A convergent subsequence of $\theta_n(x + iy)$ in a half strip

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January 20, 2024

Abstract

For $\frac{1}{2} < x < 1$, $y > 0$ and $n \in \mathbb{N}$, let $\theta_n(x + iy) = \sum_{i=1}^{n} \frac{\text{sgn} q_i q_i x + iy}{q_i}$, where $Q = \{q_1, q_2, q_3, \cdots\}$ is the set of finite product of distinct odd primes and $\text{sgn} q = (-1)^k$ if $q$ is the product of $k$ distinct primes. In this paper we prove that there exists an ordering on $Q$ such that $\theta_n(x + iy)$ has a convergent subsequence.

2020 Mathematics Subject Classification ; 11M26.

1 Introduction

Let $\mathbb{N}$ be the set of natural numbers and $P$ be the set of odd primes.

Definition 1.1. For an ordering on $P = \{p_1, p_2, p_3, \cdots\}$ and $m \in \mathbb{N}$, let

$$P_m = \{p_1, p_2, \cdots, p_m\}.$$ 

Definition 1.2. Let $Q$ be the set of finite products of distinct odd primes.

$$Q = \{p_1 p_2 \cdots p_k \mid k \in \mathbb{N} \text{ and } p_1, p_2, \cdots, p_k \text{ are distinct primes in } P\}$$

and, for each $m \in \mathbb{N}$, let

$$U_m = \{p_1 p_2 \cdots p_k \mid k \in \mathbb{N} \text{ and } p_1, p_2, \cdots, p_k \text{ are distinct primes in } P_m\}.$$ 

Notice that $U_m$ depends on the choice of ordering on $P$ and $U_m \subset U_{m+1}$.

Lemma 1.3. The number of elements of $U_m$ is $2^m - 1$. 

\[1\]
Proof. Since

\[ U_m = \{p_1, \cdots, p_m, p_1p_2, \cdots, p_{m-1}p_m, p_1p_2p_3, \cdots, p_1p_2 \cdots p_m\}, \]

the number of elements of \( U_m \) is

\[ \left( \begin{array}{c} m \\ 1 \end{array} \right) + \left( \begin{array}{c} m \\ 2 \end{array} \right) + \cdots + \left( \begin{array}{c} m \\ m \end{array} \right) = 2^m - 1. \]

\[ \square \]

**Definition 1.4.** Let

\[ Q_1 = U_1 \] and \( Q_m = U_m - U_{m-1} \) for each \( m = 2, 3, 4, \cdots \).

Notice that

\[ Q_m = \{p_m, p_mq \mid q \in U_{m-1}\}, \quad \bigcup_{i=1}^{m} Q_m = U_m \quad (1) \]

and \( Q_1, Q_2, Q_3, \cdots \) are mutually disjoint. Notice also that the number of elements of \( Q_m \) is

\[ (2^m - 1) - (2^{m-1} - 1) = 2^{m-1}. \]

**Example 1.5.** In the increasing ordering on \( P \), we have

\[ p_1 = 3, \quad p_2 = 5, \quad p_3 = 7, \cdots. \]

Therefore

\[ Q_1 = \{3\}, \quad Q_2 = \{5, 3 \cdot 5\}, \quad Q_3 = \{7, 3 \cdot 7, 5 \cdot 7, 3 \cdot 5 \cdot 7\}, \cdots. \]

**Definition 1.6.** An ordering on \( P \) and the following two conditions (C1)-(C2) induce a unique ordering on \( Q = \{q_1, q_2, q_3, \cdots\} \).

(C1) \( i < j \) if \( q_i < q_j \) and \( q_i, q_j \in Q_m \) for some \( m \).

(C2) \( i < j \) if \( q_i \in Q_m, \quad q_j \in Q_n \) for some \( m < n \)

Note that any ordering on \( P \) induces a unique ordering on \( Q \) in this way.

**Example 1.7.** Suppose that \( P \) has the increasing ordering. In the induced ordering on \( Q \), we have

\[ q_1 = 3, \quad q_2 = 5, \quad q_3 = 15, \quad q_4 = 7, \quad q_5 = 21, \quad q_6 = 35, \quad q_7 = 105, \quad q_8 = 11, \cdots. \]

**Definition 1.8.** For each \( q = p_1p_2 \cdots p_k \in Q \), let

\[ \text{sgn } q = (-1)^k \]

where \( p_1, p_2, \cdots, p_k \) are distinct odd primes.
Definition 1.9. Suppose that an ordering is given on \( Q = \{ q_1, q_2, q_3, \cdots \} \). For \( \frac{1}{2} < x < 1, y > 0 \) and \( n \in \mathbb{N} \), let
\[
\theta_n(x + iy) = \sum_{i=1}^{n} \frac{\text{sgn} q_i}{q_{i+1}^x + iy}
\]

In this paper we prove

Theorem 1.10. For each \( \frac{1}{2} < x < 1 \) and \( y > 0 \), there exists an ordering on \( P \) such that, under the induced ordering on \( Q \), \( \theta_n(x + iy) \) has a convergent subsequence.

2 Preliminary Theorems

We need the following theorem in the proof of Theorem 1.10.

Theorem 2.1 ([1]). Suppose that \( y > 0 \), \( 0 \leq \alpha < 2\pi \) and \( 0 < K < 1 \). Let \( P^+ \) be the set of primes \( p \) such that \( \cos(y \ln p + \alpha) > K \) and \( P^- \) the set of primes \( p \) such that \( \cos(y \ln p + \alpha) < -K \). Then we have
\[
\sum_{p \in P^+} \frac{1}{p} = \infty \quad \text{and} \quad \sum_{p \in P^-} \frac{1}{p} = \infty.
\]

From the argument in the proof of the Riemann rearrangement theorem, we have

Theorem 2.2 ([4],[5]). For a series \( \sum_{i=1}^{\infty} a_i \) of real numbers, suppose that
\[
\lim_{i \to \infty} a_i = 0
\]
and let
\[
a_i^+ = \max\{a_i, 0\} \quad \text{and} \quad a_i^- = -\min\{a_i, 0\}.
\]
If
\[
\sum_{i=1}^{\infty} a_i^+ = \sum_{i=1}^{\infty} a_i^- = \infty
\]
then there exists a rearrangement such that the series \( \sum_{i=1}^{\infty} a_i \) is convergent.

We need the Lévy-Steinitz theorem which is a generalization of the Riemann rearrangement theorem and Theorem 2.2.

Lévy-Steinitz theorem ([5]). The set of all sums of rearrangements of a given series of vectors
\[
\sum_{i=1}^{\infty} v_i
\]
in \( \mathbb{R}^n \) is either the empty set or a translate of subspace i.e., a set of the form \( \mathbf{v} + M \), where \( \mathbf{v} \) is a vector and \( M \) is a subspace. If the following two conditions (a)-(b) are satisfied then it is nonempty i.e., it has convergent rearrangements.
(a) \( \lim_{i \to \infty} v_i = 0 \)

(b) For all vector \( w \) in \( \mathbb{R}^n \),

\[
\sum_{i=1}^{\infty} (v_i, w)^+ \quad \text{and} \quad \sum_{i=1}^{\infty} (v_i, w)^-
\]

are either both finite or both infinite, where we use the notations in eq. (2) and \((v_i, w)\) is the Euclidean inner product of \( v_i \) and \( w \).

The Coriolis test is useful in the proof of Theorem 1.10.

**Coriolis Test** ([6]). If \( z_i \) is a sequence of complex numbers such that

\[
\sum_{i=1}^{\infty} z_i \quad \text{and} \quad \sum_{i=1}^{\infty} |z_i|^2
\]

are convergent, then

\[
\prod_{i=1}^{\infty} (1 + z_i)
\]

converges.

### 3 Proof of Theorem 1.10

**Definition 3.1.** Suppose that \( P \) has the increasing ordering. For \( \frac{1}{2} < x < 1 \) and \( y > 0 \), let

\[
\rho(x + iy) = \frac{1}{2^x + iy} + \sum_{i=1}^{\infty} \frac{1}{p_i^{x+iy}}
\]

\[
= \frac{\cos(y \ln 2) - i \sin(y \ln 2)}{2^x} + \sum_{i=1}^{\infty} \frac{\cos(y \ln p_i) - i \sin(y \ln p_i)}{p_i^{x+iy}}
\]

**Lemma 3.2.** \( \rho(x + iy) \) has a convergent rearrangement and therefore

\[
\sum_{i=1}^{\infty} \frac{1}{p_i^{x+iy}}
\]

has a convergent rearrangement, too. In other words, \( P \) has an ordering such that eq. (3) is convergent.

**Proof.** Recall that \( \frac{1}{2} < x < 1 \) and \( y > 0 \). Let

\[
v_1 = \left( \frac{\cos(y \ln 2)}{2^x}, -\frac{\sin(y \ln 2)}{2^x} \right)
\]
and, for $i \in \mathbb{N}$, let

$$v_{i+1} = \left(\frac{\cos(y \ln p_i)}{p_i^x}, -\frac{\sin(y \ln p_i)}{p_i^x}\right).$$

Since $P$ has the increasing ordering, we have

$$\lim_{i \to \infty} v_i = 0. \quad (4)$$

Let

$$w = r(\cos \alpha, \sin \alpha)$$

be a vector in $\mathbb{R}^2$, where $r \geq 0$ and $0 \leq \alpha < 2\pi$. If $r = 0$ then $(v_i, w) = 0$ for all $i \in \mathbb{N}$ and therefore

$$\sum_{i=1}^{\infty} (v_i, w)^+ = \sum_{i=1}^{\infty} (v_i, w)^- = 0. \quad (5)$$

Suppose that $r > 0$. We have

$$v_1 \cdot w = \frac{r \cos(y \ln 2) \cos \alpha - r \sin(y \ln 2) \sin \alpha}{2^x} = \frac{r \cos(y \ln 2 + \alpha)}{2^x},$$

and

$$v_{i+1} \cdot w = \frac{r \cos(y \ln p_i) \cos \alpha - r \sin(y \ln p_i) \sin \alpha}{p_i^x} = \frac{r \cos(y \ln p_i + \alpha)}{p_i^x}.$$
Therefore
\[ \sum_{i=1}^{\infty} (v_i, w)^+ = \sum_{i=1}^{\infty} (v_i, w)^- = \infty. \] (6)

From eq. (4), (5), (6) and Lévy-Steinitz theorem, we know that the series of vectors in \( \mathbb{R}^2 \)
\[ \sum_{i=1}^{\infty} v_i \]
has a convergent rearrangement, and therefore \( \rho(x + iy) \) has a convergent rearrangement. \( \square \)

**Lemma 3.3.** Let \( z = x + iy \). For all \( m \in \mathbb{N} \), we have
\[ \prod_{i=1}^{m} \left( 1 - \frac{1}{p_i^2} \right) - 1 = \sum_{q \in Q_1} \frac{\text{sgn} q}{q^2} + \sum_{q \in Q_2} \frac{\text{sgn} q}{q^2} + \cdots + \sum_{q \in Q_m} \frac{\text{sgn} q}{q^2}. \]

**Proof.** We use induction on \( m \). If \( m = 1 \), it is clear. Suppose that it is true for \( m = k - 1 \). We will show that it is true for \( m = k \). From eq. (1), we have
\[ \prod_{i=1}^{k} \left( 1 - \frac{1}{p_i^2} \right) = \left( \prod_{i=1}^{k-1} \left( 1 - \frac{1}{p_i^2} \right) \right) \left( 1 - \frac{1}{p_k^2} \right) \]
\[ = \left( 1 + \sum_{q \in Q_1} \frac{\text{sgn} q}{q^2} + \cdots + \sum_{q \in Q_{k-1}} \frac{\text{sgn} q}{q^2} \right) \left( 1 - \frac{1}{p_k^2} \right) \]
\[ = \left( 1 + \sum_{q \in Q_1} \frac{\text{sgn} q}{q^2} + \cdots + \sum_{q \in Q_{k-1}} \frac{\text{sgn} q}{q^2} \right) - \frac{1}{p_k^2} \left( 1 + \sum_{q \in Q_{k-1}} \frac{\text{sgn} q}{q^2} + \cdots + \sum_{q \in Q_k} \frac{\text{sgn} q}{q^2} \right) \]
\[ = 1 + \sum_{q \in Q_1} \frac{\text{sgn} q}{q^2} + \cdots + \sum_{q \in Q_{k-1}} \frac{\text{sgn} q}{q^2} + \sum_{q \in Q_k} \frac{\text{sgn} q}{q^2} \]
\[ \square \]

Now we can prove Theorem 1.10.

**Proof of Theorem 1.10**
By Lemma 3.2, we can choose an ordering on $P$ such that

$$\sum_{i=1}^{\infty} \frac{1}{p_i^{x+iy}}$$

is convergent. From now on, we assume that $P$ has the chosen ordering, and $Q$ has the induced ordering.

Since $\frac{1}{2} < x < 1$,

$$\sum_{i=1}^{\infty} \left| \frac{1}{p_i^{x+iy}} \right|^2 = \sum_{i=1}^{\infty} \frac{1}{p_i^{2x}}$$

is convergent. Therefore, by the Coriolis test,

$$\prod_{i=1}^{\infty} \left( 1 - \frac{1}{p_i^{x+iy}} \right)$$

is convergent. By Lemma 3.3, Lemma 1.3 and eq. (1), we have

$$\prod_{i=1}^{m} \left( 1 - \frac{1}{p_i^{x+iy}} \right) - 1 = \sum_{q \in Q_1} \frac{\text{sgn } q}{q^{x+iy}} + \sum_{q \in Q_2} \frac{\text{sgn } q}{q^z} + \cdots + \sum_{q \in Q_m} \frac{\text{sgn } q}{q^{x+iy}}$$

$$= \sum_{q \in U_m} \frac{\text{sgn } q}{q^{x+iy}}$$

$$= \sum_{i=1}^{2^m-1} \frac{\text{sgn } q_i}{q_i^{x+iy}}.$$  

Therefore

$$\theta_{2^m-1}(x + iy) = \sum_{q \in U_m} \frac{\text{sgn } q}{q^{x+iy}}$$  \hspace{1cm} (7)

is a convergent subsequence of $\theta_n(x + iy)$. \hspace{1cm} \boxed{} 

References


