity), an integer number in the case of a weak field. This integer number is what we mean with a set subset fields or a set of curvature components obeying the topological quantum condition.

In the case of a weak field and the Newtonian limit, by substituting eq.(20) into (21), we obtain

$$S_{CS} = \frac{\sqrt{c}}{2\pi i}\int_M \varepsilon^{\mu\nu\rho}\ f^a\ \partial_\rho q\ \partial_\alpha(f^a\ \partial_\alpha q)\ d^3r \tag{23}$$

The action, S_{CS}, (23) takes a role as a gravitational knot in the case of a weak field and the Newtonian limit.

VIII. DISCUSSION AND CONCLUSION

The proposal that curvature i.e. Ricci curvature tensor has a set of subset fields or a set of curvature components, complex scalar fields (scalar potentials) has deep and far-reaching consequences. One of the consequences is that we can formulate the Ricci curvature tensor in non-linear form using the scalar field and its conjugate complex (12).

In the case of an empty space-time or weak field, the non-linear Ricci curvature tensor (12) reduces to the linear Ricci curvature tensor (13) where Newton's theory of gravitation in the form of a subset field, a scalar field, could be derived from eq.(13). The linearized Ricci curvature tensor (13) is locally equivalent to eq.(1), but globally different. Eq.(1) is no longer valid globally.

We assume that a subset field, a scalar field, or a component of Ricci curvature tensor, as a map of gravitational theory in (3+1) to (2+1)-dimensional space-time. It implies there exists (one) dimensional reduction in such a map. We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a subset field, a scalar field, for an infinite r i.e. for infinite distance from the source the gravitational field is weak. It implies also that the linearized Ricci curvature tensor and its derived Newton's theory of gravitation can be formulated in (2+1)-dimensional space-time.

The remarkable one, as we mentioned that the (2+1)-dimensional general theory of relativity could be interpreted as a Chern-Simons (topological gauge theory) three form, it has a consequence that we could relate and interpret (2+1)-dimensional linearized Ricci curvature tensor (13) and its derived Newton's theory of gravitation as Chern-Simons three form in (2+1)-dimensional space-time where its action is related to a gravitational knot, an integer number (22). It means that the subset fields obey the topological quantum condition.

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