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Geometric product of two oriented points in conformal geometric algebra

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Abstract. We compute and explore the full geometric product of two oriented points in conformal geometric algebra $Cl(4,1)$ of three-dimensional Euclidean space. We comment on the symmetry of the various components, and state for all expressions also a representation in terms of point pair center and radius vectors.

Keywords. Conformal geometric algebra, oriented points, point geometry.

1. Introduction

This work is a substantial extension of [15, 17], where only the scalar part of the geometric product (also called inner product) was considered. In this work we apply conformal geometric algebra (CGA) to the description of points, including a planar orientation. An excellent general reference on Clifford's geometric algebras is [19], a short engineering oriented tutorial is [12], and [22] describes a free software extension for a standard industrial computer algebra system (MATLAB), which was also used for validation in the current work. Alternatively, all computations could be done in the optimized geometric algebra algorithm software GAALOP [7]. Introductions to CGA are given in [3, 5] and efficient computational implementations are described in [7]. CGA has found wide ranging applications in physics, quantum computing, molecular geometry, engineering, signal and image processing, neural networks, computer graphics and vision, encryption, robotics, electronic and power engineering, etc. Up to date surveys are [1, 10, 14]. An introduction to the notion of oriented point can be found in [6]. Prominent applications

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Soli Deo Gloria. This work is dedicated to virologist Takayuki Miyazawa for courageously publishing his findings about SARS-CoV-2 variants [23], for which Kyoto University appears to have terminated his professorship. Please note that this research is subject to the Creative Peace License [11].

could be to LIDAR terrain strip adjustment [13], protein geometry modeling [20, 21], and machine learning.

The example of the full geometric product of two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$

$$\boldsymbol{a}\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \wedge \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|(\cos\varphi + \sin\varphi\, \mathbf{i}_{ab}), \tag{1}$$

clearly demonstrates that it includes more information, than only the inner product, about the relative geometry of the two factors, in this case the cosine and the sine of the angle φ enclosed by the two vectors, and the oriented unit bivector $\mathbf{i}_{ab}$ of the plane spanned by the two vectors. Even though in this work we do not exhaustively analyze the relative geometric information of two oriented points contained in their full geometric product, the current study provides important foundations for this purpose.

In the following, we begin with the CGA expression for oriented points in three Euclidean dimensions (Section 2) and then fully compute their geometric product (Section 3). The computations have been checked with The Clifford Multivector Toolbox for MATLAB [22] using a representative example (Section 4).

2. The notion of oriented point in conformal geometric algebra

An *oriented point* is given by the trivector expression of a *circle with radius zero* ($r = 0$) in CGA,

$$Q = \mathbf{i}_q \wedge \boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \mathbf{i}_q)]\boldsymbol{e}_\infty + \mathbf{i}_q\boldsymbol{e}_0 + \mathbf{i}_q \cdot \boldsymbol{q}E, \tag{2}$$

where the three-dimensional position vector of Q is the vector $\boldsymbol{q} \in \mathbb{R}^3$, the unit oriented bivector[1] of the plane (orthogonal to the unit normal vector $\boldsymbol{n}_q$ of the plane) is $\mathbf{i}_q \in Cl^2(3, 0)$, $\boldsymbol{e}_0$ is the vector for the origin dimension, $\boldsymbol{e}_\infty$ is the vector for the infinity dimension, and the origin-infinity bivector is $E = \boldsymbol{e}_\infty \wedge \boldsymbol{e}_0$, with

$$\begin{aligned} &\boldsymbol{e}_0^2 = \boldsymbol{e}_\infty^2 = 0, \quad \boldsymbol{e}_0 \cdot \boldsymbol{e}_\infty = -1, \quad \boldsymbol{e}_0\boldsymbol{e}_\infty = -E - 1, \quad \boldsymbol{e}_\infty\boldsymbol{e}_0 = E - 1, \\ &\boldsymbol{e}_0 E = -\boldsymbol{e}_0, \quad E\boldsymbol{e}_0 = \boldsymbol{e}_0, \quad \boldsymbol{e}_\infty E = \boldsymbol{e}_\infty, \quad E\boldsymbol{e}_\infty = -\boldsymbol{e}_\infty, \end{aligned} \tag{3}$$

and $\boldsymbol{e}_0$ and $\boldsymbol{e}_\infty$ are both orthogonal to $\mathbb{R}^3$. This means, e.g. that

$$\begin{aligned} &\boldsymbol{q}\boldsymbol{e}_0 = -\boldsymbol{e}_0\boldsymbol{q}, \quad \boldsymbol{q}\boldsymbol{e}_\infty = -\boldsymbol{e}_\infty\boldsymbol{q}, \quad \mathbf{i}_q\boldsymbol{e}_0 = \boldsymbol{e}_0\mathbf{i}_q, \quad \mathbf{i}_q\boldsymbol{e}_\infty = \boldsymbol{e}_\infty\mathbf{i}_q, \\ &\boldsymbol{n}_q\boldsymbol{e}_0 = -\boldsymbol{e}_0\boldsymbol{n}_q, \quad \boldsymbol{n}_q\boldsymbol{e}_\infty = -\boldsymbol{e}_\infty\boldsymbol{n}_q, \end{aligned} \tag{4}$$

all relations which are frequently used in the computations later in this paper.

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[1] I thank an anonymous reviewer for the following valuable comment: Changing the sign of Q and therefore of $\mathbf{i}_q$ and $\boldsymbol{n}_q$ means to give the oriented point Q exactly the opposite orientation.

The central pseudoscalar of CGA $I = e_{123}E = i_3E = Ei_3$, $I^{-1} = -i_3E$, leads to the dual (bivector) form[2] of the oriented point

$$\begin{aligned} Q^* &= QI^{-1} = -Qi_3E \\ &= -(\mathbf{i}_q \wedge \boldsymbol{q})i_3E + [\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q i_3 - \boldsymbol{q}(\boldsymbol{q}\cdot\mathbf{i}_q)i_3]\boldsymbol{e}_\infty E + \mathbf{i}_q i_3\boldsymbol{e}_0 E - (\mathbf{i}_q\cdot\boldsymbol{q})i_3E^2 \\ &= \mathbf{i}_q^*\cdot\boldsymbol{q}E + [\frac{1}{2}\boldsymbol{q}^2(-\mathbf{i}_q^*) + \boldsymbol{q}(\boldsymbol{q}\wedge\mathbf{i}_q^*)]\boldsymbol{e}_\infty + \mathbf{i}_q^*\boldsymbol{e}_0 + \mathbf{i}_q^*\wedge\boldsymbol{q} \\ &= \mathbf{i}_q^*\cdot\boldsymbol{q}E + [-\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q^* + \boldsymbol{q}(\boldsymbol{q}\mathbf{i}_q^* - \boldsymbol{q}\cdot\mathbf{i}_q^*)]\boldsymbol{e}_\infty + \mathbf{i}_q^*\boldsymbol{e}_0 + \mathbf{i}_q^*\wedge\boldsymbol{q} \\ &= \mathbf{i}_q^*\cdot\boldsymbol{q}E + [\frac{1}{2}\boldsymbol{q}^2\mathbf{i}_q^* - \boldsymbol{q}(\boldsymbol{q}\cdot\mathbf{i}_q^*)]\boldsymbol{e}_\infty + \mathbf{i}_q^*\boldsymbol{e}_0 + \mathbf{i}_q^*\wedge\boldsymbol{q}, \\ &= \boldsymbol{n}_q\wedge\boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{n}_q)]\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 + \boldsymbol{n}_q\cdot\boldsymbol{q}E, \end{aligned} \tag{7}$$

using[3] $\boldsymbol{n}_q = \mathbf{i}_q^* = -\mathbf{i}_q i_3$, for the unit normal vector of bivector $\mathbf{i}_q$. The same expression for Q^* is found in [6], equation (4).

Note that oriented points naturally arise from the intersection of two spheres tangent in one point, or a sphere and a plane tangent in one point, see e.g. [8]. Furthermore, a dual oriented point at the origin ($\boldsymbol{q} = 0$) has the simple form $\boldsymbol{n}_q\boldsymbol{e}_0$, which is a bivector that squares to zero and can be used as generator for transversions, similar to how bivectors $\frac{1}{2}\boldsymbol{t}\boldsymbol{e}_\infty$ generate translations, see e.g. [4]. Moreover, from the oriented point at the origin $\boldsymbol{n}_q\boldsymbol{e}_0$ one can elegantly obtain the full expression of the oriented point located at $\boldsymbol{q}\in\mathbb{R}^3$ with a translation

$$\begin{aligned} &Q^* = T^{-1}(\boldsymbol{q})\boldsymbol{n}_p\boldsymbol{e}_0T(\boldsymbol{q}), \quad T(\boldsymbol{q}) = 1 + \frac{1}{2}\boldsymbol{q}\boldsymbol{e}_\infty, \\ &T^{-1}(\boldsymbol{q}) = T(-\boldsymbol{q}) = 1 - \frac{1}{2}\boldsymbol{q}\boldsymbol{e}_\infty, \end{aligned} \tag{8}$$

where the equality

$$-\boldsymbol{q}\boldsymbol{n}_q\boldsymbol{q} = \boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}\boldsymbol{n}_q\boldsymbol{q} = \boldsymbol{q}^2\boldsymbol{n}_q - 2\boldsymbol{q}(\boldsymbol{q}\cdot\boldsymbol{n}_q), \tag{9}$$

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[2]Note that the result of (7) can also be written as

$$Q^* = \boldsymbol{n}_q\wedge\boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q}\rfloor\boldsymbol{n}_q)]\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 + (\boldsymbol{q}\rfloor\boldsymbol{n}_q)E, \tag{5}$$

where $\rfloor$ is the left contraction of geometric algebra. If then the unit orientation vector $\boldsymbol{n}_q$ is formally replaced by the carrier of a conformal point, i.e. the scalar 1 (see [9]), then we get the expression for a standard conformal point Q_{no} in CGA without orientation

$$Q_{no} = \boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^2\boldsymbol{e}_\infty + \boldsymbol{e}_0, \tag{6}$$

because $1\wedge\boldsymbol{q} = \boldsymbol{q}$ and $\boldsymbol{q}\rfloor 1 = 0$.

[3]The dual Q^* of an oriented point Q in (7) is computed by division with the five-dimensional pseudoscalar I of $Cl(4,1)$, whereas the dual of entities in $Cl(3,0)\subset Cl(4,1)$ is computed by division with the three-dimensional pseudoscalar $i_3 = e_{123}$.

is also needed. It also means that the expression (7) for a dual oriented point can always be simplified to

$$Q^* = \boldsymbol{n}_q \wedge \boldsymbol{q} - \frac{1}{2}\boldsymbol{q}\boldsymbol{n}_q\boldsymbol{q}\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 + \boldsymbol{n}_q \cdot \boldsymbol{q}E, \tag{10}$$

and the factor of $\boldsymbol{e}_\infty$ is

$$-\frac{1}{2}\boldsymbol{q}\boldsymbol{n}_q\boldsymbol{q} = \frac{1}{2}\boldsymbol{q}^2(-\widehat{\boldsymbol{q}}\boldsymbol{n}_q\widehat{\boldsymbol{q}}) = \frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}'_q, \tag{11}$$

with unit vector $\widehat{\boldsymbol{q}} = \boldsymbol{q}/|\boldsymbol{q}|$, and

$$\boldsymbol{n}'_q = -\widehat{\boldsymbol{q}}\boldsymbol{n}_q\widehat{\boldsymbol{q}} = \boldsymbol{n}_{q\perp\boldsymbol{q}} - \boldsymbol{n}_{q\|\boldsymbol{q}}, \quad \boldsymbol{n}_{q\perp\boldsymbol{q}} = (\boldsymbol{n}_q \wedge \boldsymbol{q})\boldsymbol{q}^{-1}, \quad \boldsymbol{n}_{q\|\boldsymbol{q}} = (\boldsymbol{n}_q \cdot \boldsymbol{q})\boldsymbol{q}^{-1}, \tag{12}$$

the orientation vector $\boldsymbol{n}_q$ reflected at the plane orthogonal to $\widehat{\boldsymbol{q}}$, respectively its two components orthogonal and parallel to $\widehat{\boldsymbol{q}}$. Note that

$$\boldsymbol{n}_q \wedge \boldsymbol{q} = \boldsymbol{n}_{q\perp\boldsymbol{q}}\boldsymbol{q} = \boldsymbol{n}'_q \wedge \boldsymbol{q}, \quad \boldsymbol{n}_q \cdot \boldsymbol{q} = \boldsymbol{n}_{q\|\boldsymbol{q}}\boldsymbol{q} = -\boldsymbol{n}'_q \cdot \boldsymbol{q}. \tag{13}$$

Using $\boldsymbol{n}'_q$ and its above properties allows to write the dual oriented point[4] as

$$\begin{aligned} Q^* &= \boldsymbol{n}_q \wedge \boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^2(-\widehat{\boldsymbol{q}}\boldsymbol{n}_q\widehat{\boldsymbol{q}})\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 + \boldsymbol{n}_q \cdot \boldsymbol{q}E \\ &= \boldsymbol{n}'_q \wedge \boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}'_q\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 - \boldsymbol{n}'_q \cdot \boldsymbol{q}E \\ &= \boldsymbol{n}'_q \wedge \boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}'_q\boldsymbol{e}_\infty - \widehat{\boldsymbol{q}}\boldsymbol{n}'_q\widehat{\boldsymbol{q}}\boldsymbol{e}_0 - \boldsymbol{n}'_q \cdot \boldsymbol{q}E. \end{aligned} \tag{16}$$

Comparing lines one and three of (16), we see that a dual oriented point can be freely expressed with the original orientation vector $\boldsymbol{n}_q$ or with the reflected vector $\boldsymbol{n}'_q$. When using $\boldsymbol{n}_q$, the factor of $\boldsymbol{e}_\infty$ will include the reflection operation applied to $\boldsymbol{n}_q$ explicitly, and when using $\boldsymbol{n}'_q$ (as in line three of (16)), then the factor of $\boldsymbol{e}_0$ will include the same reflection operation applied to $\boldsymbol{n}'_q$, because

$$\boldsymbol{n}'_q = -\widehat{\boldsymbol{q}}\boldsymbol{n}_q\widehat{\boldsymbol{q}}, \quad \boldsymbol{n}_q = -\widehat{\boldsymbol{q}}\boldsymbol{n}'_q\widehat{\boldsymbol{q}}, \tag{17}$$

as reflections are involutions.

It is now also easy to see that the orientation vector[5] $\boldsymbol{n}_q$ can be directly obtained from Q^* by

$$\boldsymbol{n}_q = -(Q^* \wedge \boldsymbol{e}_\infty)\lfloor E, \tag{19}$$

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[4]The bivector expression for a dual oriented point

$$Q^* = \boldsymbol{n}_q \wedge \boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}'_q\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 + \boldsymbol{n}_q \cdot \boldsymbol{q}E \tag{14}$$

also shows similarity to that of a standard conformal point Q_{no}, a vector in $\mathbb{R}^{4,1}$, without orientation

$$Q_{no} = \boldsymbol{q} + \frac{1}{2}\boldsymbol{q}^2\boldsymbol{e}_\infty + \boldsymbol{e}_0. \tag{15}$$

[5]Because the representation is homogenous, it may be necessary for obtaining a unit vector to compute

$$\boldsymbol{n}_q = -(Q^* \wedge \boldsymbol{e}_\infty)\lfloor E/\sqrt{\{(Q^* \wedge \boldsymbol{e}_\infty)\lfloor E\}^2}, \tag{18}$$

in order to remove the homogenous factor.

and the position vector[6] $\boldsymbol{q}$ by

$$\boldsymbol{q} = \boldsymbol{n}_q(\boldsymbol{n}_q \wedge \boldsymbol{q} + \boldsymbol{n}_q \cdot \boldsymbol{q}) = \boldsymbol{n}_q\Big[(Q^* \wedge E)E + Q^* \rfloor E\Big] = \boldsymbol{n}_q\Big(\big[(Q^* \wedge E) + Q^*\big] \rfloor E\Big), \tag{20}$$

where $\rfloor$ is the right contraction, which can in this case also be replaced by the inner product.

3. Computation of geometric product of oriented points

We consider the geometric product of two oriented points in conformal geometric algebra [6], as reference for practical CGA computations in this section we recommend the introductory chapter of [16] and [9]. Note that inner product and wedge product have priority over the geometric product, e.g., $\mathbf{i}_q \cdot \boldsymbol{q}E = (\mathbf{i}_q \cdot \boldsymbol{q})E$, etc. The computations have been validated (see e.g. the example in Section 4) with The Clifford Multivector Toolbox for MATLAB [22], which proved indispensable for correcting quite a number of errors.

Assume a second dual oriented point P^* to be given by

$$P^* = \boldsymbol{n}_p \wedge \boldsymbol{p} + [\frac{1}{2}\boldsymbol{p}^2\boldsymbol{n}_p - \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{n}_p)]\boldsymbol{e}_\infty + \boldsymbol{n}_p\boldsymbol{e}_0 + \boldsymbol{n}_p \cdot \boldsymbol{p}E, \tag{21}$$

where the three-dimensional position vector of P is the vector $\boldsymbol{p} \in \mathbb{R}^3$ and the unit oriented bivector of the plane (orthogonal to the unit normal vector $\boldsymbol{n}_p = \mathbf{i}_p^*$ of the plane) is $\mathbf{i}_p \in Cl^2(3,0)$.

Now we compute the full geometric product of the two dual oriented points.

$$\begin{aligned}
P^*Q^* = &\big(\boldsymbol{n}_p \wedge \boldsymbol{p} + [\frac{1}{2}\boldsymbol{p}^2\boldsymbol{n}_p - \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{n}_p)]\boldsymbol{e}_\infty + \boldsymbol{n}_p\boldsymbol{e}_0 + \boldsymbol{n}_p \cdot \boldsymbol{p}E\big) \\
&\big(\boldsymbol{n}_q \wedge \boldsymbol{q} + [\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{n}_q)]\boldsymbol{e}_\infty + \boldsymbol{n}_q\boldsymbol{e}_0 + \boldsymbol{n}_q \cdot \boldsymbol{q}E\big) \\
= &(\boldsymbol{n}_p \wedge \boldsymbol{p})(\boldsymbol{n}_q \wedge \boldsymbol{q}) + (\boldsymbol{n}_p \wedge \boldsymbol{p})[\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{n}_q)]\boldsymbol{e}_\infty \\
&+ (\boldsymbol{n}_p \wedge \boldsymbol{p})\boldsymbol{n}_q\boldsymbol{e}_0 + (\boldsymbol{n}_p \wedge \boldsymbol{p})\boldsymbol{n}_q \cdot \boldsymbol{q}E \\
&+ [\frac{1}{2}\boldsymbol{p}^2\boldsymbol{n}_p - \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{n}_p)](\boldsymbol{n}_q \wedge \boldsymbol{q})\boldsymbol{e}_\infty - [\frac{1}{2}\boldsymbol{p}^2\boldsymbol{n}_p - \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{n}_p)]\boldsymbol{n}_q\boldsymbol{e}_\infty\boldsymbol{e}_0 \\
&+ [\frac{1}{2}\boldsymbol{p}^2\boldsymbol{n}_p - \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{n}_p)]\boldsymbol{n}_q \cdot \boldsymbol{q}\boldsymbol{e}_\infty E + \boldsymbol{n}_p(\boldsymbol{n}_q \wedge \boldsymbol{q})\boldsymbol{e}_0 \\
&- \boldsymbol{n}_p[\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{n}_q)]\boldsymbol{e}_0\boldsymbol{e}_\infty + \boldsymbol{n}_p\boldsymbol{n}_q \cdot \boldsymbol{q}\boldsymbol{e}_0E \\
&+ \boldsymbol{n}_p \cdot \boldsymbol{p}(\boldsymbol{n}_q \wedge \boldsymbol{q})E + \boldsymbol{n}_p \cdot \boldsymbol{p}[\frac{1}{2}\boldsymbol{q}^2\boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{n}_q)]E\boldsymbol{e}_\infty \\
&+ \boldsymbol{n}_q(\boldsymbol{n}_p \cdot \boldsymbol{p})E\boldsymbol{e}_0 + (\boldsymbol{n}_p \cdot \boldsymbol{p})(\boldsymbol{n}_q \cdot \boldsymbol{q}).
\end{aligned} \tag{22}$$

[6]Division with $\sqrt{\{(Q^* \wedge \boldsymbol{e}_\infty)\rfloor E\}^2}$ will again remove any homogeneous factor.

This result constitutes a linear combination of the four conformal blades $\{1, \boldsymbol{e}_0, \boldsymbol{e}_\infty, E\}$ with $Cl(3,0)$ multivector coefficients

$$P^*Q^* = M + M_0\boldsymbol{e}_0 + M_\infty\boldsymbol{e}_\infty + M_E E, \quad M, M_0, M_\infty, M_E \in Cl(3,0), \tag{23}$$

after the relationships (3) are taken into account for the products $\boldsymbol{e}_0\boldsymbol{e}_\infty, \boldsymbol{e}_\infty\boldsymbol{e}_0$, $\boldsymbol{e}_0 E, E\boldsymbol{e}_0, \boldsymbol{e}_\infty E$ and $E\boldsymbol{e}_\infty$. We call the four $Cl(3,0)$ multivector coefficients real part M, $\boldsymbol{e}_0$-part M_0, $\boldsymbol{e}_\infty$-part M_∞ and E-part M_E, respectively. Because P^* and Q^* are both bivectors, the grades occurring in the geometric product P^*Q^* are limited[7] to scalars $(2-2)$ (symmetric inner product part $\langle P^*Q^*\rangle = \langle Q^*P^*\rangle$), bivectors $(2+0)$ (antisymmetric commutator product part $\langle P^*Q^*\rangle_2 = \frac{1}{2}(P^*Q^* - Q^*P^*)$) and 4-vectors $(2+2)$ (symmetric outer product part $\langle P^*Q^*\rangle_4 = P^* \wedge Q^* = Q^* \wedge P^*$). This in turn means that the $Cl(3,0)$ multivector coefficients of the real part and the E-part will be even grade linear combinations of scalars and bivectors,

$$M = M_s + M_b, \quad M_E = M_{Es} + M_{Eb}, \tag{24}$$

and the $Cl(3,0)$ multivector coefficients of the $\boldsymbol{e}_0$-part and the $\boldsymbol{e}_\infty$-part will be odd grade vectors and trivectors,

$$M_0 = M_{0v} + M_{0t}, \quad M_\infty = M_{\infty v} + M_{\infty t}, \tag{25}$$

respectively.

The symmetric part of the geometric product of two oriented points is then

$$\begin{aligned}\langle P^*Q^*\rangle_{sy} &= \frac{1}{2}(P^*Q^* + Q^*P^*) = \langle P^*Q^*\rangle + \langle P^*Q^*\rangle_4 \\ &= M_s + M_{0t}\boldsymbol{e}_0 + M_{\infty t}\boldsymbol{e}_\infty + M_{Eb}E,\end{aligned} \tag{26}$$

and the antisymmetric part is the bivector part

$$\langle P^*Q^*\rangle_{as} = \frac{1}{2}(P^*Q^* - Q^*P^*) = \langle P^*Q^*\rangle_2 = M_b + M_{0v}\boldsymbol{e}_0 + M_{\infty v}\boldsymbol{e}_\infty + M_{Es}E, \tag{27}$$

respectively.

According to what has been pointed out about the symmetry of the various product parts, we therefore expect that M_s, M_{Eb}, M_{0t} and $M_{\infty t}$ will be symmetric under changing the order of factors P^* and Q^*, whereas M_b, M_{Es}, M_{0v} and $M_{\infty v}$ will be antisymmetric, respectively. This means that every of the four $Cl(3,0)$ multivector coefficients in (24) and (25) comprises exactly one symmetric and one antisymmetric blade part, and the two parts always have grade difference two.

We conveniently define the three-dimensional Euclidean distance vector from $\boldsymbol{p}$ to $\boldsymbol{q}$ as

$$\boldsymbol{d} = \boldsymbol{q} - \boldsymbol{p}, \tag{28}$$

and we introduce the three-dimensional mid point position

$$\boldsymbol{c} = \frac{1}{2}(\boldsymbol{p} + \boldsymbol{q}) \tag{29}$$

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[7] Note that in geometric algebra the symmetry of products depends critically on the grades of the factors.

and the three-dimensional distance vector $\boldsymbol{r}$ connecting $\boldsymbol{c}$ with $\boldsymbol{q}$ as

$$\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{c} = \frac{1}{2}\boldsymbol{d}, \tag{30}$$

and can then express the two Euclidean point positions as

$$\boldsymbol{p} = \boldsymbol{c} - \boldsymbol{r}, \quad \boldsymbol{q} = \boldsymbol{c} + \boldsymbol{r}. \tag{31}$$

In the following we list and explain all eight multivector coefficient parts separately in the order of $M_s, M_b, M_{Es}, M_{Eb}, M_{0v}, M_{0t}, M_{\infty v}$, and $M_{\infty t}$.

3.1. Real scalar part

The real scalar part is also known as the inner product of the two dual oriented points

$$\begin{aligned} M_s = \langle P^* Q^* \rangle &= (\boldsymbol{n}_p \wedge \boldsymbol{p}) \cdot (\boldsymbol{n}_q \wedge \boldsymbol{q}) + [\frac{1}{2}\boldsymbol{p}^2 \boldsymbol{n}_p - \boldsymbol{p}(\boldsymbol{p} \cdot \boldsymbol{n}_p)] \cdot \boldsymbol{n}_q \\ &\quad + \boldsymbol{n}_p \cdot [\frac{1}{2}\boldsymbol{q}^2 \boldsymbol{n}_q - \boldsymbol{q}(\boldsymbol{q} \cdot \boldsymbol{n}_q)] + (\boldsymbol{n}_p \cdot \boldsymbol{p})(\boldsymbol{n}_q \cdot \boldsymbol{q}) \\ &= (\boldsymbol{n}_p \wedge \boldsymbol{p}) \cdot (\boldsymbol{n}_q \wedge \boldsymbol{q}) + \frac{1}{2}\boldsymbol{p}^2 \boldsymbol{n}_p' \cdot \boldsymbol{n}_q + \frac{1}{2}\boldsymbol{q}^2 \boldsymbol{n}_q' \cdot \boldsymbol{n}_p + (\boldsymbol{n}_p \cdot \boldsymbol{p})(\boldsymbol{n}_q \cdot \boldsymbol{q}), \end{aligned} \tag{32}$$

with

$$\boldsymbol{n}_p' = -\widehat{\boldsymbol{p}} \boldsymbol{n}_p \widehat{\boldsymbol{p}}, \quad \boldsymbol{n}_q' = -\widehat{\boldsymbol{q}} \boldsymbol{n}_q \widehat{\boldsymbol{q}}. \tag{33}$$

The real scalar part M_s and can also be expressed[8] with (28) and (30) as

$$M_s = \frac{1}{2}\boldsymbol{d}^2 \boldsymbol{n}_p \cdot \boldsymbol{n}_q - \boldsymbol{d} \cdot \boldsymbol{n}_p \boldsymbol{d} \cdot \boldsymbol{n}_q = 4\boldsymbol{r}^2 (\frac{1}{2}\boldsymbol{n}_p \cdot \boldsymbol{n}_q - \widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_p \widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_q), \tag{34}$$

where the unit direction vector $\widehat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$. Note that M_s is independent of the absolute Euclidean positions of P and Q, i.e., only the distance vector $\boldsymbol{r}$, and the point orientations $\boldsymbol{n}_q$, $\boldsymbol{n}_q$, matter for the real scalar part. Furthermore, M_s is symmetric with respect to interchanging the oriented points P and Q, and it is also symmetric with respect to only interchanging the two point orientations $\boldsymbol{n}_p \leftrightarrow \boldsymbol{n}_q$.

The real scalar part M_s has already been extensively discussed in [15,17], and applied in [20, 21].

3.2. Real bivector part

By straightforward computation we express the real bivector part in three different forms. First in terms of $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{n}_p$ and $\boldsymbol{n}_q$:

$$\begin{aligned} M_b = \langle (\boldsymbol{n}_p \wedge \boldsymbol{p})(\boldsymbol{n}_q \wedge \boldsymbol{q}) \rangle_2 &+ \frac{1}{2}\boldsymbol{p}^2 (\boldsymbol{n}_p \wedge \boldsymbol{n}_q) - (\boldsymbol{p} \wedge \boldsymbol{n}_q)(\boldsymbol{p} \cdot \boldsymbol{n}_p) \\ &+ \frac{1}{2}\boldsymbol{q}^2 (\boldsymbol{n}_p \wedge \boldsymbol{n}_q) - (\boldsymbol{n}_p \wedge \boldsymbol{q})(\boldsymbol{q} \cdot \boldsymbol{n}_q). \end{aligned} \tag{35}$$

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[8] I thank an anonymous reviewer for the following valuable observation: the use of (34), compared to (32), exemplifies how the numerical error can be reduced.

Note further that by straightforward computation

$$\langle(\boldsymbol{n}_p \wedge \boldsymbol{p})(\boldsymbol{n}_q \wedge \boldsymbol{q})\rangle_2 \\ = (\boldsymbol{n}_p \wedge \boldsymbol{q})(\boldsymbol{p} \cdot \boldsymbol{n}_q) + (\boldsymbol{n}_p \cdot \boldsymbol{q})(\boldsymbol{p} \wedge \boldsymbol{n}_q) - (\boldsymbol{p} \cdot \boldsymbol{q})(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) - (\boldsymbol{p} \wedge \boldsymbol{q})(\boldsymbol{n}_p \cdot \boldsymbol{n}_q). \tag{36}$$

The definition (28) allows to simplify the real bivector part to

$$M_b = \frac{1}{2}\boldsymbol{d}^2(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) + (\boldsymbol{d} \cdot \boldsymbol{n}_p)(\boldsymbol{p} \wedge \boldsymbol{n}_q) + (\boldsymbol{d} \cdot \boldsymbol{n}_q)(\boldsymbol{q} \wedge \boldsymbol{n}_p) - (\boldsymbol{p} \wedge \boldsymbol{q})(\boldsymbol{n}_p \cdot \boldsymbol{n}_q). \tag{37}$$

Inserting (31) the real bivector part can further be expressed as

$$M_b = 2\Big(\boldsymbol{r}^2(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) + (\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{c} \wedge \boldsymbol{n}_q) - (\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{r} \wedge \boldsymbol{n}_q) \\ + (\boldsymbol{r} \cdot \boldsymbol{n}_q)(\boldsymbol{c} \wedge \boldsymbol{n}_p) + (\boldsymbol{r} \cdot \boldsymbol{n}_q)(\boldsymbol{r} \wedge \boldsymbol{n}_p) - (\boldsymbol{n}_p \cdot \boldsymbol{n}_q)(\boldsymbol{c} \wedge \boldsymbol{r})\Big). \tag{38}$$

We can split the real bivector part M_b into a symmetric part M_{b+} and an antisymmetric part M_{b-} with respect to exchanging[9] the two point orientations $\boldsymbol{n}_p \leftrightarrow \boldsymbol{n}_q$. We obtain

$$M_b = M_{b+} + M_{b-}, \tag{39}$$

with

$$M_{b+} = 2\Big((\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{c} \wedge \boldsymbol{n}_q) + (\boldsymbol{r} \cdot \boldsymbol{n}_q)(\boldsymbol{c} \wedge \boldsymbol{n}_p) - (\boldsymbol{n}_p \cdot \boldsymbol{n}_q)(\boldsymbol{c} \wedge \boldsymbol{r})\Big) \tag{40}$$

and

$$M_{b-} = 2\Big(\boldsymbol{r}^2(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) - (\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{r} \wedge \boldsymbol{n}_q) + (\boldsymbol{r} \cdot \boldsymbol{n}_q)(\boldsymbol{r} \wedge \boldsymbol{n}_p)\Big) \\ = 2\boldsymbol{r}^2\Big(\boldsymbol{n}_p \wedge \boldsymbol{n}_q - (\widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_p)(\widehat{\boldsymbol{r}} \wedge \boldsymbol{n}_q) + (\widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_q)(\widehat{\boldsymbol{r}} \wedge \boldsymbol{n}_p)\Big), \tag{41}$$

Note that M_{b-} is identical to the full real bivector part M_b, when the point pair is centered around the origin, i.e., with $\boldsymbol{c} = 0$.

3.3. Scalar E-part

The scalar E-part is found to be

$$M_{Es} = \frac{1}{2}\boldsymbol{p}^2(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) + (\boldsymbol{p} \cdot \boldsymbol{n}_p)(\boldsymbol{p} \cdot \boldsymbol{n}_q) + \frac{1}{2}\boldsymbol{q}^2(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) - (\boldsymbol{q} \cdot \boldsymbol{n}_p)(\boldsymbol{q} \cdot \boldsymbol{n}_q) \\ = \frac{1}{2}(\boldsymbol{q}^2 - \boldsymbol{p}^2)(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) + (\boldsymbol{p} \cdot \boldsymbol{n}_p)(\boldsymbol{p} \cdot \boldsymbol{n}_q) - (\boldsymbol{q} \cdot \boldsymbol{n}_p)(\boldsymbol{q} \cdot \boldsymbol{n}_q). \tag{42}$$

Using definition (31) the scalar E-part can be simplified to

$$M_{Es} = 2\Big((\boldsymbol{c} \cdot \boldsymbol{r})(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) - (\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{c} \cdot \boldsymbol{n}_q) - (\boldsymbol{c} \cdot \boldsymbol{n}_p)(\boldsymbol{r} \cdot \boldsymbol{n}_q)\Big). \tag{43}$$

Note that, as expected, M_{Es} is antisymmetric with respect to interchanging the two oriented points P and Q, in marked contrast to the above symmetry of the real scalar part M_s.

For a pair of points centered at the origin ($\boldsymbol{c} = 0$), M_{Es} vanishes

$$M_{Es} = 0. \tag{44}$$

[9]Note that when exchanging not only the two point orientations, but also the positions, then $\boldsymbol{c}$ is invariant, but $\boldsymbol{r} \to -\boldsymbol{r}$, which means that, as expected, M_b as a whole is antisymmetric with respect to changing the order of P and Q in the geometric product.

3.4. Bivector E-part

The bivector E-part is found to be

$$M_{Eb} = (\boldsymbol{n}_q \cdot \boldsymbol{q})(\boldsymbol{n}_p \wedge \boldsymbol{p}) + (\boldsymbol{n}_p \cdot \boldsymbol{p})(\boldsymbol{n}_q \wedge \boldsymbol{q}) - \frac{1}{2}\boldsymbol{p}^2(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) + (\boldsymbol{n}_p \cdot \boldsymbol{p})(\boldsymbol{p} \wedge \boldsymbol{n}_q) + \frac{1}{2}\boldsymbol{q}^2(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) - (\boldsymbol{q} \cdot \boldsymbol{n}_q)(\boldsymbol{n}_p \wedge \boldsymbol{q}). \tag{45}$$

Using definition (31) the bivector E-part can be reexpressed as

$$M_{Eb} = 2\Big((\boldsymbol{c} \cdot \boldsymbol{r})(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) - (\boldsymbol{n}_q \cdot \boldsymbol{c})(\boldsymbol{n}_p \wedge \boldsymbol{r}) - (\boldsymbol{n}_q \cdot \boldsymbol{r})(\boldsymbol{n}_p \wedge \boldsymbol{r}) + (\boldsymbol{n}_p \cdot \boldsymbol{c})(\boldsymbol{n}_q \wedge \boldsymbol{r}) - (\boldsymbol{n}_p \cdot \boldsymbol{r})(\boldsymbol{n}_q \wedge \boldsymbol{r})\Big), \tag{46}$$

which is symmetric under the exchange of the two oriented points P and Q. We can split the bivector E-part M_{Eb} into a symmetric part M_{Eb+} and an antisymmetric part M_{Eb-} with respect to exchanging the two point orientations $\boldsymbol{n}_p \leftrightarrow \boldsymbol{n}_q$. We obtain

$$M_{Eb} = M_{Eb+} + M_{Eb-}, \tag{47}$$

with

$$M_{Eb-} = 2\Big((\boldsymbol{c} \cdot \boldsymbol{r})(\boldsymbol{n}_p \wedge \boldsymbol{n}_q) - (\boldsymbol{n}_q \cdot \boldsymbol{c})(\boldsymbol{n}_p \wedge \boldsymbol{r}) + (\boldsymbol{n}_p \cdot \boldsymbol{c})(\boldsymbol{n}_q \wedge \boldsymbol{r})\Big) \tag{48}$$

and

$$M_{Eb+} = -2\Big((\boldsymbol{n}_q \cdot \boldsymbol{r})(\boldsymbol{n}_p \wedge \boldsymbol{r}) + (\boldsymbol{n}_p \cdot \boldsymbol{r})(\boldsymbol{n}_q \wedge \boldsymbol{r})\Big) = -2\boldsymbol{r}^2\Big((\boldsymbol{n}_q \cdot \widehat{\boldsymbol{r}})(\boldsymbol{n}_p \wedge \widehat{\boldsymbol{r}}) + (\boldsymbol{n}_p \cdot \widehat{\boldsymbol{r}})(\boldsymbol{n}_q \wedge \widehat{\boldsymbol{r}})\Big) \tag{49}$$

Note that M_{Eb+} is identical to the full bivector E-part M_{Eb}, when the point pair is centered around the origin, i.e., with $\boldsymbol{c} = 0$. Furthermore, note the striking similarity with the symmetry behavior of the real bivector part M_b under the exchange of orientation $\boldsymbol{n}_p \leftrightarrow \boldsymbol{n}_q$, see (39) to (41), although the roles of the symmetric and antisymmetric parts are interchanged.

3.5. Vector $\boldsymbol{e}_0$-part

The vector $\boldsymbol{e}_0$-part is found to be

$$M_{0v} = (\boldsymbol{n}_p \wedge \boldsymbol{p}) \cdot \boldsymbol{n}_q + \boldsymbol{n}_p \cdot (\boldsymbol{n}_q \wedge \boldsymbol{q}) - (\boldsymbol{n}_q \cdot \boldsymbol{q})\boldsymbol{n}_p + (\boldsymbol{n}_p \cdot \boldsymbol{p})\boldsymbol{n}_q = 2\Big((\boldsymbol{n}_p \cdot \boldsymbol{n}_q)\boldsymbol{r} - (\boldsymbol{r} \cdot \boldsymbol{n}_q)\boldsymbol{n}_p - (\boldsymbol{r} \cdot \boldsymbol{n}_p)\boldsymbol{n}_q\Big), \tag{50}$$

where we have applied definition (31) in the final step. The above expression for the vector $\boldsymbol{e}_0$-part M_{0v} shows that it is independent of the position of the center $\boldsymbol{c}$ of the pair of points.

Note that the vector $\boldsymbol{e}_0$-part M_{0v} is antisymmetric when exchanging the two oriented points P and Q, but it is symmetric when only interchanging the two point orientations $\boldsymbol{n}_p \leftrightarrow \boldsymbol{n}_q$.

Note further that we have the relationship

$$M_{Eb+} = M_{0v} \wedge \boldsymbol{r}. \tag{51}$$

3.6. Trivector $\boldsymbol{e}_0$-part

The trivector $\boldsymbol{e}_0$-part is found to be

$$M_{0t} = \boldsymbol{n}_p \wedge \boldsymbol{p} \wedge \boldsymbol{n}_q + \boldsymbol{n}_p \wedge \boldsymbol{n}_q \wedge \boldsymbol{q} = (\boldsymbol{q} - \boldsymbol{p}) \wedge \boldsymbol{n}_p \wedge \boldsymbol{n}_q = 2\boldsymbol{r} \wedge \boldsymbol{n}_p \wedge \boldsymbol{n}_q, \quad (52)$$

is manifestly independent of the position of the center $\boldsymbol{c}$ of the pair of points, and is indeed symmetric under the interchange of the two oriented points P and Q.

Remark 1. *Altogether we have thus found five constituents of the $Cl(3,0)$ multivector coefficients of the geometric product P^*Q^* of two oriented points that are independent of the position of the center $\boldsymbol{c}$ of the pair of points, namely M_s of (34), M_{b-} of (41), M_{Eb+} of (49), M_{0v} of (50), and M_{0t} of (52).*

3.7. Vector $\boldsymbol{e}_\infty$-part

The vector $\boldsymbol{e}_\infty$-part is found to be

$$\begin{aligned} M_{\infty v} = &\frac{1}{2}\boldsymbol{q}^2[\boldsymbol{n}_q \cdot (\boldsymbol{p} \wedge \boldsymbol{n}_p)] - (\boldsymbol{q} \cdot \boldsymbol{n}_q)[\boldsymbol{q} \cdot (\boldsymbol{p} \wedge \boldsymbol{n}_p)] + \frac{1}{2}\boldsymbol{p}^2[\boldsymbol{n}_p \cdot (\boldsymbol{n}_q \wedge \boldsymbol{q})] \\ &- (\boldsymbol{p} \cdot \boldsymbol{n}_p)[\boldsymbol{p} \cdot (\boldsymbol{n}_q \wedge \boldsymbol{q})] + \frac{1}{2}\boldsymbol{p}^2(\boldsymbol{n}_q \cdot \boldsymbol{q})\boldsymbol{n}_p - (\boldsymbol{n}_q \cdot \boldsymbol{q})(\boldsymbol{p} \cdot \boldsymbol{n}_p)\boldsymbol{p} \\ &- \frac{1}{2}\boldsymbol{q}^2(\boldsymbol{n}_p \cdot \boldsymbol{p})\boldsymbol{n}_q + (\boldsymbol{n}_p \cdot \boldsymbol{p})(\boldsymbol{q} \cdot \boldsymbol{n}_q)\boldsymbol{q} \\ = &[\frac{1}{2}\boldsymbol{q}^2(\boldsymbol{p} \cdot \boldsymbol{n}_q) - (\boldsymbol{p} \cdot \boldsymbol{q})(\boldsymbol{q} \cdot \boldsymbol{n}_q) + \frac{1}{2}\boldsymbol{p}^2(\boldsymbol{q} \cdot \boldsymbol{n}_q)]\boldsymbol{n}_p \\ &+ [-\frac{1}{2}\boldsymbol{p}^2(\boldsymbol{q} \cdot \boldsymbol{n}_p) + (\boldsymbol{p} \cdot \boldsymbol{q})(\boldsymbol{p} \cdot \boldsymbol{n}_p) - \frac{1}{2}\boldsymbol{q}^2(\boldsymbol{p} \cdot \boldsymbol{n}_p)]\boldsymbol{n}_q \\ &+ [-\frac{1}{2}\boldsymbol{q}^2(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) + (\boldsymbol{q} \cdot \boldsymbol{n}_p)(\boldsymbol{q} \cdot \boldsymbol{n}_q) - (\boldsymbol{p} \cdot \boldsymbol{n}_p)(\boldsymbol{q} \cdot \boldsymbol{n}_q)]\boldsymbol{p} \\ &+ [\frac{1}{2}\boldsymbol{p}^2(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) - (\boldsymbol{p} \cdot \boldsymbol{n}_p)(\boldsymbol{p} \cdot \boldsymbol{n}_q) + (\boldsymbol{p} \cdot \boldsymbol{n}_p)(\boldsymbol{q} \cdot \boldsymbol{n}_q)]\boldsymbol{q}. \end{aligned} \quad (53)$$

Using definition (31) leads to the expression

$$\begin{aligned} M_{\infty v} = &\boldsymbol{r}[(\boldsymbol{r}^2 + \boldsymbol{c}^2)(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) - 4(\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{r} \cdot \boldsymbol{n}_q) \\ &- 2(\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{c} \cdot \boldsymbol{n}_q) + 2(\boldsymbol{c} \cdot \boldsymbol{n}_p)(\boldsymbol{r} \cdot \boldsymbol{n}_q)] \\ &+ 2\boldsymbol{c}[-(\boldsymbol{c} \cdot \boldsymbol{r})(\boldsymbol{n}_p \cdot \boldsymbol{n}_q) + (\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{c} \cdot \boldsymbol{n}_q) + (\boldsymbol{c} \cdot \boldsymbol{n}_p)(\boldsymbol{r} \cdot \boldsymbol{n}_q)] \\ &+ \boldsymbol{n}_p[2\boldsymbol{r}^2(\boldsymbol{c} \cdot \boldsymbol{n}_q) + (-\boldsymbol{c}^2 - 2\boldsymbol{c} \cdot \boldsymbol{r} + \boldsymbol{r}^2)(\boldsymbol{r} \cdot \boldsymbol{n}_q)] \\ &- \boldsymbol{n}_q[2\boldsymbol{r}^2(\boldsymbol{c} \cdot \boldsymbol{n}_p) + (\boldsymbol{c}^2 - 2\boldsymbol{c} \cdot \boldsymbol{r} - \boldsymbol{r}^2)(\boldsymbol{r} \cdot \boldsymbol{n}_p)]. \end{aligned} \quad (54)$$

Note that the vector $\boldsymbol{e}_\infty$-part $M_{\infty v}$ is indeed antisymmetric when exchanging the two oriented points P and Q.

For a pair of points centered at the origin ($\boldsymbol{c} = 0$), $M_{\infty v}$ reduces to

$$\begin{aligned} M_{\infty v} &= [-4(\boldsymbol{r} \cdot \boldsymbol{n}_p)(\boldsymbol{r} \cdot \boldsymbol{n}_q) + \boldsymbol{r}^2(\boldsymbol{n}_p \cdot \boldsymbol{n}_q)]\boldsymbol{r} + \boldsymbol{r}^2(\boldsymbol{r} \cdot \boldsymbol{n}_q)\boldsymbol{n}_p + \boldsymbol{r}^2(\boldsymbol{r} \cdot \boldsymbol{n}_p)\boldsymbol{n}_q \\ &= |\boldsymbol{r}|^3\Big([-4(\widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_p)(\widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_q) + (\boldsymbol{n}_p \cdot \boldsymbol{n}_q)]\widehat{\boldsymbol{r}} + (\widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_q)\boldsymbol{n}_p + (\widehat{\boldsymbol{r}} \cdot \boldsymbol{n}_p)\boldsymbol{n}_q\Big), \end{aligned} \quad (55)$$

which is symmetric when only interchanging the two point orientations $\boldsymbol{n}_p \leftrightarrow \boldsymbol{n}_q$.

3.8. Trivector $\boldsymbol{e}_\infty$-part

The trivector $\boldsymbol{e}_\infty$-part is found to be

$$\begin{aligned} M_{\infty t} &= \frac{1}{2}\boldsymbol{q}^2(\boldsymbol{n}_p \wedge \boldsymbol{p} \wedge \boldsymbol{n}_q) + \frac{1}{2}\boldsymbol{p}^2(\boldsymbol{n}_p \wedge \boldsymbol{n}_q \wedge \boldsymbol{q}) \\ &\quad - (\boldsymbol{q}\cdot\boldsymbol{n}_q)(\boldsymbol{n}_p \wedge \boldsymbol{p} \wedge \boldsymbol{q}) - (\boldsymbol{p}\cdot\boldsymbol{n}_p)(\boldsymbol{p} \wedge \boldsymbol{n}_q \wedge \boldsymbol{q}) \\ &= -2(\boldsymbol{c}\cdot\boldsymbol{r})(\boldsymbol{c} \wedge \boldsymbol{n}_p \wedge \boldsymbol{n}_q) + (\boldsymbol{c}^2 + \boldsymbol{r}^2)(\boldsymbol{r} \wedge \boldsymbol{n}_p \wedge \boldsymbol{n}_q) \\ &\quad - 2(\boldsymbol{c}\cdot\boldsymbol{n}_q + \boldsymbol{r}\cdot\boldsymbol{n}_q)(\boldsymbol{c} \wedge \boldsymbol{r} \wedge \boldsymbol{n}_p) + 2(\boldsymbol{c}\cdot\boldsymbol{n}_p - \boldsymbol{r}\cdot\boldsymbol{n}_p)(\boldsymbol{c} \wedge \boldsymbol{r} \wedge \boldsymbol{n}_q), \end{aligned} \tag{56}$$

which is indeed seen to be symmetric under the exchange of the two oriented points P and Q.

For a pair of points centered at the origin ($\boldsymbol{c} = 0$), $M_{\infty t}$ reduces to

$$M_{\infty t} = \boldsymbol{r}^2(\boldsymbol{r} \wedge \boldsymbol{n}_p \wedge \boldsymbol{n}_q) = |\boldsymbol{r}|^3(\widehat{\boldsymbol{r}} \wedge \boldsymbol{n}_p \wedge \boldsymbol{n}_q). \tag{57}$$

Note that for $\boldsymbol{c} = 0$, the ratio of $M_{\infty t}$ of (56) and M_{0t} of (52) allows to directly compute the scalar point pair radius

$$\frac{M_{\infty t}}{M_{0t}} = \frac{1}{2}\boldsymbol{r}^2. \tag{58}$$

4. Explicit example of geometric product of two conformal points

This section presents a numerical example, computed with The Clifford Multivector Toolbox for MATLAB [22] for the full geometric product of two conformal points. We define the position and unit orientation vectors of the two points as

$$\begin{aligned} &\boldsymbol{p} = 3\boldsymbol{e}_1 - 4\boldsymbol{e}_2 + 5\boldsymbol{e}_3, \quad \boldsymbol{n}_p = -0.2\boldsymbol{e}_1 + 0.4\boldsymbol{e}_2 - 0.8944\boldsymbol{e}_3, \\ &\boldsymbol{q} = \boldsymbol{e}_1 + 2\boldsymbol{e}_2, \quad \boldsymbol{n}_q = 0.5\boldsymbol{e}_1 + 0.3\boldsymbol{e}_2 + 0.8124\boldsymbol{e}_3. \end{aligned} \tag{59}$$

The two corresponding oriented points in CGA are then

$$\begin{aligned} P^* &= -0.4e_{12} + 1.6833e_{13} - 1.5777e_{23} + (15.0164\boldsymbol{e}_1 - 16.6885\boldsymbol{e}_2 + 11\boldsymbol{e}_3)\boldsymbol{e}_\infty \\ &\quad + (-0.2\boldsymbol{e}_1 + 0.4\boldsymbol{e}_2 - 0.8944\boldsymbol{e}_3)\boldsymbol{e}_0 - 6.6721E, \\ Q^* &= 0.7e_{12} - 0.8124e_{13} - 1.6248e_{23} + (0.15\boldsymbol{e}_1 - 1.4500\boldsymbol{e}_2 + 2.0310\boldsymbol{e}_3)\boldsymbol{e}_\infty \\ &\quad + (0.5\boldsymbol{e}_1 + 0.3\boldsymbol{e}_2 + 0.8124\boldsymbol{e}_3)\boldsymbol{e}_0 + 1.1E. \end{aligned} \tag{60}$$

Their full geometric product is

$$\begin{aligned} P^*Q^* &= 0.7562 + 17.0959e_{12} + 8.1817e_{13} - 16.4890e_{23} \\ &\quad + (42.1361\boldsymbol{e}_1 - 2.7920\boldsymbol{e}_2 + 38.0273\boldsymbol{e}_3 - 28.8649e_{123})\boldsymbol{e}_\infty \\ &\quad + (-2.8752\boldsymbol{e}_1 - 5.1166\boldsymbol{e}_2 - 5.2924\boldsymbol{e}_3 - 1.5950e_{123})\boldsymbol{e}_0 \\ &\quad + (-13.8647 - 17.7297e_{12} + 0.3007e_{13} + 25.4788e_{23})E \end{aligned} \tag{61}$$

The eight $Cl(3,0)$ multivector components can then be identified as

$$\begin{aligned} M_s &= 0.7562, \quad M_b = 17.0959e_{12} + 8.1817e_{13} - 16.4890e_{23}, \\ M_{Es} &= -13.8647, \quad M_{Eb} = -17.7297e_{12} + 0.3007e_{13} + 25.4788e_{23}, \\ M_{0v} &= -2.8752\boldsymbol{e}_1 - 5.1166\boldsymbol{e}_2 - 5.2924\boldsymbol{e}_3, \quad M_{0t} = -1.5950e_{123}, \\ M_{\infty v} &= 42.1361\boldsymbol{e}_1 - 2.7920\boldsymbol{e}_2 + 38.0273\boldsymbol{e}_3, \quad M_{\infty t} = -28.8649e_{123}. \end{aligned} \tag{62}$$

Symmetric and antisymmetric parts of the bivectors M_b and M_{Eb} under exchange of the orientation vectors $\boldsymbol{n}_p \leftrightarrow \boldsymbol{n}_q$ are

$$\begin{aligned} M_{b+} &= 13.1085e_{12} + 3.3967e_{13} - 18.084e_{23}, \\ M_{b-} &= 3.9874e_{12} + 4.7849e_{13} + 1.5950e_{23}, \\ M_{Eb+} &= -13.7422e_{12} + 1.8956e_{13} + 28.6687e_{23}, \\ M_{Eb-} &= -3.9874e_{12} - 1.595e_{13} - 3.19e_{23}. \end{aligned} \tag{63}$$

Centering the point pair at the origin ($\boldsymbol{c} = 0$) gives

$$\begin{aligned} T(\boldsymbol{c})P^*Q^*T(-\boldsymbol{c}) &= 0.7562 + 3.9874e_{12} + 4.7849e_{13} + 1.5950e_{23} \\ &+ (22.6048\boldsymbol{e}_1 + 43.8413\boldsymbol{e}_2 + 41.1101\boldsymbol{e}_3 - 12.9592e_{123})\boldsymbol{e}_\infty \\ &+ (-2.8752\boldsymbol{e}_1 - 5.1166\boldsymbol{e}_2 - 5.2924\boldsymbol{e}_3 - 1.5950e_{123})\boldsymbol{e}_0 \\ &+ (-13.7422e_{12} + 1.8956e_{13} + 28.6687e_{23})E \end{aligned} \tag{64}$$

Comparison of (61) and (64) illustrates the invariance of the parts M_s and M_0 under translation.

We furthermore list for this special case ($\boldsymbol{c} = 0$) the symmetric and antisymmetric parts of the bivectors M_{b0} and M_{Eb0}

$$\begin{aligned} M_{b0+} &= 0, \\ M_{b0-} &= 3.9874e_{12} + 4.7849e_{13} + 1.5950e_{23}, \\ M_{Eb0+} &= -13.7422e_{12} + 1.8956e_{13} + 28.6687e_{23}, \\ M_{Eb0-} &= 0, \end{aligned} \tag{65}$$

which illustrates that for $\boldsymbol{c} = 0$:

$$M_{b0+} = 0, \quad M_{b0-} = M_{b-}, \quad M_{Eb0+} = M_{Eb+}, \quad M_{Eb0-} = 0. \tag{66}$$

5. Conclusion

In this work we have computed all parts of the full geometric product of two oriented points in conformal geometric algebra (CGA) $Cl(4,1)$ of three-dimensional Euclidean geometry. The computations have been validated with The Clifford Multivector Toolbox for MATLAB [22], using a representative example. Only the scalar part has previously been computed, analyzed [15, 17], and applied [20,21]. The symmetry of all eight resulting parts was stated and an important alternative representation in terms of the center position and the radius vector of the pair of oriented points was given. We expect that this theoretical work provides the foundation for better understanding the geometry of oriented points, which is likely to lead to further concrete applications.

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Conflict of interest The author declares that he has no conflict of interest.

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