Topology of the Newtonian limit in (2+1)-dimensional empty space-time

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We propose the existence of a topological object i.e. a Newtonian knot of a weak field with a small positive cosmological constant and very slow motion compared to the velocity of light (Newtonian limit) in (2+1)-dimensional empty space-time. The Ricci curvature tensor consists of a set of curvature components, complex scalar potentials, where their properties could be described by the non-trivial Hopf maps. The very slow motion is shown by the space components only of the Abelian Ricci curvature tensor. The Abelian Chern-Simons action is interpreted as such a knot.

Keywords: *general relativity, Newtonian limit, (2+1)-dimensional empty space-time, cosmological constant, Abelian Chern-Simons action, knot.*

I. INTRODUCTION

It is widely believed that topological objects are impossible to exist in linear theories. Topological theories are inherently non-linear[1]. How, then, could a topological object, like a gravitational knot, exist in the linear (Abelian) theory? We will work in (2+1)-dimensional space-time gravity theory instead of (3+1)-dimensional space-time throughout this article for a reason that will be clear later.

In analogy to the linearized Ricci curvature tensor in (3+1)-dimensional space-time, we assume that the Abelian Ricci curvature tensor as the weak-field limit with a small positive cosmological constant of the gravity theory is also valid in (2+1)-dimensional space-time. The existence of a topological structure in Maxwell's linear theory of vacuum space is similar to the existence of a topological structure in Abelian gravity theory in (2+1)-dimensional empty space-time.

In analogy to Maxwell's linear theory of vacuum space where the field strength could consist of a set of subset fields[1,2], we propose that the Ricci curvature tensor (the set of the solutions of Einstein field equations) in (3+1)-dimensional empty space-time could consist of a set of curvature components, complex scalar potentials. We assume that it is also valid in the case of (2+1)-dimensional empty space-time. Both, a set of subset fields and a set of curvature components satisfy the non-trivial Hopf maps. It means that the properties of a set of subset fields and a set of curvature components could be described by the non-trivial Hopf maps.

A set of curvature components is locally equal to the linearized Ricci curvature tensor, i.e. the linearized Ricci curvature tensor can be obtained by patching together a set of curvature components (except in a zero-measure set) but globally different. The difference between a set of curvature components and the linearized Ricci curvature tensor in an empty space-time is global, instead of local, since a set of curvature components obeys the topological quantum condition, but the linearized Ricci curvature tensor does not.

The linearized Ricci curvature tensor satisfies the linear Ricci theory, but a set of curvature components satisfies the non-linear Ricci theory. Both, the linearized Ricci curvature tensor and a set of curvature components, satisfy the linear Ricci theory in the case of a weak-field limit. It means that, in the case of a weak-field limit, the non-linear Ricci theory reduces to the linear Ricci theory.

Maxwell's theory of electromagnetism and Einstein's theory of gravitation (general relativity) are identical where the gauge potential and the field strength tensor in Maxwell's theory (in general, a non-Abelian gauge theory, such as Yang-Mills theory) are identical to the connection (Christoffel symbols) and the curvature in general relativity, respectively[3]. Both theories are the gauge theories, where Maxwell's theory is an Abelian $U(1)$ gauge theory of internal space and general relativity is the gauge theory of translation of (3+1)-dimensional (external) space-time[4].

The vierbein formalism of general relativity makes general relativity similar to a gauge theory[5]. Nevertheless, in the case of (3+1)-dimensional space-time, general relativity and a gauge theory are definitely not equivalent. But, in (2+1)-dimensional space-time, general relativity and a gauge theory are precisely equivalent[5].

Roughly speaking, general relativity in (2+1)-dimensional space-time is the simpler model than general relativity in (3+1)-dimensional space-time that shares the important conceptual features of general relativity while avoiding some of the computational difficulties[6]. As a generally covariant theory of space-time geometry, (2+1)-dimensional gravity has the same conceptual foundation as realistic (3+1)-dimensional general relativity. With a few exceptions, (2+1)-dimensional solutions are physically quite different from those in 3+1 dimensions. The 2+1 dimensional model is not very helpful for understanding the dynamics of realistic quantum gravity. But for the analysis of conceptual problems - the nature of time, the construction of states and observable, the role of topology and topology change - the model has proven highly instructive[6].

In (2+1)-dimensional space-time of general theory relativity, the dynamics is topology[7]. The (2+1)-dimensional general relativity could be interpreted as a Chern-Simons three form[5] where Chern-Simons theory is topological gauge theory in (2+1)-dimensional space-time[7]. The Chern-Simons action precisely coincides with the (2+1)-dimensional space-time of the Einstein-Hilbert action[5,8]. The Chern-Simons theory was discovered in the context of anomalies and used as a rather exotic toy model for gauge systems in (2+1)-dimensional space-time[9].

The Einstein-Hilbert action in (2+1)-dimensional space-time, without a cosmological constant, is equivalent to a gauge theory with gauge group ISO(2,1) and a pure Chern-Simons action[5]. If we include a cosmological constant in (2+1) general relativity, then Minkowski (flat) space-time is replaced by space-time with a constant curvature: de Sitter or anti-de Sitter depending on the sign of a cosmological constant, and gauge group ISO(2,1) is replaced by SO(3,1) or SO(2,2)[5].

If the relation between general relativity and Chern-Simons gauge theory is valid at the quantum level, then there is a close relationship between general relativity and knot theory, at least in (2+1)-dimensional space-time, since Chern-Simons gauge theory in (2+1)-dimensional space-time is intimately connected with knot theory[5]. We consider the quantum level here to be related to topology.

The formulation of a gravitational knot for a non-Abelian Chern-Simons action in (2+1)-dimensional empty space-time has been done[5,7,8,10,11]. In this article, we propose that there exists an Abelian gravitational knot of the weak field in (2+1)-dimensional empty space-time with a small positive cosmological constant written using the Clebsch variables. This Abelian gravitational knot is formulated by an Abelian Chern-Simons action. To the best of our knowledge[1,5,7–17], the formulation of such knot has not been done yet.

This article is organized as follows. In Sect. II, we discuss in brief gravity theory in (3+1)-dimensional space-time. In Sect. III, gravity theory in (2+1)-dimensional space-time. In Sect. IV, we try to identify the relation between the Einstein-Hilbert and the Chern-Simons actions in (2+1)-dimensional space-time. In Sect. V, the relation between a set of subset fields and the non-trivial Hopf maps $S^3 \to S^2$ is discussed. In Sect. VI, we discuss the relations between the Hopf maps, the Hopf invariant, the Hopf index, and the Chern-Simons action. In Sect. VII, the concept of small metric perturbations is described and we interpret these small metric perturbations as the scalar potentials. The related vector potential is written using the Clebsch scalar variables. In Sect. VIII, we propose that in analogous to a set of subset fields, we have a set of curvature components. In Sect. IX, we show that as the consequences of the Hopf maps for the set of curvature components: we could construct the non-linear and linear Ricci theories. In Sect. X, we treat the vector potential as the gauge potential and this gauge potential could be related to the gauge fields (the vierbein and the spin connection). In Sect. XI, the gravitational knot is formulated as the Abelian Chern-Simons action. We give a discussion and conclusion in Sect. XII.

II. (3+1) GRAVITY

The Einstein field equation in (3+1)-dimensional space-time can be written as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi G\ T_{\mu\nu} \tag{1}$$

where

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\ R \tag{2}$$

$G_{\mu\nu}$ is Einstein tensor, $R_{\mu\nu}$ is Ricci curvature tensor, $R_{\mu\nu} = R^{\alpha}{}_{\mu\alpha\nu}$, $R^{\alpha}{}_{\mu\alpha\nu}$ is Riemann curvature tensor, $g_{\mu\nu}$ is metric tensor, R is Ricci scalar curvature, Λ is a cosmological constant, G is the gravitational coupling constant - the generalization to other dimensions of Newton's constant[10], $T_{\mu\nu}$ is the energy-momentum tensor of matter.

A. $T_{\mu\nu} = 0$, $\Lambda = 0$

What we mean with an empty space-time is a vacuum space-time, $R_{\mu\nu} = 0$, where there is no matter source present, $T_{\mu\nu} = 0$, and there exists no physical fields except the gravitational field. The gravitational field does not disturb the emptyness, but other fields do[18].

In the absence of matter and without cosmological constant, the Einstein field equation read[11]

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\ R = 0 \tag{3}$$

In general, the vanishing of $G_{\mu\nu}$, hence of $R_{\mu\nu}$ and R, does not imply that the Riemann curvature tensor is zero, i.e. the space-time need not be flat[11]. Einstein made the assumption that in (3+1)-dimensional empty space-time, it constitutes his law of gravitation[18]. However, in (2+1)-dimensional space-time the situation is different.

B. $T_{\mu\nu} = 0$, $\Lambda \neq 0$

In this article, we will not discuss the gravity theory with a non-zero cosmological constant in (3+1)-dimensional empty space-time.

C. Weak-field limit of (3+1) gravity

In the case of a weak-field limit, the linearization (assume that we ignore the non-linear terms of connection[19]) of the Ricci curvature tensor yields[20]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} \tag{4}$$

This equation of the linearized Ricci curvature tensor is identical to the equation of an Abelian field strength tensor in Maxwell's theory as written below

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5}$$

By comparing eqs.(4),(5), we see that the curvature (the Ricci tensor), $R_{\mu\nu}$, is identical to the field strength tensor, $F_{\mu\nu}$, and the connection (Christoffel symbols), $\Gamma^\alpha{}_{\mu\alpha}$, is identical to the gauge potential, A_μ. The identical form of eqs.(4),(5) shows the beautiful relationship between mathematics and physics.

III. (2+1) GRAVITY

In (2+1)-dimensional space-time manifold, M, Einstein-Hilbert action for gravity coupled to matter can be written as[6,21]

$$I_{\text{EH}} = \frac{1}{16\pi G} \int_M d^{2+1}x \, \sqrt{-g} \, (R - 2\Lambda) + I_{\text{matter}} \tag{6}$$

Equation of motion for the action (6) are[6,21]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \, R + \Lambda g_{\mu\nu} = -8\pi G \, T_{\mu\nu} \tag{7}$$

We see that the equation of motion (7) is the same as (1). Eq. (7) are generally covariant, i.e. they are invariant under the action of the group of diffeomorphisms of the space-time, which can be viewed as a gauge group[6].

In (2+1)-dimensional space-time, the relation between Einstein tensor and Riemann curvature tensor can be written as[11]

$$G^\mu_\nu = -\frac{1}{4} \varepsilon^{\mu\alpha\beta} \, \varepsilon_{\nu\gamma\delta} \, R_{\alpha\beta}{}^{\gamma\delta} \tag{8}$$

Eq.(8) may be inverted as[10]

$$R^{\alpha\mu}{}_{\beta\nu} = \varepsilon^{\alpha\mu\gamma} \, \varepsilon_{\beta\nu\delta} \, G^\delta_\gamma \tag{9}$$

A. $T_{\mu\nu} = 0$, $\Lambda = 0$

Equation (9) without a cosmological constant implies that if the Einstein tensor vanishes (as a consequence of the absence of matter) then the Riemann curvature tensor vanishes. In turn, the vanishing Riemann tensor implies that the Ricci tensor and Ricci scalar are equal to zero. So, the solution of eq.(9) is flat space-time. The theory is trivial[11]. The theory does not possess any propagating degrees of freedom[11].

B. $T_{\mu\nu} = 0$, $\Lambda \neq 0$

In the case of empty space-time and a non-zero cosmological constant, eq.(1) can be replaced by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{10}$$

and by substituting eq.(10) into eq.(9), we obtain[11]

$$R_{\alpha\mu\beta\nu} = -\Lambda (g_{\alpha\beta} \, g_{\mu\nu} - g_{\alpha\nu} \, g_{\beta\mu}) \tag{11}$$

which shows that without sources, all spaces that solve (10) are of constant curvature: a closed $R_1 \times S_2$ de Sitter space for $\Lambda > 0$ or a hyperbolic anti-de Sitter space for $\Lambda < 0$[11]. We consider that the constant curvature indicates that the geometry of space-time is locally homogeneous[5] and isotropic in the sense that curvature is the same or uniform everywhere.

Eq.(10) implies that the Ricci curvature tensor can be written as

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \, R - \Lambda g_{\mu\nu} \tag{12}$$

It means that the Ricci curvature tensor is not simply proportional to the metric tensor $g_{\mu\nu}$ scaled by a constant R, but also has an additional term involving the cosmological constant, Λ.

IV. COULD (2+1) EINSTEIN-HILBERT ACTION BE INTERPRETED AS (2+1) CHERN-SIMONS ACTION?

A. The Einstein-Hilbert action without a cosmological constant

For a (2+1)-dimensional space-time manifold, the Einstein-Hilbert action without a cosmological constant would be[5]

$$I_{\text{EH}} = \frac{1}{2} \int_M \varepsilon^{\mu\nu\rho} \, \varepsilon_{abc} \left\{ e^a_\mu \left(\partial_\nu \omega^{bc}_\rho - \partial_\rho \omega^{bc}_\nu + [\omega_\nu, \omega_\rho]^{bc} \right) \right\} d^{2+1}x \tag{13}$$

where e^a_μ is a vierbein, ω^{bc}_ρ is a spin connection, $\varepsilon^{\mu\nu\rho}$, ε_{abc} are the Levi-Civita symbols in the space-time and internal space, respectively. If e^a_μ and ω^{bc}_ρ are interpreted as gauge fields, it might conceivably be interpreted as a Chern-Simons three form[5].

From eq.(13), we could define the Ricci curvature tensor as

$$R_{\nu\rho}{}^{bc} = \partial_\nu \omega_\rho{}^{bc} - \partial_\rho \omega_\nu{}^{bc} + [\omega_\nu, \omega_\rho]^{bc} \tag{14}$$

Eq.(14) is a non-linear equation. The nonlinearity is shown by the commutation relation in the third term of the right-hand side (14). This term $[\omega_\nu, \omega_\rho]^{bc}$ represents the self-interaction of the spin connection. The non-zero value of this term contributes to the curvature of space-time.

B. The Chern-Simons action (without cosmological constant)

An integral on a (2+1)-dimensional space-time manifold of the Chern-Simons action can be written as[5,10]

$$I_{\rm CS} = \int_M \varepsilon^{\mu\nu\rho} \left\{ e_{\mu a} \left(\partial_\nu {\omega_\rho}^a - \partial_\rho {\omega_\nu}^a + \varepsilon_{abc}\ {\omega_\nu}^b {\omega_\rho}^c \right) \right\} d^{2+1}x \tag{15}$$

We see that the ISO(2,1) Chern-Simons action (15) precisely coincides with the Einstein-Hilbert action in (2+1)-dimensional space-time (13)[5].

C. The Einstein-Hilbert action with a cosmological constant

If we include a non-zero cosmological constant, the generalized Einstein-Hilbert action in (2+1)-dimensional space-time is[5]

$$I_{\rm EH} = \int_M \varepsilon^{\mu\nu\rho} \left\{ e_{\mu a} \left(\partial_\nu \omega_\rho^a - \partial_\rho \omega_\nu^a \right) + \varepsilon_{abc}\ e_\mu^a\ \omega_\nu^b \omega_\rho^c + \frac{\Lambda}{3}\ \varepsilon_{abc}\ e_\mu^a\ e_\nu^b e_\rho^c \right\} d^{2+1}x \tag{16}$$

The equations of motion (16) now say not that space-time is flat but locally homogeneous, with curvature proportional to Λ[5].

The simply connected covering space of such a space-time is not a portion of Minkowski space, but a portion of de Sitter or anti-de Sitter space. The spaces of de Sitter and anti-de Sitter have for their symmetries SO(3,1) and SO(2,2), respectively, not ISO(2,1) as in Minkowski flat space-time[5]. Thus, it is reasonable to guess that if the gravity theory without a cosmological constant in (2+1)-dimensional space-time is related to the gauge theory of ISO(2,1), then the gravity theory with a cosmological constant in (2+1)-dimensional space-time will be related to gauge theory of SO(3,1) and SO(2,2)[5].

We see from eq.(16), the Ricci curvature tensor could be written as[8]

$$R^a_{\nu\rho} = \partial_\nu \omega_\rho^a - \partial_\rho \omega_\nu^a + \omega_\nu^a\ \omega_\rho^a + \frac{\Lambda}{3}\ e_\nu^a\ e_\rho^a \tag{17}$$

In terms of the spin connection, eq.(17) is a non-linear equation due to there exists $\omega_\nu^a\ \omega_\rho^a$ term as in eq.(14).

D. Weak-field limit and small positive cosmological constant

In the case of weak-field limit and a small positive cosmological constant, $\Lambda > 0$, $|\Lambda| << 1$, eq.(17) reduces to Abelian Ricci curvature tensor written below

$$R^a_{\nu\rho} = \partial_\nu \omega_\rho^a - \partial_\rho \omega_\nu^a + \frac{\Lambda}{3}\ e_\nu^a\ e_\rho^a \tag{18}$$

At first sight, eq.(18) looks like a non-linear equation, because we see there exists a quadratic form (as a product of the vierbein components) in the third term of eq.(18). Although there exists such a quadratic form, eq.(18) is a linear equation. It is because the vierbein components, e_ν^a, e_ρ^a, can be viewed as fixed fields. Fixed fields here refer to fields that are considered given or fixed externally, parameters. They are not variables being solved for. The fixed vierbein fields due to the cosmological constant introduce a source term that is imposed on the curvature. So, in terms of the spin connection, the equation (18) remains linear.

E. Newtonian limit

In the case of weak field, a small positive cosmological constant, $\Lambda > 0$, $|\Lambda| << 1$, and very slow motion, eq.(18) reduces to

$$R^a_{jk} = \partial_j \omega_k^a - \partial_k \omega_j^a + \frac{\Lambda}{3}\ e_j^a\ e_k^a \tag{19}$$

where $j, k = 1, 2$, denote the spatial indices. Here, ∂_j and ∂_k represent derivatives with respect to the spatial coordinates only, and ω_j^a and e_j^a are the spatial components of the spin connection and the vierbein, respectively. The assumption of slow motion implies that the terms involving time derivatives are negligible compared to the spatial derivatives. Eq.(19) is the Ricci curvature tensor in slow-motion which we could treat it as the time-independent curvature.

F. The Chern-Simons action with a cosmological constant

Without proof, the Chern-Simons action with non-zero cosmological constant could be written as[5,8]

$$I_{\rm CS} = \int_M \varepsilon^{\mu\nu\rho}\ e_{\mu a} \times \left\{ \partial_\nu \omega_\rho^a - \partial_\rho \omega_\nu^a + \varepsilon^a_{bc} \left(\omega_\nu^b \omega_\rho^c + \frac{\Lambda}{3}\ e_\nu^b e_\rho^c \right) \right\} d^{2+1}x \tag{20}$$

From eq.(20), the non-Abelian curvature can be written as

$$R^a_{\nu\rho} = \partial_\nu \omega_\rho^a - \partial_\rho \omega_\nu^a + \varepsilon^a_{bc} \left(\omega_\nu^b \omega_\rho^c + \frac{\Lambda}{3}\ e_\nu^b e_\rho^c \right) \tag{21}$$

Here, the Levi-Civita symbol appears in the term involving the spin connection and the vierbein. It differs from eq.(17) where there is no such the Levi-Civita symbol in the term involving the spin connection and the vierbein. It shows that the related Chern-Simons action (20) describes the gauge fields with an internal symmetry. In the case of the Einstein-Hilbert action (16), the Levi-Civita symbol is used to contract the indices of the spin connection and the vierbein.

In the case of a weak gauge field limit and a small value of the positive cosmological constant, eq.(21) reduces to Abelian curvature written below

$$R^a_{\nu\rho} = \partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu + \frac{\Lambda}{3}\ \varepsilon^a_{bc}\ e^b_\nu e^c_\rho \qquad (22)$$

Eq.(22) is a linear equation in terms of the spin connection. The reason is analogous to eq.(18).

In the case of Newtonian limit where the motion is very slow compared to the velocity of light, eq.(22) reduces to

$$R^a_{jk} = \partial_j \omega^a_k - \partial_k \omega^a_j + \frac{\Lambda}{3}\ \varepsilon^a_{bc}\ e^b_j e^c_k \qquad (23)$$

We see that eq.(23) is similar to eq.(19). The only different is there exists Levi-Civita symbol in the second term of eq.(23) but there is not in eq.(19). This Levi-Civita symbol shows that Chern-Simons theory works in internal space.

G. The Abelian Chern-Simons action with a cosmological constant

The Abelian Chern-Simons action with non-zero cosmological constant can be obtained from eq.(20) by replacing the non-Abelian curvature (21) with Abelian curvature (22), then we have

$$I_{\rm CS} = \int_M \varepsilon^{\mu\nu\rho}\ e_{\mu a} \left(\partial_\nu \omega^a_\rho - \partial_\rho \omega^a_\nu + \frac{\Lambda}{3}\ \varepsilon^a_{bc}\ e^b_\nu e^c_\rho\right) d^{2+1}x \qquad (24)$$

In the case of Newtonian limit, eq.(24) reduces to

$$I_{\rm CS} = \int_M \varepsilon^{ijk}\ e_{ia} \left(\partial_j \omega^a_k - \partial_k \omega^a_j + \frac{\Lambda}{3}\ \varepsilon^a_{bc}\ e^b_j e^c_k\right) d^{2+1}x \qquad (25)$$

We will use this Abelian Chern-Simons action (26) to formulate the Newtonianl knot.

In the following, we need to reformulate the gauge potential written using the Clebsch scalar variables related to the vierbein and the spin connection. First, it is necessary to show that a set of subset fields satisfies the non-trivial Hopf maps $S^3 \to S^2$. Physically, it is because the properties of a set of subset fields could be described by Hopf maps. Analogous to a set of subset fields that satisfies the non-trivial Hopf maps, we have a set of curvature components that also satisfy the non-trivial Hopf maps. The time-independent problems of a set of curvature components, as in the case of a set of subset fields, could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values.

V. A SET OF SUBSET FIELDS AND HOPF MAPS $S^3 \to S^2$

Let us consider maps of a set of subset fields consisting of the complex scalar fields as a function of the position vector, $\phi(\vec{r}),\ \phi^*(\vec{r})$, from a finite radius r to an infinite r implies from the stronger field to the weak field. A scalar field has properties that, by definition, its value for a finite r depends on the magnitude and the direction of the position vector, but for an infinite r it is well-defined[2] (it depends on the magnitude only). In other words, for an infinite r, a scalar field is isotropic. Throughout this article, we will work with the classical scalar field.

The properties of the complex scalar fields can be interpreted as the non-trivial Hopf maps written below[1]

$$\phi(\vec{r}),\ \phi^*(\vec{r}) : S^3 \to S^2 \qquad (26)$$

where S^3 and S^2 are three and two dimensional spheres, respectively[1]. These non-trivial Hopf maps can be classified in homotopy classes labeled by the value of the corresponding Hopf indexes, integer numbers, and the topological invariants[1,2]. The other names of the topological invariants are the topological charge, and the winding number (the degree of a continuous mapping)[22]. The topological charge is independent of the metric tensor, it can be interpreted as energy[23].

We see from eq.(26) that the complex scalar fields in the non-trivial Hopf maps are time-independent. This problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[24].

There exists (one) dimensional reduction in the non-trivial Hopf maps (26). The problems in the higher dimensional space can often be more complex than the problems in the lower dimensional space. By mapping onto the lower dimensional space, as in the Hopf maps, the problem becomes simpler, without losing the information about the non-trivial topological properties of space.

Physically, we interpret this dimensional reduction as a consequence of the isotropic (well-defined) property of the complex scalar fields for an infinite r. In the infinite r, the value of the complex scalar fields only depends on its magnitude. The direction of the position vector, as the complex scalar fields are the function of the position vector, does not matter. The property of the complex scalar fields as a function of space (the position vector) seems likely in harmony with the property of space-time itself. Space-time could be locally anisotropic but globally isotropic (the distribution of matter energy in the universe is assumed to be homogeneous).

A little we would like to say about the non-trivial Hopf maps. Roughly speaking, the Hopf maps or the Hopf fibration, $S^3 \to S^2$, is a non-trivial fibration where the fibre is S^1. Here, S^3 is the total space (domain), S^2 is base space (codomain) and each point on S^2 corresponds to a circle S^1 in S^3. It means that S^3 can be seen as a bundle of circles (fibres) over S^2.

VI. HOPF INVARIANT, HOPF INDEX, AND CHERN-SIMONS ACTION

These non-trivial Hopf maps (26) are related to the Hopf invariant[16], $\mathcal{H}$, expressed as an integral[16,25,26]

$$\mathcal{H} = \int_{S^3} \omega \wedge d\omega \tag{27}$$

where ω is a 1-form on S^3[16] and $d\omega$ is a 2-form. We see eq.(27) is identical to the formulation of circulation in hydrodynamics[27] where circulation is identical to Hopf invariant, ω and $d\omega$ are identical to velocity field and vorticity, respectively.

The relation between the Hopf invariant and the Hopf index, h, can be written explicitly as[1]

$$\mathcal{H} = h\ \gamma^2 \tag{28}$$

where γ is the total strength of the field which is the sum of the strengths of all the tubes formed by the integral lines of electric and magnetic fields[1].

Related to gauge theory and magnetohydrodynamics (self-helicity), it can be interpreted naturally that the Hopf invariant has a deep relationship with the Chern-Simons action (the Chern-Simons integral)[16]. We will see that the Hopf invariant is identical to the Chern-Simons action itself. The Hopf invariant is just the winding number of Gauss mapping[16] (so probably, there exists a relationship between Gauss mapping and non-trivial Hopf maps).

The Hopf invariant or the Chern-Simons integral is an important topological invariant to describe the topological characteristics of the knot family[16,17]. In a more precise expression, the Hopf invariant or the Chern-Simons integral is the total sum of all the self-linking and all the linking numbers of the knot family[16,17]. The self-linking and linking numbers by themselves have a topological structure.

VII. SMALL METRIC PERTURBATIONS, SCALAR AND VECTOR POTENTIALS

The linearized metric perturbations take a role as "potentials" in the linearized gravitation[28]. We consider a set of curvature components, $e^a(\vec{r},t)$, $e^{a^*}(\vec{r},t)$, as scalar potentials and they could be interpreted similarly to the linearized metric perturbations. The linearized (small) metric perturbations can be written as

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \tag{29}$$

or[28]

$$h_{\mu\nu} = \rho_{\mu\nu}\ e^{i\vec{k}\cdot\vec{r}} \tag{30}$$

where $g_{\mu\nu}$ is the metric tensor, $\eta_{\mu\nu}$ is the metric of Minkowski (flat) space-time, $\rho_{\mu\nu}$ is the amplitude, $\vec{k}\cdot\vec{r}$ is the phase, $\vec{k}$ is the wave vector, and $\vec{r}$ is the position vector. The subscript indices, μ, ν, represent space-time coordinates. In an empty space-time, the amplitude is constant. The small metric perturbations means that $|h_{\mu\nu}| << 1$ for all μ and ν. Eq.(30) shows us that the linearized metric perturbations can be understood in terms of the wave.

In analogy to eq.(30), we propose that the scalar and the related vector potentials could be written in terms of the wave, respectively as[29]

$$e^a = \rho^a e^{iq} \tag{31}$$

where ρ^a is the amplitude, q is the phase, the notation e in e^{iq} refers to the exponential ($e^{iq} = \exp(iq)$), and

$$e^a_\rho = f^a\ \partial_\rho q \tag{32}$$

where the subscript index ρ in e^a_ρ represents space-time coordinates, and the superscript index a represents a set of indices that label the curvature components, $\partial_\rho q$ means the gradient of the phase, f^a is the function of amplitude written as below

$$f^a = -1/\left\{2\pi[1 + (\rho^a)^2]\right\} \tag{33}$$

where $f^a(\vec{r},t)$ and $q(\vec{r},t)$ the Clebsch variables[24] or Gaussian potentials[10,27], scalars.

These Clebsch variables are related to any divergenceless vector field[1]. An example of a divergenceless vector field is the vorticity, $\vec{\omega}$, in hydrodynamics[27] or the magnetic field, $\vec{B}$, where $\vec{\nabla}\cdot\vec{B} = 0$. The Clebsch variables are not uniquely defined (many different choices are possible for them)[1].

We consider that e^a_ρ (32) is not a total derivative, otherwise it would be a pure gauge[24]. A pure gauge in this context means that the field configuration does not produce any observable curvature or field strength. For example, in electromagnetism, a pure gauge vector potential can be written as the gradient of a scalar function λ, $A_\mu = \partial_\mu \lambda$. In this case, the corresponding field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is zero, indicating no physical electromagnetic field is present. Since e^a_ρ is not a total derivative, it is not a pure gauge and therefore represents a physical, non-trivial field configuration.

We will see that this vector potential, e^a_ρ (32), could be viewed as the gauge potential and in turn, could be related to, in terms of Cartan gravity, the gauge fields (the vierbein and the spin connection).

VIII. A SET OF CURVATURE COMPONENTS

As the field strength in Maxwell's theory could consist of the complex scalar fields, we assume that the curvature could consist of the complex scalar potentials. The scalar fields are precisely equivalent to the scalar potentials (the linearized small metric perturbations). It is very clear if we see the notation of the scalar potentials (31) which involve the imaginary number, i.

The properties of the complex scalar potentials, as the complex scalar fields, could also be described by the Hopf maps written below

$$e^a(\vec{r}),\ e^{a^*}(\vec{r}) : S^3 \to S^2 \tag{34}$$

If we relate the Hopf maps with the space we are working, these three and two-dimensional spheres could be interpreted as the constant curvature spheres.

The complex scalar potentials in the non-trivial Hopf maps (34) are time-independent. Analogous to time-independent complex scalar fields, this problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[24].

IX. NON-LINEAR AND LINEAR RICCI THEORIES

These maps (34) have a consequence (by considering that the field strength is identical to the curvature) that we could write the Ricci curvature tensor as

$$R^a_{\mu\nu} = \frac{\sqrt{c}}{2\pi i}\left(\frac{\partial_\mu e^{a^*}\partial_\nu e^a - \partial_\nu e^{a^*}\partial_\mu e^a}{(1+e^{a^*}e^a)^2}\right) \tag{35}$$

where e^a is a set of components of the Ricci curvature tensor, the scalar potentials, and e^{a^*} is the complex conjugate of e^a. In analogy to non-linear field theory in electromagnetism[1], we consider c as an action constant, introduced so that the Ricci curvature tensor will have suitable dimensions for the curvature. Eq.(35) is the non-linear equation where the nonlinearity is shown by the $e^{a^*}e^a$ term in the denominator.

In the case of a weak-field limit, the complex scalar potentials are very small, $|e^{a^*}e^a| << 1$, eq.(35) reduces to a linear equation as written below

$$R^a_{\mu\nu} = \frac{\sqrt{c}}{2\pi i}\left(\partial_\mu e^{a^*}\partial_\nu e^a - \partial_\nu e^{a^*}\partial_\mu e^a\right) \tag{36}$$

This linear Ricci curvature tensor equation (36) is equivalent to eq.(4). It means that the linearized Ricci curvature tensor (4) could be interpreted the same as the linear Ricci theory (36) in the case of a weak-field limit.

By using the vector potential (32), the linear Ricci theory (36) could be written as[24]

$$R^a_{\mu\nu} = \frac{\sqrt{c}}{2\pi i}\ \{\partial_\mu(f^a\ \partial_\nu q) - \partial_\nu(f^a\ \partial_\mu q)\} \tag{37}$$

This is the linear Ricci theory written in terms of the Clebsch variables. We see that the vector potential written using the Clebsch scalar variables is equivalent to the Levi-Civita connection (the Christoffel symbols) in eq.(4).

X. GAUGE POTENTIAL AND GAUGE FIELDS

In gauge theory, there exists only gauge (vector) potential, but in general relativity, we have the gauge fields (the vierbein and the spin connection). These gauge fields could be viewed identically to the gauge potential. In this case, the gauge potential can be written as[5,8,10]

$$A_\mu = e^a_\mu P_a + \omega^a_\mu J_a \tag{38}$$

where e^a_μ is the vierbein (translational part), ω^a_μ is the spin connection (rotational part), P_a, J_a are the generators of translation and Lorentz rotation of ISO(2,1) Poincare group, respectively.

In analogy to (32), we could write the gauge potential A_μ as

$$A_\mu = f\ \partial_\mu q \tag{39}$$

where the function of amplitude, f, is analogous to (33), can be written as

$$f = -1/\left\{2\pi[1+(\rho)^2]\right\} \tag{40}$$

ρ is amplitude, q is phase.

By substituting eq.(39) into (38), we obtain that

$$f\ \partial_\mu q = e^a_\mu P_a + \omega^a_\mu J_a \tag{41}$$

where

$$e^a_\mu = f^a_e\ \partial_\mu q \tag{42}$$

and

$$\omega^a_\mu = f^a_\omega\ \partial_\mu q \tag{43}$$

f^a_e and f^a_ω are amplitude functions of the vierbein and the spin connection, respectively. We see from eqs.(42),(43), the only difference between the vierbein and the spin connection is their amplitude functions.

By substituting eqs.(42),(43), into (41), the gauge potential becomes

$$f\ \partial_\mu q = f^a_e\ \partial_\mu q\ P_a + f^a_\omega\ \partial_\mu q\ J_a \tag{44}$$

or,

$$f\ \partial_\mu q = (f^a_e\ P_a + f^a_\omega\ J_a)\ \partial_\mu q \tag{45}$$

Eq.(45) has a consequence that

$$f = f^a_e\ P_a + f^a_\omega\ J_a \tag{46}$$

Roughly speaking, the amplitude function, f, of the gauge potential decomposes into the amplitude functions of the vierbein and the spin connection.

XI. A NEWTONIAN KNOT

By substituting eqs.(42),(43) into eq.(26) we obtain the Abelian Chern-Simons action in (2+1)-dimensional space-time as written below

$$I_{\rm CS} = \int_M \varepsilon^{ijk}\ f_{ea}\ \partial_i q\left\{\partial_j(f^a_\omega\ \partial_k q) - \partial_k(f^a_\omega\ \partial_j q) + \frac{\Lambda}{3}\ \varepsilon^a_{bc}\ f^b_e\ \partial_j q\ f^c_e\ \partial_k q\right\} d^{2+1}x \tag{47}$$

The action, $I_{\rm CS}$, (47) is related to a topological object i.e. a Newtonian knot, an integer number. This integer number is what we mean by a set of curvature components, as a set of subset fields, obeying the topological quantum condition.

XII. DISCUSSION AND CONCLUSION

It has been realized that the role of topology has become more and more important in recent days and in the future of physics. But to understand topology is complicated enough because topology is inherently related to nonlinearity. It is widely believed that topological objects are impossible to exist in linear theories, such as an Abelian Chern-Simons action in the topological quantum field theory. But the belief is no longer can be maintained. The discovery of the electromagnetic knot in vacuum Maxwell's theory more than thirty years ago (by the paper of Ranada[1]) has shown that the topological object could exist in the linear theory. In that paper, Ranada describes the electromagnetic knot by assuming that the field strength tensor consists of a set of subset fields, complex scalar fields. A set of subset fields is locally equal to the field strength tensor, but globally different since a set of subset fields obeys the topological quantum condition but the field strength tensor does not.

We adopt the idea of Ranada's electromagnetic knot and apply it to the case of gravity theory. This is because electromagnetism and gravity are similar. The electromagnetic theory or Maxwell's theory is the gauge theory and the gravity theory (the general theory of relativity) could be treated as the gauge theory. Maxwell's theory is an Abelian $U(1)$ local gauge theory of internal space and general relativity, a non-linear theory, which could be linearized in the case of the weak-field limit, is the gauge theory of translation in (3+1)-dimensional (external) space-time. The gauge potential and the field strength tensor in electromagnetism are identical to the connection and the curvature in gravity theory, respectively. So, we are interested in the case of the weak-field limit in vacuum or empty space-time.

We propose that the curvature i.e. the Ricci curvature tensor has a set of curvature components consisting of complex scalar potentials. A set of curvature components is locally equal to the Ricci curvature tensor i.e. the Ricci curvature tensor can be obtained by patching together a set of curvature components (except in a zero-measure set) but globally different. The difference between a set of curvature components and the Ricci curvature tensor in an empty space-time is global instead of local since a set of curvature components obeys the topological quantum condition but the Ricci curvature tensor does not.

The linear Ricci curvature tensor satisfies the linear Ricci theory, but a set of curvature components satisfies the non-linear Ricci theory. Both, the linear Ricci curvature tensor and a set of curvature components, satisfy the linear Ricci theory in the case of a weak field of gravitation. It means that, in the case of a weak field, the non-linear Ricci theory reduces to the linear Ricci theory.

The problem arises in the case of (3+1)-dimensional space-time when general relativity is not equivalent to gauge theory. Fortunately, in the case of lower dimensions i.e. (2+1)-dimensional space-time, general relativity is equivalent to gauge theory. So, we turn our attention to the weak-field limit in (2+1)-dimensional empty space-time. But, gravity theory in (2+1)-dimensional empty space-time, without a cosmological constant, gives rise to flat space-time. The theory is trivial. How about gravity theory in (2+1)-dimensional empty space-time with a non-zero cosmological constant? Luckily, the space-time is not flat: space-time has constant curvature i.e. de Sitter space if the cosmological constant is positive and anti-de Sitter space if the cosmological constant is negative. The theory is not trivial. We are interested in a small positive cosmological constant (de Sitter space with a small constant curvature).

For the physical system of slow-moving or slow temporal variations compared to the velocity of light (Newtonian limit), the time derivative of the system is negligible. We can treat the physical system as quasi-static, meaning it evolves so slowly that for many practical purposes, it can be considered nearly constant over short time intervals. But, it can still have significant spatial variations. These variations determine how the physical system varies in space and contribute to geometry, i.e. the local curvature. In turn, the local curvature effects on the geodesic of test particle. For slow motion, the time components of the derivatives of the spin connection can be approximated as negligible, leaving us with the spatial components only. This interpretation is rooted in the nature of space-time in general relativity, where time and space are intertwined, but their effects can be treated differently.

We find that in the case of the Newtonian limit in (2+1)-dimensional empty space-time with a small positive cosmological constant, a non-Abelian Ricci curvature tensor reduces to an Abelian Ricci curvature tensor. This Abelian Ricci curvature tensor is a linear theory in terms of the spin connection. We also find an Abelian Ricci curvature tensor in gauge theory which is precisely equivalent to an Abelian Ricci curvature tensor in gravity theory. The Levi-Civita symbol in Abelian Ricci tensor in gauge theory shows that we work in internal space.

It is necessary to show that a set of subset fields satisfies the non-trivial Hopf maps $S^3 \to S^2$. Physically, it is because the properties of a set of subset fields could be described by Hopf maps. Analogous to a set of subset fields, we have a set of curvature components that also satisfy the non-trivial Hopf maps. The time-independent problems of a set of curvature components, as in the case of a set of subset fields, could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values.

A set of curvature components could consist of complex scalar potentials. It is because, in the linearized gravitation, the linearized (small) metric perturbations take a role as potentials. We interpret these linearized metric perturbations as the scalar potentials. The notation for both is precisely equivalent. They could be denoted as the amplitude times the exponential of iq, where q is the phase, and i is the imaginary number. The scalar potentials could be complex because there exists i there.

The related gauge (vector) potential can be written using the Clebsch scalar variables, f^a and q, where f^a is the function of amplitude. In this way, the gauge potential can be understood simply. These Clebsch variables are related to any divergenceless vector field, i.e. the divergence of any vector field gives the zero result. The Clebsch variables are not uniquely defined (many different choices are possible for them). Here, the gauge potential is not a total derivative, otherwise it would be a pure gauge. A pure gauge in this context means that the field configuration does not produce any observable curvature or field strength. Since the gauge potential is not a total derivative, it is not a pure gauge and therefore represents a physical, non-trivial field configuration.

Cartan gravity or the vierbein formalism of general relativity makes general relativity similar to a gauge theory. For this reason, we need to reformulate the gauge potential in gauge theory related to the gauge fields, i.e. the vierbein and the spin connection. The remarkable one is the vierbein and the spin connection have the similar forms described using Clebsch variables. The only difference between the vierbein and the spin connection is their amplitude functions. It has the consequence that the amplitude function of the gauge potential decomposes into the amplitude functions of the vierbein and the spin connection.

We substitute the gauge potential in terms of the vierbein and the spin connection written using the Clebsch variables to the Abelian Chern-Simons action with the small positive cosmological constant to obtain an Newtonian knot in (2+1)-dimensional empty space-time.

We could say that the empirical or the observational evidence to support the existence of an Newtonian knot in (2+1)-dimensional empty space-time is guaranteed by the formal equivalence between an Newtonian knot and the electromagnetic knot formulations in vacuum Maxwell's theory for which knot solutions had been known to exist as shown by Ranada[1,12].

There exists (one) dimensional reduction in the non-trivial Hopf maps (26). The problems in the higher dimensional space can often be more complex than the problems in the lower dimensional space. By mapping onto the lower dimensional space, as in the Hopf maps, the problem becomes simpler, without losing the information about the non-trivial topological properties of space.

Physically, we interpret this dimensional reduction as a consequence of the isotropic (well-defined) property of the complex scalar fields for an infinite r. In the infinite r, the value of the complex scalar fields only depends on its magnitude. The direction of the position vector, as the complex scalar fields are the function of the position vector, does not matter. The property of the complex scalar fields as a function of space (the position vector) seems likely in harmony with the property of space-time itself. Space-time could be locally anisotropic but globally isotropic (the distribution of matter energy in the universe is assumed to be homogeneous). The dimensional reduction from (3+1) to (2+1) dimensions is a natural consequence of the weak-field properties of gravity.

A little we would like to say about the non-trivial Hopf maps. Roughly speaking, the Hopf maps or the Hopf fibration, $S^3 \to S^2$, is a non-trivial fibration where the fibre is S^1. Here, S^3 is the total space (domain), S^2 is base space (codomain) and each point on S^2 corresponds to a circle S^1 in S^3. It means that S^3 can be seen as a bundle of circles (fibres) over S^2.

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