# An Explicit Expression for the Bernoulli Numbers

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Abstract

In this paper we firstly introduce Stirling numbers of the second kind, and give their explicit expression, then we prove an identity that relates Stirling numbers of the second kind to the Bernoulli numbers, finally we give an explicit expression for the Bernoulli numbers.

## I-Stirling numbers of the second kind

Let $Y$ be an arbitrary function and set :

$$D^1Y = x\frac{d}{dx}Y$$

$$D^2Y = x\frac{d}{dx}D^1Y$$

$$D^3Y = x\frac{d}{dx}D^2Y$$

$$\dots$$

$$D^nY = x\frac{d}{dx}D^{n-1}Y$$

If we develop the first of these functions we find :

$D^1Y = xY'$

$D^2Y = xY' + x^2Y''$

$D^3Y = xY' + 3x^2Y'' + x^3Y^{(3)}$

$D^4Y = xY' + 7x^2Y'' + 6x^3Y^{(3)} + x^4Y^{(4)}$

...

We conjecture that :

$$D^nY = S_n^0Y + S_n^1xY' + S_n^2x^2Y'' + \cdots + S_n^nx^nY^{(n)} \qquad (1)$$

The coefficients $S_n^k$ are called Stirling numbers of the second kind. They can be represented in a triangle similar to Pascal's triangle.

The triangle of the numbers $S_n^k$ is the following :

| | k=0 | k=1 | k=2 | k=3 | k=4 | … |
|---|---|---|---|---|---|---|
| n=0 | 1 | | | | | |
| n=1 | 0 | 1 | | | | |
| n=2 | 0 | 1 | 1 | | | |
| n=3 | 0 | 1 | 3 | 1 | | |
| n=4 | 0 | 1 | 7 | 6 | 1 | |
| … | … | … | … | … | … | … |

We observe that :

$$\begin{cases} S_0^0 = 1 \\ \forall n \geq 1, S_n^0 = 0 \end{cases}$$

The law for forming the coefficients in the above table is given by :

$$S_n^k = S_{n-1}^{k-1} + kS_{n-1}^k$$

**II-The explicit formula of Stirling numbers of the second kind**

If we put $Y = e^x$ in the formula (1) we obtain :

$$D^ne^x = e^x\sum_{k=0}^{n} S_n^kx^k$$

$$\Rightarrow e^{-x}.D^ne^x = \sum_{k=0}^{n} S_n^kx^k$$

$$\Rightarrow \left(\sum_{j=0}^{\infty}\frac{(-1)^jx^j}{j!}\right).D^n\left(\sum_{i=0}^{\infty}\frac{x^i}{i!}\right) = \sum_{k=0}^{n} S_n^kx^k$$

$$\Longrightarrow\left(\sum_{j=0}^{\infty}\frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty}\frac{D^n x^i}{i!}\right)=\sum_{k=0}^{n}S_n^k x^k$$

One can easily prove that $D^n x^i = i^n x^i$, so :

$$\left(\sum_{j=0}^{\infty}\frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty}\frac{i^n x^i}{i!}\right)=\sum_{k=0}^{n}S_n^k\,x^k$$

If we develop the left-hand side we obtain :

$$\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\frac{(-1)^{k-i}C_k^i i^n}{k!}\right)x^k=\sum_{k=0}^{n}S_n^k\,x^k$$

Comparing coefficients in both summations we conclude that :

$$S_n^k=\frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}C_k^i\,i^n \qquad (2)$$

**III-Relation between Bernoulli numbers and Stirling numbers of the second kind**

Putting $Y = x^y$ in the formula (1) we get :

$$D^n x^y=\sum_{k=0}^{n}S_n^k\,x^k(x^y)^{(k)}$$

We know that $(x^y)^{(k)} = y(y-1)\ldots(y-k+1)x^{y-k}$ and $D^n x^y = y^n x^y$so we get :

$$y^n=\sum_{k=0}^{n}S_n^k y(y-1)\ldots(y-k+1) \qquad (3)$$

The polynomial $y(y-1)\ldots(y-k+1)$ is called the falling factorial of $y$ of order $k$. Pochhammer used the symbol $(y)_k$ to denote it, so the formula (3) becomes using Pochhammer symbol :

$$y^n=\sum_{k=0}^{n}S_n^k(y)_k \qquad (3')$$

One interesting property of the falling factorial function is the following :

Proposition
Let $y$ and $n$ be non-negative integers, then :

$$(y+1)_{n+1}-(y)_{n+1}=(n+1)(y)_n$$

Proof

$$\begin{aligned}(y+1)_{n+1}-(y)_{n+1}&=(y+1)y(y-1)\ldots(y-n+1)-y(y-1)\ldots(y-n+1)(y-n)\\&=[(y+1)-(y-n)]y(y-1)\ldots(y-n+1)\\&=(n+1)(y)_n\end{aligned}$$

We are going to use this property in the proof of the following proposition.

Proposition
Let $n\in\mathbb{N}$ and $m\in\mathbb{N}^*$. We have :

$$\sum_{y=0}^{m-1}y^n=\sum_{k=0}^{n}S_n^k\frac{(m)_{k+1}}{k+1}\qquad(4)$$

Proof
If we sum for $y$ in the formula (3') we find :

$$\sum_{y=0}^{m-1}y^n=\sum_{y=0}^{m-1}\left(\sum_{k=0}^{n}S_n^k\,(y)_k\right)$$

$$\Rightarrow\sum_{y=0}^{m-1}y^n=\sum_{k=0}^{n}S_n^k\left(\sum_{y=0}^{m-1}(y)_k\right)$$

$$\Rightarrow\sum_{y=0}^{m-1}y^n=\sum_{k=0}^{n}S_n^k\left(\sum_{y=0}^{m-1}\frac{(y+1)_{k+1}-(y)_{k+1}}{k+1}\right)$$

$$\Rightarrow\sum_{y=0}^{m-1}y^n=\sum_{k=0}^{n}S_n^k\left(\frac{(m)_{k+1}-(0)_{k+1}}{k+1}\right)$$

Therefore *:*

$$\sum_{y=0}^{m-1}y^n=\sum_{k=0}^{n}S_n^k\frac{(m)_{k+1}}{k+1}$$

**IV-Bernoulli polynomials**

Let $n \in \mathbb{N}$
The Bernoulli polynomials $B_n(x)$ are defined by the following generating function :

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}$$

If we put $x = 0$ we get :

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n(0) \frac{t^n}{n!}$$

This generating function corresponds to the generating function of Bernoulli numbers $b_n$. Hence for all $n \in \mathbb{N}$, we have :

$$B_n(0) = b_n$$

Also the Bernoulli polynomials satisfy the relation :

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

Their explicit formula is given by :

$$B_n(x) = \sum_{k=0}^{n} C_n^k b_{n-k} x^k$$

If we sum for $y$ in the relation $B_{n+1}(y+1) - B_{n+1}(y) = (n+1)y^n$ we obtain :

$$\sum_{y=0}^{m-1} y^n = \frac{1}{n+1} \sum_{y=0}^{m-1} B_{n+1}(y+1) - B_{n+1}(y)$$

$$= \frac{1}{n+1}\big(B_{n+1}(m) - B_{n+1}(0)\big)$$

$$= \frac{1}{n+1}(B_{n+1}(m) - b_{n+1})$$

Thus :

$$(n+1) \sum_{y=0}^{m-1} y^n = B_{n+1}(m) - b_{n+1} \qquad (5)$$

Comparing formula (4) with formula (5) we conclude that :

$$B_{n+1}(m) - b_{n+1} = (n+1)\sum_{k=0}^{n} S_n^k \frac{(m)_{k+1}}{k+1} \qquad (6)$$

If we develop the expression of $(X)_{k+1}$ in terms of the powers of $X$ we find :

$$\begin{aligned}(X)_{k+1} &= X(X-1)\dots(X-k)\\ &= X\left(X^k - \tfrac{k(k+1)}{2}X^{k-1} + \cdots + (-1)^k k!\right)\\ &= X\sum_{j=0}^{k} c_j\, X^j\\ &= \sum_{j=0}^{k} c_j\, X^{j+1}\end{aligned}$$

Therefore :

$$(X)_{k+1} = \sum_{j=0}^{k} c_j\, X^{j+1}$$

If we apply the above formula for $(m)_{k+1}$ in the formula (6) we find :

$$B_{n+1}(m) - b_{n+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1}$$

Substituting also $B_{n+1}(m)$ by its explicit expression, we finally get :

$$\left(\sum_{k=0}^{n+1} C_{n+1}^k b_{n+1-k} m^k\right) - b_{n+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1}$$

$$\Rightarrow \sum_{k=1}^{n+1} C_{n+1}^k b_{n+1-k} m^k = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1}$$

$$\Rightarrow \sum_{j=0}^{n} C_{n+1}^{j+1} b_{n-j} m^{j+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1}\sum_{j=0}^{k} c_j\, m^{j+1}$$

$$\Rightarrow \sum_{j=0}^{n} \left(C_{n+1}^{j+1} b_{n-j}\right) m^j = \sum_{j=0}^{n}\left(\sum_{k=j}^{n} S_n^k \frac{n+1}{k+1} c_j\right) m^j$$

We have equality between two polynomials in $m$, both of degree $n$, so the coefficients of the terms of the same degree are equal. In particular for $j = 0$ we have

$$C_{n+1}^{1} b_n = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} c_0$$
$$\Longrightarrow b_n = \sum_{k=0}^{n} S_n^k \frac{(-1)^k k!}{k+1}$$

To get the explicit expression of $b_n$ in terms of $n$ we substitute the expression of $S_n^k$ in the above identity, and after simplification we obtain the following remarkable formula of Bernoulli numbers $b_n$ :

$$b_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{i=0}^{k} C_k^i (-1)^i i^n$$

References :


- Sur Une Classe de Nombres Remarquables, Maurice d'Ocagne, American Journal of Mathematics, June 1887

- Close Encounters with The Stirling Numbers of the Second Kind, Khristo Boyadzhiev, June 21, 2018