A (1.999999)-APPROXIMATION RATIO FOR VERTEX COVER PROBLEM

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ABSTRACT

The vertex cover problem is a famous combinatorial problem, and its complexity has been heavily studied. While a 2-approximation for it can be trivially obtained, researchers have not been able to approximate it better than 2-$o(1)$. In this paper, by introducing a new semidefinite programming formulation that satisfies new properties, we introduce an approximation algorithm for the vertex cover problem with a performance ratio of 1.999999 on arbitrary graphs, en route to answering an open question about the correctness of the unique games conjecture.

Keywords Combinatorial Optimization · Vertex Cover Problem · Unique Games Conjecture · Complexity Theory

1 Introduction

In complexity theory, the abbreviation *NP* refers to "nondeterministic polynomial", where a problem is in *NP* if we can quickly (in polynomial time) test whether a solution is correct. *P* and *NP*-complete problems are subsets of *NP* Problems. We can solve *P* problems in polynomial time while determining whether or not it is possible to solve *NP*-complete problems quickly (called the *P vs NP* problem) is one of the principal unsolved problems in Mathematics and Computer science.

Here, we consider the vertex cover problem (VCP) which is a famous *NP*-complete problem. It cannot be approximated within a factor of 1.36 [1], unless $P = NP$, while a 2-approximation factor for it can be trivially obtained by taking all the vertices of a maximal matching in the graph. However, improving this simple 2-approximation algorithm is a hard task [2, 3].

In this paper, based on a lower bound on the objective value of VCP feasible solutions, we introduce a (2-ε)-approximation ratio, where the value of ε is not constant. Then, we introduce a new semidefinite programming (SDP) formulation and fix the ε value equal to ε=0.000001, to produce a 1.999999-approximation ratio on arbitrary graphs.

The rest of the paper is structured as follows. Section 2 is about the vertex cover problem and introduces new properties about it. In section 3, using a new SDP model whose solution satisfies the properties, we propose a solution algorithm for VCP with a performance ratio of 1.999999 on arbitrary graphs. Finally, Section 4 concludes the paper.

2 Performance ratio based on VCP feasible solutions

In the mathematical discipline of graph theory, a vertex cover of a graph is a set of vertices such that each edge of the graph is incident to at least one vertex of the set. The problem of finding a minimum vertex cover is a typical example of an *NP*-complete optimization problem. In this section, we calculate the performance ratios of VCP feasible solutions to produce an approximation ratio of 2-ε, where the value of ε is not constant and it depends on the VCP objective value. Then, in the next section, we will fix the value of ε equal to ε=0.000001, to produce a 1.999999-approximation ratio for the vertex cover problem.

Let $G = (V, E)$ be an undirected graph on vertex set V and edge set E, where |V|$= n$. Throughout this paper, $z^*(G)$ is the optimal value for the vertex cover problem on G, and VCP feasible solutions have been introduced by a vertex partitioning $V = V_1 \cup V_0$ with an objective value $\mid V_1 \mid$.

The integer linear programming (ILP) model for VCP is as follows; i.e. $z1^* = z^*(G)$.

$$(1)\ min_{s.t.}\ z1 = \sum_{i \in V} x_i$$

$$x_i + x_j \geq 1 \quad ij \in E$$

$$x_i \in \{0, +1\} \quad i \in V$$

Lemma 1. **[4]** Let x^* be an extreme optimal solution to the linear programming (LP) relaxation of the model (1). Then $x^*_j \in \{0, 0.5, 1\}$ for $j \in V$. If we define $V^0 = \{j \in V \mid x^*_j = 0\}$, $V^{0.5} = \{j \in V \mid x^*_j = 0.5\}$ and $V^1 = \{j \in V \mid x^*_j = 1\}$, then there exists a VCP optimal solution which includes all of the vertices V^1, and it is a subset of $V^{0.5} \cup V^1$.

Theorem 1. Let x^* be an extreme optimal solution to the LP relaxation of the model (1), $V^0 = \{j \in V \mid x^*_j = 0\}$, $V^{0.5} = \{j \in V \mid x^*_j = 0.5\}$, $V^1 = \{j \in V \mid x^*_j = 1\}$, and $G_{0.5}$ be the induced graph on the vertices $V^{0.5}$. If we can introduce a vertex cover feasible partitioning $V^{0.5} = V^{0.5}_1 \cup V^{0.5}_0$ with an approximation ratio of $1 \leq \rho < 2$, for the VCP on $G_{0.5}$, then the vertex cover feasible partitioning $V = (V_1 \cup V_0) = (V^{0.5}_1 \cup V^1) \cup (V^{0.5}_0 \cup V^0)$, has an approximation ratio of $1 \leq \rho < 2$, for the VCP on G.

Proof. Based on the approximation ratio of $\frac{|V^{0.5}_1|}{z^*(G_{0.5})} \leq \rho$, we have,

$$\mid V^{0.5}_1 \mid + \mid V^1 \mid \leq \rho z^*(G_{0.5}) + \rho \mid V^1 \mid$$

Therefore, $\frac{|V_1|}{z^*(G)} = \frac{|V^{0.5}_1| + |V^1|}{z^*(G_{0.5}) + |V^1|} \leq \rho \diamond$

The Theorem (1) says that it is sufficient to produce an approximation ratio of $1 \leq \rho < 2$, on $G_{0.5}$. Then, let's assume that for the optimal solution of the LP relaxation of the model (1), we have $V^0 = V^1 = \{\}$, $V^{0.5} = V$; i.e. $G = G_{0.5}$.

We know that we can efficiently solve the following SDP formulation, as a relaxation of the VCP model (1).

$$(2)\ min_{s.t.}\ z2 = \sum_{i \in V} X_{oi}$$

$$X_{oi} + X_{oj} \geq 1 \quad ij \in E$$

$$0 \leq X_{oi} \leq +1 \quad i \in V$$

$$X \succeq 0$$

This model can be written as follows,

$$(3)\ min_{s.t.}\ z3 = \sum_{i \in V} X_{oi}$$

$$X_{oi} + X_{oj} - X_{ij} = 1 \quad ij \in E$$

$$X_{ii} = 1, \quad 0 \leq X_{ij} \leq +1 \quad i, j \in V \cup \{o\}$$

$$X \succeq 0$$

Moreover, by introducing the normal vectors $v_o, v_1, ..., v_n$, the SDP model (3) can be written as follows, where $v_i v_j = X_{ij}$, $V_1 = \{i \in V \mid v_i = v_o\}$ is a feasible vertex cover, and $V_o = V - V_1$ is the set of perpendicular vectors to v_o.

$$(4)\ min_{s.t.}\ z4 = \sum_{i \in V} v_o v_i$$

$$v_o v_i + v_o v_j - v_i v_j = 1 \quad ij \in E$$

$$v_i v_i = 1, \quad 0 \leq v_i v_j \leq +1 \quad i, j \in V \cup \{o\}$$

Theorem 2. Let $\frac{n}{2}+\frac{n}{k}$ be a lower bound on VCP optimal value ($z^*(G) \geq \frac{n}{2}+\frac{n}{k}=\frac{(k+2)n}{2k}$). Then, for all vertex cover feasible partitioning $V=V_1 \cup V_0$, we have the approximation ratio $\frac{|V_1|}{z^*(G)} \leq \frac{2k}{k+2} < 2$.
Proof. If $z^*(G) \geq \frac{(k+2)n}{2k}$, then $\frac{n}{z^*(G)} \leq \frac{2k}{k+2}$. Therefore,

$$\frac{\mid V_1 \mid}{z^*(G)} \leq \frac{n}{z^*(G)} \leq \frac{2k}{k+2} < 2$$

and this completes the Proof $\diamond$

Theorem 3. Let $z^*(G) \geq \frac{n}{2}$, and we have a VCP feasible partitioning $V=V_1 \cup V_0$, where $\mid V_1 \mid \leq \frac{kn}{k+1}$ and $\mid V_0 \mid \geq \frac{n}{k+1}$ (or $\mid V_1 \mid \leq k \mid V_0 \mid$). Then, based on such a solution, we have an approximation ratio $\frac{|V_1|}{z^*(G)} \leq \frac{2k}{k+1} < 2$.
Proof. If $\mid V_1 \mid \leq \frac{kn}{k+1}$, then $n \geq \frac{k+1}{k} \mid V_1 \mid$. Hence, $z^*(G) \geq \frac{n}{2} \geq \frac{k+1}{2k} \mid V_1 \mid$ and $\frac{|V_1|}{z^*(G)} \leq \frac{2k}{k+1} < 2 \diamond$

To find a suitable lower bound (and apply Theorem 2) or a suitable feasible solution (and apply Theorem 3), we will introduce a new SDP model, in the next section.

3 A (1.999999)-approximation algorithm on arbitrary graphs

In section 2, we introduced a (2-ε)-approximation ratio for VCP, where ε value was not constant. In this section, we fix the value of ε equal to ε=0.000001 to produce a 1.999999-approximation ratio on arbitrary graphs. To do this, we introduce the following property on a solution of the SDP model (4).

Property 1. For some vertex cover problems, after solving the SDP model (4), both of the following conditions occur.

a) For less than 0.000001n of vertices $j \in V$ and corresponding vectors we have $v_o^* v_j^* < 0.5$.

b) For less than 0.01n of vertices $j \in V$ and corresponding vectors we have $v_o^* v_j^* > 0.5004$.

Theorem 4. If $z^*(G) \geq \frac{n}{2}$ and the optimal solution of the SDP model (4) does not meet the Property (1), then we can produce a VCP solution with a performance ratio of 1.999999.
Proof. If the optimal solution of the SDP model (4) does not meet the Property (1.a), then we can introduce $V_0=\{j \in V \mid v_o^* v_j^* < 0.5\}$ and $V_1=V-V_0$, to have a VCP feasible solution with $\mid V_0 \mid \geq 0.000001n$ and $\mid V_1 \mid \leq 0.999999n \leq 999999 \mid V_0 \mid$. Therefore, for such a solution and based on the Theorem (3), we have an approximation ratio of $\frac{|V_1|}{z^*(G)} < \frac{2(999999)}{999999+1} = 1.999998 < 1.999999$.

Otherwise, if the optimal solution of the SDP model (4) meets the Property (1.a) but it does not meet the Property (1.b), then there exists the following lower bound on $z^*(G)$ value.

$$z^*(G) \geq z4^* \geq (0)(0.000001n)_{\{s.t.\ v_o^* v_j^* < 0.5\}}$$

$$+(0.5)(0.989999n)_{\{s.t.\ v_o^* v_j^* \geq 0.5\}} + (0.5004)(0.01n)_{\{s.t.\ v_o^* v_j^* > 0.5004\}}$$

$$= \frac{n}{2} + 0.0000035n$$

Note that, the Property (1.a) is met and we have less than 0.000001n of vertices $j \in V$ with $v_o^* v_j^* < 0.5$. The Property (1.b) is not met and we have more than 0.01n of vertices $j \in V$ with $v_o^* v_j^* > 0.5004$. Therefore, in the above inequality, the first summation is the lower bound on the vertices $j \in V$ with $v_o^* v_j^* < 0.5$, and the third summation is the lower bound on only 0.01n of the vertices $j \in V$ with $v_o^* v_j^* > 0.5004$. In other words, beyond the 0.01n of such vertices are considered in the second summation. Moreover, the second summation is the lower bound on the other vertices (the vertices $j \in V$ with $0.5 \leq v_o^* v_j^* \leq 0.5004$ or the vertices $j \in V$ with $v_o^* v_j^* > 0.5004$ and beyond the 0.01n of such vertices that have been considered in the third summation).

Therefore, based on the above lower bound on $z^*(G)$ value and based on the Theorem (2), for all VCP feasible solutions $V=V_1 \cup V_0$, we have the approximation ratio $\frac{|V_1|}{z^*(G)} \leq \frac{2(\frac{1}{0.0000035})}{\frac{1}{0.0000035}+2} < 1.999999 \diamond$

Definition 1. Let ε=0.0004 and $V_\varepsilon = \{j \in V \mid 0.5 \leq v_o^* v_j^* \leq 0.5 + \varepsilon\}$.

After solving the SDP model (4) on problems with $z^*(G) \geq \frac{n}{2}$,
i) If the solution of the SDP model (4) does not meet the Property (1), then we have a performance ratio of 1.999999,
ii) Otherwise (The solution of the SDP model (4) meets the Property (1)), for more than 0.989999n of vertices $j \in V$, we have $0.5 \leq v_o^* v_j^* \leq 0.5 + \varepsilon$; i.e. $\mid V_\varepsilon \mid \geq 0.989999n$. Moreover, for each edge ij in $E_\varepsilon = \{ij \in E \mid i, j \in V_\varepsilon\}$, we have $v_o^* v_i^* + v_o^* v_j^* - v_i^* v_j^* = 1$ (That is $0 \leq v_i^* v_j^* \leq 2\varepsilon = 0.0008$), and the corresponding vectors of each edge in E_ε are almost perpendicular to each other.

Therefore, to produce a VCP performance ratio of 1.999999 for problems with $z^*(G) \geq \frac{n}{2}$, we need a solution for the SDP model (4) that does not meet the Property (1). To do this, we introduce a new SDP model.

Let $G2 = (V_{new}, E_{new})$ be a new graph, where we add two adjacent vertices a and b to the graph G, and connect all vertices of G to them. Then, based on the SDP model (3), we introduce a new SDP model as follows:

$$(5)\ min_{s.t.}\ z5 = \sum_{i \in V} X_{oi}$$
$$SDP\ (3)\ constraints\ on\ G$$
$$X_{oi} + X_{oj} - X_{ij} = 1 \quad i \in V,\ j \in \{a, b\}$$
$$-0.5 \leq X_{ij} \leq +0.5 \quad i \in V,\ j \in \{a, b\}$$
$$X_{ii} = 1, \quad X_{oi} = +0.5 \quad i \in \{a, b\}$$
$$X_{ab} = 0$$
$$X \succeq 0$$

Moreover, by introducing the normal vectors $v_o, v_1, ..., v_n, v_a, v_b$, the SDP model (5) can be written as follows, where $V_{1new} = V_1 = \{i \in V_{new} \mid v_i = v_o\}$ corresponds to a feasible vertex cover on graph G, and $V_{0new} = V_0 = V - V_1$ corresponds to perpendicular vectors to v_o.

$$(6)\ min_{s.t.}\ z6 = \sum_{i \in V} v_o v_i$$
$$SDP\ (4)\ constraints\ on\ G$$
$$v_o v_i + v_o v_j - v_i v_j = 1 \quad i \in V,\ j \in \{a, b\}$$
$$-0.5 \leq v_i v_j \leq +0.5 \quad i \in V,\ j \in \{a, b\}$$
$$v_i v_i = 1, \quad v_o v_i = +0.5 \quad i \in \{a, b\}$$
$$v_a v_b = 0$$

Lemma 2. Due to the additional constraints, we have $z6^* \geq z4^*$. Moreover, to produce a feasible solution for the SDP model (6) on $G2$, we can add suitable vectors v_a and v_b to each VCP feasible partitioning $V = V_1 \cup V_0$ on G, where $v_i v_j = +0.5$ for $i \in V_1,\ j \in \{a, b\}$, and $v_i v_j = -0.5$ for $i \in V_0,\ j \in \{a, b\}$ (For example, for $v_o = v_i = [0.5, 0.5, 0.5, 0.5]^t \in V_1$ and $v_i = [-0.5, -0.5, 0.5, 0.5]^t \in V_0$, we can introduce $v_a = e_1 = [1, 0, 0, 0]^t$, and $v_b = e_2 = [0, 1, 0, 0]^t$). Therefore, $z6^* \leq z^*(G)$.

We can now prove that by solving the SDP model (6) on problems with $z^*(G) \geq \frac{n}{2}$, it is impossible to produce a solution that meets the Property (1) on G, unless the induced graph on V_ε is bipartite.

Theorem 5. For four normalized vectors v_1, v_2, v_3, v_4 which are perpendicular to each other, there exists exactly one normalized vector v with $vv_i = 0.5$ for $i = 1, 2, 3, 4$. Such a vector v satisfies the equation $v = 0.5(v_1 + v_2 + v_3 + v_4)$.
Proof.

Due to $v_1 v_2 = 0$, we have $\mid v_1 + v_2 \mid = \sqrt{\mid v_1 \mid^2 + \mid v_2 \mid^2} = \sqrt{2}$.

Due to $v_3 v_4 = 0$, we have $\mid v_3 + v_4 \mid = \sqrt{\mid v_3 \mid^2 + \mid v_4 \mid^2} = \sqrt{2}$.

Due to $(v_1 + v_2)(v_3 + v_4) = 0$, we have $\mid v_1 + v_2 + v_3 + v_4 \mid = 2$.

Moreover, we have $(v_1 + v_2 + v_3 + v_4)v = 2$. Hence, $| v_1 + v_2 + v_3 + v_4 || v | cos(\theta) = 2$ and this concludes that $\theta = 0$ and $v = 0.5(v_1 + v_2 + v_3 + v_4)$ $\diamond$

proposition 1. For four normalized vectors v_1, v_2, v_3, v_4 which are almost perpendicular to each other, a normalized vector v with $0.5 \leq vv_i \leq 0.5 + \varepsilon = 0.5004$ for $i = 1, 2, 3, 4$ is almost equal to $0.5(v_1 + v_2 + v_3 + v_4)$. In other words, there exists a vector ϵ with $| \epsilon |\leq 0.001$, where $2v = \epsilon + (v_1 + v_2 + v_3 + v_4)$

Theorem 6. By solving the SDP model (6) on $G2$, it is impossible to have an optimal solution that meets the Property (1) on G, unless the induced graph on V_ε is bipartite.
Proof. Suppose that the optimal solution of the SDP model (6) meets the Property (1) on G. Therefore, for the edge ab and any edge ij in E_ε (a complete subgraph of $G2$ on four vertices a, b, i, j) we have four normalized vectors $v^*_a, v^*_b, v^*_i, v^*_j$ which are almost perpendicular to each other.

Moreover, we have a normalized vector v^*_o for which $0.5 \leq v^*_o v^*_c \leq 0.5004$ for $c = a, b, i, j$. Hence, based on the proposition (1), the vector v^*_o is almost equal to $0.5(v^*_a + v^*_b + v^*_i + v^*_j)$ for $ij \in E_\varepsilon$, and there exists a vector ϵ_{ij}, where $v^*_a + v^*_b + v^*_i + v^*_j + \epsilon_{ij} = 2v^*_o$, and $| \epsilon_{ij} |\leq 0.001$ for $ij \in E_\varepsilon$.

By introducing $U = 2v^*_o - v^*_a - v^*_b = v^*_i + v^*_j + \epsilon_{ij}$ for $ij \in E_\varepsilon$, we have

$$| U |= \sqrt{UU} = \sqrt{4 - 1 - 1 - 1 + 1 + 0 - 1 + 0 + 1} = \sqrt{2}$$

Moreover, for each vertex k in V_ε, we have

$$v^*_o v^*_c + v^*_o v^*_k - v^*_c v^*_k = 1 \quad c \in \{a, b\}, \ \ k \in V_\varepsilon$$

Therefore, we obtain

$$v^*_c v^*_k = -0.5 + v^*_o v^*_k \quad c \in \{a, b\}, \ \ k \in V_\varepsilon$$

and

$$Uv^*_k = 2v^*_o v^*_k - v^*_a v^*_k - v^*_b v^*_k = 1 \quad k \in V_\varepsilon \qquad (1)$$

Moreover, we have

$$U(v^*_i + v^*_j + \epsilon_{ij}) = UU = 2 \quad ij \in E_\varepsilon$$

and based on (1), we have

$$U(v^*_i + v^*_j) = 2 \quad ij \in E_\varepsilon$$

which conclude

$$U\epsilon_{ij} = 0 \quad ij \in E_\varepsilon \qquad (2)$$

We can now prove that there does not exist any odd cycle in the subgraph $G_\varepsilon = (V_\varepsilon, E_\varepsilon)$ and it is bipartite. Then, suppose that we have an odd cycle on 5 vertices, in G_ε (We prove it on $2t + 1$ vertices, later).

By addition of the vectors in this cycle, we have

$$(v_1 + v_2 + \epsilon_{12}) + (v_2 + v_3 + \epsilon_{23}) + (v_3 + v_4 + \epsilon_{34}) + (v_4 + v_5 + \epsilon_{45}) + (v_5 + v_1 + \epsilon_{51}) \ = \ 5U$$

The above summation can be written as follows,

$$v_1 + 0.5\epsilon_{12} + v_2 + 0.5\epsilon_{23} + v_3 + 0.5\epsilon_{34} + v_4 + 0.5\epsilon_{45} + v_5 + 0.5\epsilon_{51} = 2.5U \qquad (3)$$

Then, by addition of $0.5\epsilon_{2l,2l+1} - 0.5\epsilon_{2l,2l+1}$ for $l = 1, 2$, to the equation (3), we obtain

$$v_1 + 0.5\epsilon_{12} + v_2 + \epsilon_{23} + v_3 - 0.5\epsilon_{23} + 0.5\epsilon_{34} + v_4 + \epsilon_{45} + v_5 - 0.5\epsilon_{45} + 0.5\epsilon_{51} \ = \ 2.5U$$

or

$$v_1 + 0.5\epsilon_{12} + U - 0.5\epsilon_{23} + 0.5\epsilon_{34} + U - 0.5\epsilon_{45} + 0.5\epsilon_{51} \ = \ 2.5U$$

Therefore, we have

$$v_1 + W_1 = v_1 + 0.5\epsilon_{12} - 0.5\epsilon_{23} + 0.5\epsilon_{34} - 0.5\epsilon_{45} + 0.5\epsilon_{51} \ = \ 0.5U$$

Moreover, by addition of $0.5\epsilon_{2l-1,2l} - 0.5\epsilon_{2l-1,2l}$ for $l = 1, 2$, to the equation (3), we obtain

$$v_5 + 0.5\epsilon_{51} + v_1 + \epsilon_{12} + v_2 - 0.5\epsilon_{12} + 0.5\epsilon_{23} + v_3 + \epsilon_{34} + v_4 - 0.5\epsilon_{34} + 0.5\epsilon_{45} \ = \ 2.5U$$

or

$$v_5 + 0.5\epsilon_{51} + U - 0.5\epsilon_{12} + 0.5\epsilon_{23} + U - 0.5\epsilon_{34} + 0.5\epsilon_{45} \;=\; 2.5U$$

Therefore, we have

$$v_5 + W_5 = v_5 + 0.5\epsilon_{51} - 0.5\epsilon_{12} + 0.5\epsilon_{23} - 0.5\epsilon_{34} + 0.5\epsilon_{45} \;=\; 0.5U$$

where,

$$-W_1 + \epsilon_{51} = W_5$$

and due to (2), we have

$$UW_1 = UW_5 = 0$$

Therefore, $v_1,\ W_1$ and $0.5U$ produce a right triangle, where $\mid W_1 \mid = 0.5\sqrt{2}$. Moreover, $v_5,\ W_5$ and $0.5U$ produce a right triangle, where $\mid W_5 \mid = 0.5\sqrt{2}$, and this is a contradiction, unless $\epsilon_{5,1}$ be a zero vector.

Due to similar approaches, all ϵ_{ij} vectors, on this cycle, are zero. Therefore, the equation (3) is as follows,

$$(v_1 + v_2) + (v_3 + v_4) + v_5 \;=\; 2.5U$$

Hence, $v_5 \;=\; 0.5U$ and $\mid v_5 \mid = \; 0.5\sqrt{2}$, which is a contradiction.

Now, suppose that we have an odd cycle on $2t + 1$ vertices in $G_\varepsilon = (V_\varepsilon, E_\varepsilon)$.

Then, by addition of the vectors in this cycle, we have

$$(v_1 + v_2 + \epsilon_{12}) + (v_2 + v_3 + \epsilon_{23}) + ...+$$
$$(v_{2t} + v_{2t+1} + \epsilon_{2t,2t+1}) + (v_{2t+1} + v_1 + \epsilon_{2t+1,1}) \;=\; (2t+1)U$$

The above summation can be written as follows,

$$\sum_{l=1}^{t}(v_{2l-1} + 0.5\epsilon_{2l-1,2l} + v_{2l} + 0.5\epsilon_{2l,2l+1}) + (v_{2t+1} + 0.5\epsilon_{2t+1,1}) \;=\; tU + 0.5U \tag{4}$$

Then, by addition of $0.5\epsilon_{2l,2l+1} - 0.5\epsilon_{2l,2l+1}$ for $l = 1, ..., t$, to the equation (4), and using the substitutions,

$$(v_{2l} + \epsilon_{2l,2l+1} + v_{2l+1}) = U \quad l = 1, ..., t$$

we obtain

$$v_1 + \sum_{l=1}^{t}(0.5\epsilon_{2l-1,2l} + U - 0.5\epsilon_{2l,2l+1}) + (0.5\epsilon_{2t+1,1}) \;=\; tU + 0.5U$$

or

$$v_1 + W_1 = v_1 + \sum_{l=1}^{t}(0.5\epsilon_{2l-1,2l} - 0.5\epsilon_{2l,2l+1}) + (0.5\epsilon_{2t+1,1}) \;=\; 0.5U$$

Similarly, by addition of $0.5\epsilon_{2l-1,2l} - 0.5\epsilon_{2l-1,2l}$ for $l = 1, ..., t$, to the equation (4), and using the substitutions,

$$v_{2l-1} + \epsilon_{2l-1,2l} + v_{2l} = U \quad l = 1, ..., t$$

we obtain

$$v_{2t+1} + \sum_{l=1}^{t}(-0.5\epsilon_{2l-1,2l} + U + 0.5\epsilon_{2l,2l+1}) + 0.5\epsilon_{2t+1,1} \;=\; tU + 0.5U$$

or

$$v_{2t+1} + W_{2t+1} = v_{2t+1} + \sum_{l=1}^{t}(-0.5\epsilon_{2l-1,2l} + 0.5\epsilon_{2l,2l+1}) + 0.5\epsilon_{2t+1,1} \;=\; 0.5U$$

where,

$$-W_1 + \epsilon_{2t+1,1} = W_{2t+1}$$

and due to (2), we have

$$UW_1 = UW_{2t+1} = 0$$

Therefore, $v_1,\ W_1$ and $0.5U$ produce a right triangle, where $\mid W_1 \mid = 0.5\sqrt{2}$. Moreover, $v_{2t+1},\ W_{2t+1}$ and $0.5U$ produce a right triangle, where $\mid W_{2t+1} \mid = 0.5\sqrt{2}$, and this is a contradiction, unless $\epsilon_{2t+1,1}$ be a zero vector.

Due to similar approaches, all ϵ_{ij} vectors, on this cycle, are zero. Therefore, the equation (4) is as follows,

$$\sum_{l=1}^{t}(v_{2l-1}+v_{2l})+(v_{2t+1}) \ = \ tU+0.5U$$

or

$$\sum_{l=1}^{t}(U)+(v_{2t+1}) \ = \ tU+0.5U$$

Hence, $v_{2t+1} \ = \ 0.5U$ and $\mid v_{2t+1} \mid = \ 0.5\sqrt{2}$, which is a contradiction.

Therefore, there is not any odd cycle in G_ε, and, if the optimal solution of the SDP model (6) on $G2$ meets the Property (1) on G, then the subgraph G_ε is bipartite $\diamond$

proposition 2. To produce a performance ratio of 1.999999 for problems with $z^*_{VCP} \geq \frac{n}{2}$, we should solve the SDP model (6) on $G2$. Then, if the solution does not meet the Property (1), we have a performance ratio of 1.999999. Otherwise, we can solve the VCP problem on the bipartite graph G_ε, where $\mid V_\varepsilon \mid \geq 0.989999n$, to produce a performance ratio of 1.999999.

Moreover, based on the Theorem (1) and the proposition (2), to produce a performance ratio of 1.999999 for problems with $z^*_{VCP} < \frac{n}{2}$, it is sufficient to produce an extreme optimal solution for the LP relaxation of the model (1), to introduce $G2$ based on $G_{0.5}$.

Theorem 7. The Optimal solution of the following LP model corresponds to an extreme optimal solution of the LP relaxation of the model (1).

$$(7)\ min_{s.t.} z7 = \sum_{i=1}^{n}(0.1)^i x_i$$
$$x_i + x_j \geq 1 \quad ij \in E$$
$$\sum_{i \in V} x_i = z^*_{LP\ relaxation\ of\ the\ model\ (1)}$$
$$0 \leq x_i \leq +1 \quad i \in V$$

Proof. The feasible region of the model (7) is an optimal face of the feasible region of the LP relaxation of the model (1), and based on the priority weights of the decision variables, its optimal solution corresponds to the solution of the following algorithm.
Step 0. Let k=1 and z^* be the optimal value of the LP relaxation of the model (1).
Step k. Solve the following LP model.

$$(8)\ min_{s.t.} z(k) = x_k$$
$$x_i + x_j \geq 1 \quad ij \in E$$
$$\sum_{i \in V} x_i = z^*$$
$$x_i = x_i^* = z(k)^* \quad i = 1, \cdots, k-1$$
$$0 \leq x_i \leq +1 \quad i \in V$$

Let k=k+1. If $k < n$ repeat this step, otherwise, the solution x^* is an extreme optimal solution of the LP relaxation of the model (1) $\diamond$

Therefore, our algorithm to produce an approximation ratio of 1.999999, for arbitrary vertex cover problems, is as follows:

Mahdis Algorithm (To produce a vertex cover solution on graph G with a ratio factor $\rho = 1.999999$)
Step 1. Let $V^1 = V^0 = \{\}$ and solve the LP relaxation of the model (1) on G.

Step 2. If $z^*_{LP\ relaxation\ of\ the\ model\ (1)} < \frac{n}{2}$, then solve the model (7) to produce an extreme optimal solution of the LP relaxation of the model (1), and based on the solution ($x^*_j \in \{0, 0.5, 1\}$ $j \in V$), introduce $V^0 = \{j \in V \mid x^*_j = 0\}$, $V^{0.5} = \{j \in V \mid x^*_j = 0.5\}$, $V^1 = \{j \in V \mid x^*_j = 1\}$, and let $G = G_{0.5}$ as the induced graph on the vertex set $V^{0.5}$.
Step 3. Produce $G2$ based on G and solve the SDP (6) model.
Step 4. If $\mid \{j \in V \mid v^*_o v^*_j < 0.5\} \mid > 0.000001n$, then produce a suitable solution $V_1 \cup V_0$, correspondingly, where $V_0 = \{j \in V \mid v^*_o v^*_j < 0.5\}$ and $V_1 = V - V_0$ and go to Step 7. Hence, the solution does not meet the Property (1.a) and we have $\frac{|V_1|}{z^*(G)} \leq 1.999999$. Otherwise, go to Step 5.
Step 5. If $\mid \{j \in V \mid v^*_o v^*_j > 0.5004\} \mid > 0.01n$, then it is sufficient to produce an arbitrary VCP feasible solution $V = V_1 \cup V_0$ to have $\frac{|V_1|}{z^*(G)} \leq 1.999999$ and go to Step 7. Otherwise, go to Step 6.
Step 6. The solution meets the Property (1) and based on the Theorem (6), graph G_ε is bipartite, and $\mid V_\varepsilon \mid \geq 0.989999n$. Therefore, solve the VCP problem on bipartite subgraph G_ε and add all vertices of $V - V_\varepsilon$ to the solution (That is $V_1 = V_1 \cup (V - V_\varepsilon)$), to produce a feasible solution $V_1 \cup V_0$ for which we have $\frac{|V_1|}{z^*(G)} \leq 1.999999$. Then, go to Step 7.
Step 7. The partitioning $(V_1 \cup V^1) \cup (V_0 \cup V^0)$ produces a VCP feasible solution on the original graph G with an approximation ratio factor $\rho = 1.999999$.

proposition 3. Based on the proposed 1.999999-approximation algorithm for the vertex cover problem, the unique games conjecture is not true.

4 Conclusions

One of the open problems about the vertex cover problem is the possibility of introducing an approximation algorithm within any constant factor better than 2. Here, we proposed a new algorithm to produce a 1.999999-approximation ratio for the vertex cover problem on arbitrary graphs, and this led to the conclusion that the unique games conjecture is not true.

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Competing Interest and Data Availability
The authors have no relevant financial or non-financial interests to declare relevant to this article's content. Data sharing does not apply to this article as no data sets were generated or analyzed during the current study.

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