I. Abstract:
In his 1859 article "On the number of prime numbers less than a given quantity", Bernhard Riemann formulated the hypothesis that all non-trivial zeros of the Zeta function have the real part 1/2.

This assertion, known as the "Riemann Hypothesis", remains unproven to this day.

The present paper is an attempt at a direct demonstration.

II. Demonstration:
The demonstration proposed here is based on two well-known results:

1. Zeta function as Hadamard product on one side:
\[
\zeta(s) = \frac{e^{\left(\ln(2\pi) - \frac{1}{2}\right)s}}{2(s - 1)\Gamma\left(1 + \frac{s}{2}\right)} \prod\rho \left(1 - \frac{s}{\rho}\right) e^{s/\rho}
\]
(1)

Where the \( \rho \) are the non-trivial zeros of the Zeta function.

2. The value of \( \zeta(-1) \) on the other hand:
\[
\zeta(-1) = -\frac{1}{12}
\]
(2)

Replacing \( s \) by -1 in expression (1) and equating (1) and (2), we obtain:
\[
\zeta(-1) = \frac{e^{-\left(\ln(2\pi) - \frac{1}{2}\right)}}{2(-2)\sqrt{\pi}} \prod_{\rho} \left(1 + \frac{1}{\rho}\right) e^{-1/\rho} = -\frac{1}{12}
\]

Now, the calculation shows that
\[
\frac{e^{-\left(\ln(2\pi) - \frac{1}{2}\right)}}{2(-2)\sqrt{\pi}} \approx -\frac{1}{12}
\]

And so
\[
\prod_{\rho} \left(1 + \frac{1}{\rho}\right) e^{-1/\rho} = 1
\]
(3)
Since the number of zeros is infinite, the number of factors \((1 + \frac{1}{\rho})e^{-1/\rho}\) is also infinite, so that equality (3) can only be verified if each factor \((1 + \frac{1}{\rho})e^{-1/\rho}\) is equal to 1 and therefore for any zero \(\rho\) of the Zeta function :

\[
(1 + \frac{1}{\rho})e^{-1/\rho} = 1 \quad (4)
\]

**NB:** This equality has been verified numerically for the first 100 zeros of the zeta function.

Posing \(\rho = \sigma + it\), then:

\[
(1 - \frac{\sigma-it}{\sigma^2+t^2})e^{\frac{-\sigma}{\sigma^2+t^2}}(\cos \frac{t}{\sigma^2+t^2} + isin \frac{t}{\sigma^2+t^2}) = 1 \quad (5)
\]

Now, \(e^{\frac{-\sigma}{\sigma^2+t^2}} \approx 1\) because \(0 < \sigma < 1\) (located inside the critical strip) and therefore \(-\sigma \ll \sigma^2 + t^2\) when \(t^2\) tends to infinity.

So \((1 - \frac{\sigma-it}{\sigma^2+t^2})(\cos \frac{t}{\sigma^2+t^2} + isin \frac{t}{\sigma^2+t^2}) = 1\)

\[
\Rightarrow \left[(1 - \frac{\sigma}{\sigma^2+t^2}) + i\frac{t}{\sigma^2+t^2}\right](\cos \frac{t}{\sigma^2+t^2} + isin \frac{t}{\sigma^2+t^2}) = 1
\]

\[
\Rightarrow \left[(1 - \frac{\sigma}{\sigma^2+t^2})\cos \frac{t}{\sigma^2+t^2} - \left(\frac{t}{\sigma^2+t^2}\right)(\sin \frac{t}{\sigma^2+t^2})\right] + i\left[(1 - \frac{\sigma}{\sigma^2+t^2})(\sin \frac{t}{\sigma^2+t^2}) + \left(\frac{t}{\sigma^2+t^2}\right)(\cos \frac{t}{\sigma^2+t^2})\right] = 1
\]

Equalizing the real and imaginary parts, we obtain the equations :

\[
\left[(1 - \frac{\sigma}{\sigma^2+t^2})\cos \frac{t}{\sigma^2+t^2} - \left(\frac{t}{\sigma^2+t^2}\right)(\sin \frac{t}{\sigma^2+t^2})\right] = 1 \quad (6)
\]

and

\[
\left[(1 - \frac{\sigma}{\sigma^2+t^2})\sin \frac{t}{\sigma^2+t^2} + \left(\frac{t}{\sigma^2+t^2}\right)(\cos \frac{t}{\sigma^2+t^2})\right] = 0 \quad (7)
\]

Squaring (6) and (7) gives :

\[
\left(1 - \frac{\sigma}{\sigma^2+t^2}\right)^2\left(\cos \frac{t}{\sigma^2+t^2}\right)^2 - 2\left(1 - \frac{\sigma}{\sigma^2+t^2}\right)\left(\cos \frac{t}{\sigma^2+t^2}\right)\left(\sin \frac{t}{\sigma^2+t^2}\right) + \left(\frac{t}{\sigma^2+t^2}\right)^2\left(\sin \frac{t}{\sigma^2+t^2}\right)^2 = 1 \quad (8)
\]

and

\[
\left(1 - \frac{\sigma}{\sigma^2+t^2}\right)^2\left(\sin \frac{t}{\sigma^2+t^2}\right)^2 + 2\left(1 - \frac{\sigma}{\sigma^2+t^2}\right)\left(\cos \frac{t}{\sigma^2+t^2}\right)\left(\sin \frac{t}{\sigma^2+t^2}\right) + \left(\frac{t}{\sigma^2+t^2}\right)^2\left(\cos \frac{t}{\sigma^2+t^2}\right)^2 = 0 \quad (9)
\]
Summing (8) and (9), it remains \( \left( 1 - \frac{\sigma}{\sigma^2 + t^2} \right)^2 + \left( \frac{t}{\sigma^2 + t^2} \right)^2 = 1 \)

\[
\Rightarrow 1 - 2 \frac{\sigma}{\sigma^2 + t^2} + \frac{\sigma^2}{(\sigma^2 + t^2)^2} + \frac{t^2}{(\sigma^2 + t^2)^2} = 1
\]

\[
\Rightarrow \frac{\sigma^2}{(\sigma^2 + t^2)^2} + \frac{t^2}{(\sigma^2 + t^2)^2} = 2 \frac{\sigma}{\sigma^2 + t^2}
\]

\[
\Rightarrow \frac{\sigma^2 + t^2}{(\sigma^2 + t^2)^2} = 2 \frac{\sigma}{\sigma^2 + t^2}
\]

\[
\Rightarrow 1 = 2 \sigma \quad \text{and therefore}
\]

\[
\sigma = \frac{1}{2}
\]

Vincent KOCH, November 24th 2023

III. Bibliography and videography

1. Books:

2. Articles:
   - François De Marçay, *Fonction Gamma d’Euler et fonction zêta de Riemann*.

3. Videos:
   - Factorials, prime numbers, and the Riemann Hypothesis
     [https://www.youtube.com/watch?v=oVaSA_b938U&t=132s](https://www.youtube.com/watch?v=oVaSA_b938U&t=132s)
   - The Basel Problem Part 1: Euler-Maclaurin Approximation
     [https://www.youtube.com/watch?v=nxJl4Uk4i00&t=490s](https://www.youtube.com/watch?v=nxJl4Uk4i00&t=490s)
   - The Basel Problem Part 2: Euler’s Proof and the Riemann Hypothesis
     [https://www.youtube.com/watch?v=FCpRl0NzVu4&t=781s](https://www.youtube.com/watch?v=FCpRl0NzVu4&t=781s)
   - Analytic Continuation and the Zeta Function
     [https://www.youtube.com/watch?v=CjSKmcWRFzE&t=21s](https://www.youtube.com/watch?v=CjSKmcWRFzE&t=21s)
   - Complex Integration and Finding Zeros of the Zeta Function
     [https://www.youtube.com/watch?v=uKqC5uHjE4g](https://www.youtube.com/watch?v=uKqC5uHjE4g)
   - But what is the Riemann zeta function? Visualizing analytic continuation
     [https://www.youtube.com/watch?v=sD0NjbwqlYw&t=1166s](https://www.youtube.com/watch?v=sD0NjbwqlYw&t=1166s)