

## The Elementary Solution of the Navier-Stokes Equations: Existence and Smoothness with Uniqueness

Masatoshi Ohrui

### Abstract

This is an elementary argument in the sense that there are no long or complicated calculations, and the theory of evolution equations is not used at all.

This is written as mathematical conjecture. It is a challenge to build an elementary theory without semi-group theory or apriori estimates of the Navier-Stokes equation. If you have any ideas or questions, please contact to [MasatoshiOhrui1993@gmail.com](mailto:MasatoshiOhrui1993@gmail.com) . I'm also looking for people to study together.

This is an elementary argment in the sense that there are no long or complicated calculations, and the theory of evolution equations is not used at all. The existence of the solution is actually known, and the proof that already exists is very wonderful. For example, Fujita-Kato Theory, Shibata Theory: Takayoshi Ogawa [26], Yoshihiro Shibata [22], Shibata-Kubo [24], Kakita-Shibata [3], Okamoto [20]. But these are not elementary. We apply "Fundamental theorem of distributions with compact support":

"The fundamental solution of any linear partial differential operator with constant coefficients  $L$  on  $\mathbb{R}^N$  , that is, for  $E \in \mathcal{D}'$  that satisfies  $LE = \delta$  , for  $f \in \mathcal{D}'$  , one of the solutions of the equation  $Lu = f$  on  $\Omega$  is  $u = E * \chi_\Omega f \in \mathcal{D}'(\Omega)$  .

Here if  $f \in \mathcal{E}'$  then  $\langle E * f, \varphi \rangle = \langle E(x), \langle f(y), \varphi(x + y) \rangle \rangle$  .

The policy is, let  $L$  be the heat operator  $\partial_t - \Delta$  in the Navier-Stokes equations

$$\begin{cases} \partial_t u - \Delta u = f - \nabla p - (u \cdot \nabla)u \\ \operatorname{div} u = 0 \end{cases} ,$$

erase the pressure  $\mathbf{p}$  and to approximate the nonlinear term  $(\mathbf{u} \cdot \nabla)\mathbf{u}$  by a sequence of smooth functions, use the fundamental theorem for the difference between the external force  $\mathbf{f}$  and the approximation term, and show that the limit in Sobolev space is the solution.

[definition of symbols]

For convenience, write the index of the component of the vector in the upper right corner.

"Function space" and "space" are abbreviations for "linear topological space" (of functions or distributions), other than pressure  $\mathbf{p}$  are  $\mathbb{R}^3$ -values. The absolute value of the function in the norm of normal function space is interpreted as the length of the number vector (the absolute value of  $\mathbb{R}^3$ ) in the norm of the space of the  $\mathbb{R}^3$ -value function. We write the space of the real numeric function and the space of the  $\mathbb{R}^3$ -value function in the same symbol to make the symbol easy. For any positive number

$\delta$ , let  $B_\delta(\mathbf{0}, \mathbf{y})$  be the  $\delta$ -neighborhood of point  $(\mathbf{0}, \mathbf{y})$ . Let  $\Omega$  be a bounded open set contained in  $\mathbb{R} \times \mathbb{R}^3$  whose for any  $\mathbf{y} \in \mathbb{R}^3$ , there exists  $\delta$  such that  $B_\delta(\mathbf{0}, \mathbf{y}) \cap \Omega = \emptyset$  and have smooth boundary. Let  $t_0 = \inf\{s \in \mathbb{R} : \exists \mathbf{y} \in \mathbb{R}^3, (s, \mathbf{y}) \in \overline{\Omega}\}$ . Let  $|\Omega|$  be its Lebesgue measure. Let  $\chi_\Omega$  be the characteristic function on  $\Omega$ , the support compact and the divergence for special valuables  $\mathbf{0}$ . For any natural number  $m > \max\{0 + 4/1, 0 + 4/2\} = 4$ ,  $p = 1, 2$ , let  $V_\sigma^{m,p}(\Omega) = \{u \in C^\infty(\Omega) : \|u\|_{W^{m,p}(\Omega)} < \infty, \text{div } u = 0\}$ ,

$W_\sigma^{m,p}(\Omega)$  be the Sobolev space defined by

$V_\sigma^{m,p}$ 's completion by norm of  $W_\sigma^{m,p}(\Omega) = \overline{V_\sigma^{m,p}(\Omega)}^{\|\cdot\|_{W^{m,p}(\Omega)}}$ . Let  $\mathcal{D}(\Omega)$  be the space of

the test functions ( $C_0^\infty(\Omega)$  as a set),  $\mathcal{D}_\sigma(\Omega)$  is the space of the test functions where the divergence is  $\mathbf{0}$  for spatial variables (see [Supplement 1]). Let  $P : L^2(\Omega) \rightarrow L_\sigma^2(\Omega)$  be the

projection. Let  $C^{k,\varepsilon}(\overline{\Omega})$  be the Hölder space.  $\langle w, \varphi \rangle = (w, \varphi)_{L^2(\Omega)}$

$$= \int_\Omega \sum_{i=1}^3 w^i(t, x) \varphi^i(t, x) dt dx$$

$$= \int_\Omega w(t, x) \cdot \varphi(t, x) dt dx$$

$(w = (w^1, w^2, w^3), \varphi = (\varphi^1, \varphi^2, \varphi^3))$ . In general, if for two Banach spaces  $X, Y$ , there exists linear Hausdorff space  $Z$  such that  $X, Y \subset Z$ , then  $X \cap Y$  is a Banach space with the norm given by  $\|u\|_X + \|u\|_Y$  or  $\max\{\|u\|_X, \|u\|_Y\}$ .  $\max\{\|u\|_X, \|u\|_Y\} \leq \|u\|_X + \|u\|_Y \leq 2 \max\{\|u\|_X, \|u\|_Y\}$  so these are equivalent.

$X = \bigcap_{m=5}^\infty W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)$ .  $X$  is a Banach space with the norm given by

$$\|u\|_X = \sum_{m=5}^\infty \frac{1}{m!^5} \|u\|_{W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)}.$$

[Proof]

Let  $\{u_n\}$  be the Cauchy sequence in  $X$ . Then,  $\{u_n\}$  is the Cauchy sequence of  $W_\sigma^{m,1}(\Omega) \cap$

$W_\sigma^{m,2}(\Omega) \cdot W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)$  is a Banach space, so  $\{u_n\}$  converges. Let the limit be  $u$ . If  $u \notin X$ , for any positive number  $R$ , there exists natural number  $m' \geq 5$  such that  $\sum_{m=5}^{m'} \frac{1}{m!^5} \|u\|_{W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)} > R$ . Then  $\|u\|_{W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)} > CR$ . This is a contradiction, so  $u \in X$ . If  $\lim_{n \rightarrow \infty} \|u_n - u\|_X = 0$  does not hold, there exists positive number  $R'$  such that for any natural number  $N$ , there exists  $n > N$ ,  $M' \geq 5$  such that  $\sum_{m=5}^{M'} \frac{1}{m!^5} \|u_n - u\|_{W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)} > R'$ . Then  $\|u_n - u\|_{W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)} > C'R'$ . This is a contradiction, too. So  $\lim_{n \rightarrow \infty} \|u_n - u\|_X = 0$ .  
(END)

$\chi_\Omega \in X$  so  $X \neq \{0\}$ .

A constant  $C > 0$  exists such that

$$\|u^i v^i\|_X \leq C \|u^i\|_X \|v^i\|_X$$

(Separation of the product)

and

$$\|\partial_{x^j} u\|_X \leq C \|u\|_X$$

(absorption of differential)

holds for  $u \in X$ .

[Proof]

For binomial coefficients  $c_{\alpha,\beta}$ , let

$$c_\alpha = \sum_{\beta \leq \alpha} c_{\alpha,\beta}.$$

There is a continuous embedding  $X \subset C^{k,\varepsilon}(\bar{\Omega})$  for any natural number  $k$ , because  $\|u_n - u\|_X \rightarrow 0$

$$\Rightarrow \|u_n - u\|_{W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)} \rightarrow 0$$

$\Rightarrow \|u_n - u\|_{C^{k,\varepsilon}(\bar{\Omega})} \rightarrow 0$ , so there exists constant  $c' > 0$  such that if  $|\alpha| \leq k$ , by Leibniz'

formula,

$$\|\partial^\alpha (u^i v^i)\|_{L^p(\Omega)}$$

$$\leq c_\alpha \|u^i\|_{C^{k,\varepsilon}(\bar{\Omega})} \|v^i\|_{C^{k,\varepsilon}(\bar{\Omega})} |\Omega|^{1/p}$$

$$\leq c_\alpha c' |\Omega|^{1/p} \|u^i\|_X c' \|v^i\|_X$$

$$\leq c_\alpha c'^2 |\Omega|^{1/p} \|u^i\|_X \|v^i\|_X. \text{ Therefore,}$$

$\|\partial^\alpha (u^i v^i)\|_{L^p(\Omega)} \leq c_\alpha c'^2 |\Omega|^{1/p} \|u^i\|_X \|v^i\|_X$ , so there exists a constant  $C > 0$  such that

$$\|u^i v^i\|_X \leq C \|u^i\|_X \|v^i\|_X.$$

Let  $\{u_n\} \subset X$  satisfies  $u_n \rightarrow u, \partial_{x^j} u_n \rightarrow v$ . From the Hölder's inequality, we have  $|\langle \partial_{x^j} u_n - v, \varphi \rangle| \leq \|\partial_{x^j} u_n - v\|_{L^p(\Omega)} \|\varphi\|_{L^q(\Omega)} \rightarrow 0$  and the weak differentiation is continuous in  $\mathcal{D}'_\sigma(\Omega)$ , so  $\partial_{x^j} u_n \rightarrow \partial_{x^j} u$  in  $\mathcal{D}'_\sigma(\Omega)$ . From  $v = \partial_{x^j} u \in X, \{u \in X : \partial_{x^j} u \in X\} = X$ , therefore the absorption of differentiation is true by the closed graph theorem.

(END)

$X \ni u \mapsto E * (\chi_\Omega u) \in X$  is a bounded operator and a constant  $C > 0$  exists such that for any  $u \in X$ ,

$$\left\| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) u(t - s, x - y) ds dy \right\|_X \leq C \|u\|_X$$

holds.

[Proof]

As a function of  $(s, y)$ , for any  $(t, x) \in \Omega$ ,

$$\begin{aligned} & \text{supp}(E^i(s, y) \chi_\Omega(t - s, x - y) u^i(t - s, x - y)) \\ & \subseteq \overline{-\Omega + (t, x)} \end{aligned}$$

$$= \overline{\{(s, y) \in \mathbb{R} \times \mathbb{R}^3 : (t - s, x - y) \in \Omega\}}$$

is the translation of reverse of  $\overline{\Omega}$ , so it is compact, and

$$|\partial_{t,x}^\alpha (E^i(s, y) \chi_\Omega(t - s, x - y) u^i(t - s, x - y))| \leq E^i(s, y) \sup\{|\partial_{t,x}^\alpha u^i(t - s, x - y)| : (t - s, x - y) \in \Omega\} \in L^1_{s,y}(\Omega),$$

so combine the theorem of differentiation under the integral sign, the Hölder's inequality and assumption of

$\Omega$ , we have

$$\begin{aligned} & \|\partial^\alpha (E * (\chi_\Omega u))\|_{L^p(\Omega)} \\ & \leq \|E * (\partial^\alpha (\chi_\Omega u))\|_{L^p(\Omega)} \\ & \leq \left\| \|E(s, y)\|_{L^2_{s,y}(-\Omega+(t,x))} \|\partial^\alpha u(t - s, x - y)\|_{L^2_{s,y}(-\Omega+(t,x))} \right\|_{L^p_{t,x}(\Omega)} \\ & \leq \sup_{(t,x) \in \Omega} \|E\|_{L^2(-\Omega+(t,x))} \|\partial^\alpha u\|_{L^2(\Omega)} |\Omega|^{1/p} \\ & \leq c \|\partial^\alpha u\|_{L^1(\Omega) \cap L^2(\Omega)} \\ & < \infty. \end{aligned}$$

So we have

$$\|E * (\chi_\Omega u)\|_X \leq C \|u\|_X.$$

(END)

For a constant  $M$ , let  $S$  be a subspace of  $X$ :

$$S = \{u \in X : \|u\|_X \leq M\}. \text{ We take } M \text{ the smaller one while satisfying } 2C^3 M <$$

1,  $C(1 + 3C^2)M \leq 1$ . Let the external force  $f \in S$  and  $\|f\|_X \leq M^2$ .

We solve

(N-S)'  $\partial_t u - \Delta u = f - (u \cdot \nabla)u$ , that is,  $u(t_0, x) \in L^\infty(\Omega_0)$ ,  $u \in W_\sigma^{m,1}(\Omega) \cap W_\sigma^{m,2}(\Omega)$ ,

$p \in L_{\text{loc}}^2(\Omega)$ , for any  $\varphi \in \mathcal{D}_\sigma(\Omega)$ ,

$\langle \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla p - f, \varphi \rangle = 0$ ,

for any  $\varphi \in \mathcal{D}(\Omega)$ ,

$\langle \text{div } u, \varphi \rangle = - \sum_{j=1}^3 \langle u^j, \partial_{x^j} \varphi \rangle = 0$ .

$\Phi : S \rightarrow S$  can be defined as

$\Phi[u](t, x)$

$= \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (Pf(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y)) ds dy$ . We take the function sequence  $\{u_n\} \subset S$  as  $u_0 \in S$ , if  $n \geq 0$  then

$u_{n+1}(t, x) = \Phi[u_n](t, x)$

$= \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (Pf(t - s, x - y) - P((u_n \cdot \nabla)u_n)(t - s, x - y)) ds dy$ . If  $X$  is a complete metric space, then  $S$  is complete because it is a closed subspace that is not empty, and if it can be said that  $\Phi$  is a contraction mapping, according to the Banach's

fixed point theorem, the uniqueness and the existence of a fixed point of  $\Phi$ :

Some  $u \in S$  exists uniquely and  $\Phi[u] = u$ .

Then, due to the uniqueness of the fixed point in Banach's fixed point theorem, it can be said that  $u$  is a unique weak solution. If  $f \neq 0$  then  $u \neq 0$ .  $\Omega$  can be arbitrary large, so  $u, p$  are time global.

[Proof of the possibility that  $\Phi$  can be defined as a contraction mapping]

$u \in S \Rightarrow \|E * (\chi_\Omega(Pf - P((u \cdot \nabla)u)))\|_X < \infty$

holds. Therefore

$\|\Phi[u]\|_X \leq M$ .

$\|P\| = 1$ , so

$\|\chi_\Omega(Pf - P((u \cdot \nabla)u))\|_X$

$$\begin{aligned} &\leq \|f\|_X + \|u^1 \partial_{x^1} u + u^2 \partial_{x^2} u + u^3 \partial_{x^3} u\|_X \\ &\leq M^2 + 3C^2 M^2 < \infty. \end{aligned}$$

If

$$\begin{aligned} &\|\Phi[u]\|_X \\ &\leq CM^2 + 3C^3 M^2 \\ &\leq M, M \text{ must be } C(1 + 3C^2)M \leq 1. \end{aligned}$$

(END)

$\Phi : S \rightarrow S$  is Lipschitz continuous: there is a constant  $L > 0$  such that

$$\begin{aligned} &\left\| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (P((v \cdot \nabla)v)(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right\|_X \\ &\leq L \|u - v\|_X \end{aligned}$$

may be possible. If the Lipschitz continuity established,

$$\begin{aligned} &\|\Phi[u] - \Phi[v]\|_X \\ &\leq \left\| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (P((v \cdot \nabla)v)(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right\|_X \\ &\leq L \|u - v\|_X \end{aligned}$$

follows. Here, if

[  $\Phi$  may be a contraction mapping]

$$L < 1$$

holds, the argument is justified.

[Proof of Lipschitz continuity]

$$\begin{aligned} &(v \cdot \nabla)v(t - s, x - y) - (u \cdot \nabla)u(t - s, x - y) \\ &= \sum_{j=1}^3 v^j (\partial_{x^j} v(t - s, x - y) - \partial_{x^j} u(t - s, x - y)) + (v^j \partial_{x^j} u(t - s, x - y)) - (u^j \partial_{x^j} u(t - s, x - y)), \text{ so we have} \end{aligned}$$

$$\left\| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (P((v \cdot \nabla)v)(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right\|_X$$

$$\begin{aligned}
& \|s, x - y) ds dy\|_X \\
& \leq C^2 \|v\|_X \max_j (\|\partial_{x^j} v - \partial_{x^j} u\|_X) + C^2 \|v - u\|_X \max_j (\|\partial_{x^j} u\|_X) \\
& \leq C^3 M \|v - u\|_X + C^3 M \|v - u\|_X \\
& = 2C^3 M \|u - v\|_X.
\end{aligned}$$

Therefore, we can make it  $L = 2C^3 M$ .

(END)

[Proof of the possibility that  $\Phi$  is a contraction mapping]

$$\begin{aligned}
& \text{From the above argument } \left\| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (P((v \cdot \nabla)v(t - s, x - y)) - P((u \cdot \nabla)u(t - s, x - y))) ds dy \right\|_X \\
& \leq 2C^3 M \|u - v\|_X
\end{aligned}$$

and

$$2C^3 M < 1.$$

(END)

[Solvability of the Navier-Stokes equations]

When taking  $f \in S$  to  $\|f\|_X \leq M^2$ , the fixed point of  $\Phi : S \rightarrow S$  will be the solution of (N-S)'.

If  $f \neq 0$  then  $u \neq 0$ .  $\Omega$  can be arbitrary large, so  $u, p$  are time global.

[Proof]

$u$  satisfies  $\operatorname{div} u = 0$  in the sense of a distribution belonging to  $\mathcal{D}'(\Omega)$  (See [28]). That is, for

$$\text{any } \varphi \in \mathcal{D}(\Omega), \langle \operatorname{div} u, \varphi \rangle = - \sum_{j=1}^3 \langle u^j, \partial x^j \varphi \rangle = 0.$$

For any  $\varphi \in \mathcal{D}_\sigma(\Omega)$ ,

$\operatorname{div}(\varphi) = 0$ , so by integration by parts

$$\begin{aligned}
& \langle \nabla p, \varphi \rangle \\
& = \int_\Omega \sum_{i=1}^3 (\nabla p)^i(t, x) \varphi^i(t, x) dt dx \\
& = - \int_\Omega p(t, x) \operatorname{div}(\varphi)(t, x) dt dx = 0.
\end{aligned}$$

Therefore, boundness of  $u, \partial_{x^j} u$  by the Sobolev's embedding theorem and  $|\Omega| < \infty$ , we have

$(u \cdot \nabla)u \in L^2(\Omega)$ , so by the Helmholtz decomposition,  
if we let  $f = Pf + \nabla f$ ,  $(u \cdot \nabla)u = P((u \cdot \nabla)u) + \nabla u$   
then

$$\langle f, \varphi \rangle = \langle Pf, \varphi \rangle, \langle (u \cdot \nabla)u, \varphi \rangle = \langle P((u \cdot \nabla)u), \varphi \rangle, \text{ hence we solve}$$

$$(N-S)' \partial_t u - \Delta u = f - (u \cdot \nabla)u \text{ in } \mathcal{D}'_0(\Omega).$$

The solution of the approximate equation on  $\Omega$

$$(N-S)'' \partial_t v_n - \Delta v_n = (Pf - P((u_n \cdot \nabla)u_n))$$

$$(v_n = u_{n+1})$$

is

$$v_n = E * \chi_\Omega(Pf - P((u_n \cdot \nabla)u_n)) \in V_\sigma^{m-1,p}(\Omega).$$

Therefore, the solution of (N-S)''

$$v_n(t, x) = \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (Pf(t - s, x - y) - P((u_n \cdot \nabla)u_n)(t - s, x - y)) ds dy.$$

We show that  $u = v$  is the solution of (N-S)' :

$$v_n(t, x)$$

$$= \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_\Omega(t - s, x - y) (Pf(t - s, x - y) - P((u_n \cdot \nabla)u_n)(t - s, x - y)) ds dy,$$

$$u_n \rightarrow u = v \leftarrow v_n.$$

$$\partial_t v_n(t, x) - \Delta v_n(t, x)$$

$$= \langle (\partial_t E(t - s, x - y) - \Delta E(t - s, x - y)), \chi_\Omega(s, y) (Pf(s, y) - P((u_n \cdot \nabla)u_n)(s, y)) \rangle$$

$$= \langle \delta(\tau) \otimes \delta(z), \chi_\Omega(t - \tau, x - z) (Pf(t - \tau, x - z) - P((u_n \cdot \nabla)u_n)(t - \tau, x - z)) \rangle$$

$$= Pf(t, x) - P((u_n \cdot \nabla)u_n)(t, x).$$

Therefore, the above calculation and the continuity of the heat operator on  $\mathcal{D}'_0(\Omega)$  :

$$|\langle \partial_t v_n - \Delta v_n, \varphi \rangle - \langle \partial_t u - \Delta u, \varphi \rangle| \rightarrow 0, \text{ and from the Hölder's inequality, } \|P\| = 1,$$

and product of the function  $L^2(\Omega) \times L^2(\Omega) \ni (u, v) \mapsto uv \in L^1(\Omega)$  is continuous (See [Supplement 2]), so



$$\begin{aligned}
& \left| \int_{\Omega} (P((u_n \cdot \nabla)u_n)(t, x) - P((u \cdot \nabla)u)(t, x)) \cdot \varphi(t, x) dt dx \right| \\
& \leq \|((u_n \cdot \nabla)u_n)(t, x) - ((u \cdot \nabla)u)(t, x)\|_{L^1(\Omega)} \|\varphi(t, x)\|_{L^\infty(\Omega)} \rightarrow 0 \quad (n \rightarrow \infty),
\end{aligned}$$

hence

$$\begin{aligned}
& \partial_t u - \Delta u = Pf - P((u \cdot \nabla)u) \text{ holds, so we have} \\
& u(t, x) = \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \chi_{\Omega}(t - s, x - y) (Pf(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y)) ds dy.
\end{aligned}$$

It has been shown that it is a solution in the sense of a distribution in  $\mathcal{D}'_{\sigma}(\Omega)$  of (N-S)' (See [Supplement 3]).

$$" \varphi \in \mathcal{D}_{\sigma}(\Omega) \Rightarrow \langle U, \varphi \rangle = 0 "$$

$\iff$

$$" \text{there exist } \mathfrak{p} \text{ such that } U = \nabla \mathfrak{p} "$$

(See [14]), therefore there exist  $\mathfrak{p}$  such that  $\partial_t u + (u \cdot \nabla)u - \Delta u - f = -\nabla \mathfrak{p}$  holds.

$u(t, x) \in W^{m,p}(\Omega) \subset C^{(m-4/p)-1,\varepsilon}(\overline{\Omega})$ , and if the function is bounded as variables  $(t, x)$  then it is also bounded as variable  $x$ , therefore  $u(t_0, x)$  is bounded.

(END)

[Smoothness and boundness of elementary weak solutions]

Solution  $(u, \mathfrak{p})$  are  $C^\infty$ -functions and bounded.

[Proof]

$m$  can be arbitrarily large, so the embedding theorem to Hölder space (See [18] theorem 6.12)

"if  $\mathbb{N} \ni m - 4/p > 0$  then  $W^{m,p}(\Omega) \subset C^{(m-4/p)-1,\varepsilon}(\overline{\Omega})$  for  $\varepsilon \in (0, 1)$ ", in the sense of existence of suitable representative elements,  $u$  is bounded on  $\overline{\Omega}$  and  $C^\infty$ -function.

$f$  is smooth and  $\partial_t u + (u \cdot \nabla)u - \Delta u - f = -\nabla \mathfrak{p}$  because  $-\nabla \mathfrak{p}$  is smooth, so  $\mathfrak{p}$  is also smooth.

(END)

[The uniqueness of elementary weak solutions]

Let the solutions are  $u, v$ .

If  $\partial_t u + (u \cdot \nabla)u - \Delta u - f = \partial_t v + (v \cdot \nabla)v - \Delta v - f$  then  $u = v$ .

[Proof]

$u, v$  are smooth, so if  $u \neq v$ ,

$\partial_t u + (u \cdot \nabla)u - \Delta u - f \neq \partial_t v + (v \cdot \nabla)v - \Delta v - f$ . This is a contradiction.

Therefore  $u = v$ .

(END)

[Supplement 1]

As functions  $\varphi$  that diverges for spatial variables  $\operatorname{div} \varphi = \nabla \cdot \varphi = 0$ , it is sufficient to take any  $\psi \in \mathcal{D}(\Omega)$  and set to  $\varphi = \operatorname{curl} \psi$ . (See [10])

[Supplement 2]

Let  $\|u_n - u\|_{L^2(\Omega)} \rightarrow 0, \|v_n - v\|_{L^2(\Omega)} \rightarrow 0$ . By the triangle inequality, we have

$|\|u_n\|_{L^2(\Omega)} - \|u\|_{L^2(\Omega)}| \leq \|u_n - u\|_{L^2(\Omega)}$  for any sufficiently large  $n$ . On the other hand,

$\|u_n\|_{L^2(\Omega)} < \|u\|_{L^2(\Omega)} + 1$ . Therefore

$\|u_n v_n - uv\|_{L^1(\Omega)} \leq \|u_n\|_{L^2(\Omega)} \|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \|u_n - u\|_{L^2(\Omega)} <$

$(\|u\|_{L^2(\Omega)} + 1) \|v_n - v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} \|u_n - u\|_{L^2(\Omega)} \rightarrow 0$ .

[Supplement 3]

Let  $|\alpha| \leq m - 1$ .

$\int_{\Omega} \left| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \partial^\alpha (\chi_{\Omega}(t - s, x - y) (Pf(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y))) ds dy \right|^p dt dx$

$= \int_{\Omega} \left| \int_{\mathbb{R} \times \mathbb{R}^3 - B_{\delta}(0,0)} E(s, y) \partial^\alpha (\chi_{\Omega}(t - s, x - y) (Pf(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y))) ds dy \right|^p dt dx$

$+ \int_{\Omega} \left| \int_{B_{\delta}(0,0)} E(s, y) \partial^\alpha (\chi_{\Omega}(t - s, x - y) (Pf(t - s, x - y) - P((u \cdot \nabla)u)(t - s, x - y))) ds dy \right|^p dt dx$ .

$E^i(t, x) = \begin{cases} \frac{1}{\sqrt{4\pi t}^3} e^{-\frac{|x|^2}{4t}} & (t > 0) \\ 0 & (t \leq 0) \end{cases}$ , so  $E^i(s, y)$  is a locally integrable function, therefore

$$\int_{\Omega} \left| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \partial^{\alpha} (\chi_{\Omega}(t - s, x - y) P f(t - s, x - y)) ds dy \right|^p dt dx$$

is a finite value.

$$\int_{\Omega} \left| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \partial^{\alpha} (\chi_{\Omega}(t - s, x - y) P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right|^p dt dx$$

is also finite.

$$\begin{aligned} & \int_{\Omega} \left| \int_{\mathbb{R} \times \mathbb{R}^3} E(s, y) \partial^{\alpha} (\chi_{\Omega}(t - s, x - y) P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right|^p dt dx \\ &= \int_{\Omega} \left| \int_{\mathbb{R} \times \mathbb{R}^3 - B_{\delta}(0,0)} E(s, y) \partial^{\alpha} (\chi_{\Omega}(t - s, x - y) P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right|^p dt dx \\ &+ \int_{\Omega} \left| \int_{B_{\delta}(0,0)} E(s, y) \partial^{\alpha} (\chi_{\Omega}(t - s, x - y) P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right|^p dt dx. \end{aligned}$$

This first term is a finite value:

$$\begin{aligned} & \int_{\Omega} \left| \int_{\mathbb{R} \times \mathbb{R}^3 - B_{\delta}(0,0)} E(s, y) \partial^{\alpha} (\chi_{\Omega}(t - s, x - y) P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right|^p dt dx \\ & \leq \sup \{ E^i(s, y) : (s, y) \in \mathbb{R} \times \mathbb{R}^3 - B_{\delta}(0,0) \}^p \int_{\Omega} \left| \int_{\{(s,y):(t-s,x-y) \in \Omega\}} \partial^{\alpha} (P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right|^p dt dx \\ & \leq \sup \{ E^i(s, y) : (s, y) \in \mathbb{R} \times \mathbb{R}^3 - B_{\delta}(0,0) \}^p \sup \{ |\partial^{\alpha} (P((u \cdot \nabla)u))(s, y)| : (s, y) \in \Omega \}^p |\Omega|^{1+p} \\ & < \infty. \end{aligned}$$

Also, the second term is also a finite value:by

Hölder's inequality,

$$\begin{aligned} & \int_{\Omega} \left| \int_{B_{\varepsilon}(0,0)} E(s, y) \partial^{\alpha} (\chi_{\Omega}(t - s, x - y) P((u \cdot \nabla)u)(t - s, x - y)) ds dy \right|^p dt dx \\ & \leq \|E\|_{L^1(B_{\varepsilon}(0,0))}^p \|\partial^{\alpha} (P((u \cdot \nabla)u))\|_{L^{\infty}(B_{\varepsilon}(0,0))}^p |\Omega| \\ & < \infty. \end{aligned}$$

(END)

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