Canvas theory

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Abstract

We overview the formal aspect of canvas theory and universality.

1 A formal approach on physics

To construct QFT we start by defining physics in categorical language. This is a metaphysical set-up that we draw our ideas on. Before we go further, let us clarify main building blocks of this construction.

1. Canvas theory : The language that we use to show the existence result. It is simply a categorical representation of physics.

2. Condensed set : The elaboration of the language that we are going to deploy for the definition of first quantization.

3. Universal relativity : Physical principle that we are going to give the meaning of our new formal construction.

In this paper, we introduce the big picture. The goal is to construct first quantization and second quantization via canvas theory with universality. Technically, we want to show the existence of free and interacting scalar quantum field theory. After the definition of our main conceptual ingredients, we show examples to make the idea appreciable. Those examples will reappear with more details one by one.

2 Introduction to canvas theory

Let us compress our ideas in one page. We define physics as a functor from *Idea* to *Canvas*.

$$Idea \Rightarrow Canvas$$

Category of *Idea* is what the universe is all about. Simply, the objects of *Idea* is inputs and outputs and morphisms are change. As we will see, it is hard to define *Idea* with concrete notions. We assume there is appreciable physical reality beyond our mathematical description of nature.

$$\begin{tikzcd} (input) \arrow[loop left, "\text{no change}"] \arrow[r, "change"] & (output) \arrow[loop right, "\text{no change}"] \end{tikzcd}$$

Suppose that every phenomenon of the universe is all about this cycle of changes from one object to the other. And if physical principle of causality holds, then the composability and associativity of changes in Idea also holds. So *Idea* forms a category. One can say that more pathologies may be hidden in Idea. However what we observe on observable set, which we call canvas, satisfies causality and basic categorical rules. So we can set physics functor to sort out those pathologies. There is philosophical question whether those pathologies we never observed is even part of our observable universe. So we just restrict study on the Idea that forms category.

Canvas is a category of observable set. We will elaborate the observable set up to the style of physics. Terminology is from topos, which is categorical set up for studying general topological space. And *Idea* is from Plato's philosophy. We see the physics as a diagram, or picture in our terminology, on observable set. So we use terminology of art sometimes. We will define terminologies one by one by showing some examples.

Then physics is a forgetful functor from *Idea* to category of *Canvas*. We want to discuss what it means to define physical theory by this functorial approach. Since we define physics as a map from sub-diagrams of Idea to canvas, *Idea* being big or ill defined does not affect that much. Since by the definition of physics, we can focus on what is seeable on the canvas we choose then forget what not.

The advantage of defining physics in this abstract language is to bring more freedom of what observable set we can choose. Classically, observables are in real numbers. Observables such as mass or energy or distances are represented by real number $\mathbb{R}$. Calculus is available when dealing with real functions. However, quantum observables such as energy level, spin of a particle state are represented by some constant times integers $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$, etc. We want observable set of discrete number system and more. Since the function is not real and smooth any more we work with new calculus, which we call condensed calculus. Also defining quantum or classical field theory over smooth manifold suffers many singularities. There are difficulties for showing the existence. To overcome that, we want to define spectral spacetime and use the principle of universal relativity. We want to introduce new foundations for quantum field theory.

3 Canvas theory

We define nomenclatures of canvas theory.

Definition 1. Canvas is a condensed observable set, which is a functor from category of essence to set.

$$\underline{\mathrm{X}} : \{Essence\}^{op} \to X$$

For construction, we take Essence to be only profinite set, which is inverse limit of finite set, throughout the construction of first quantization. The essence for classical mechanics is trivial null-set, so that the condensed set is isomorphic to the set X. Essence is one of mathematical entities to be selected. According to which essence we choose, it determines how and how much we describe the Idea.

Definition 2. Idea is a category of every phenomenon of universe.

The definition of Idea is not concrete, as it is all about the universe that change and static. We will treat Idea as a small category containing practical data flow such as change of experimental results. For further imagination, we open the definition of Idea. In our construction of QFT, Idea is mostly known facts about quantum and classical part of observable universe whose unknown part does not break down the whole structure of it.

Definition 3. ∼ism or ∼ mechanics, or physics is a functor from Idea to Canvas

$$\sim ism : Idea \to Canvas$$

This is how we see physics in general. It is defined upto the canvas one choose. For example, Newtonian mechanics is a functor to Newtonian canvas whose objects are vector space and morphisms to be transformations between them. See the examples followed.

Definition 4. Relativity or universality is a *relativistic functor* from Idea to Canvas. Relativistic functor is a physics functor that has natural isomorphism when canvas is universal to transformation functor T.

$$Relativity : \quad \begin{tikzcd} Idea \arrow[r] \arrow[dr, "\equiv"] & Canvas \arrow[d, "T"] \\ & Canvas \end{tikzcd}$$

For example, taking the transformation as Lorentz transformation and canvas to be spacetime one gets formal definition of special relativity. The interpretation is that according to the reference frame, the physics shown to the observer is determined. But the Idea or phenomenon itself doesn't change. The physics functor or physical law applies to the same. Look at *Einsteinism* below for further description.

Canvas should be both representable and universal for physical reason. Since universal functor is representable and vice versa, universality is all that one wants to check for the representable theory of physics. One thing we want to emphasize is that the universality is used for both relativity and first quantization. In this sense we call this formality as principle of universal relativity. We use the principle of universal relativity to pull the definition of first quantization and interpret measurement problem.

As one can see, without the language of category theory, one may assume that canvas to be an alternative of spacetime which has certain non-trivial local geometry. We are going to show more intuitional and physical side of canvas theory in *Universal relativity*.

4 Classical examples

In this section, we provide examples to make the definition appreciable. Each examples will be revisited with more details later. Here they are shown to give a sense of the definitions of canvas theory and universality. In order to differentiate nomenclatures of this construction from classical terminology, we will use notions such as Newtonism instead of Newtonian mechanics to emphasize the functorial definition of the physics.

4.1 Newtonism

For example, Newtonian mechanics can be defined as a functor, which we call *Newtonism*, from Idea to the category of Newtonian canvas.

$$x \xrightarrow{f_t} y$$

$$x, y \in \text{3-d vector space}, \quad f_t \in \text{time translation function}$$

So we define physics as depicting small part of *Idea* by category of Newtonian *observable* set. We know that there are different but equivalent ways of describing the *Idea* by Hamiltonian mechanics or Lagrangian mechanics.

4.2 Lagrangeism

We define *Lagrangeism* as a functor from *Idea* to the category of Lagrangian observable set. It's *picture* is as below

$$x \xrightarrow{\delta S=0} y$$

$$x, y \in \text{configuration space}, \quad \delta S = 0 \in \text{least action trajectory}$$

We can consider naturality between Newtonism and Lagrangeism by taking $\delta S = 0 \Rightarrow$ *e.o.m.*[1]

[1] equation of motion or f_t

4.3 *Hamiltonism*

Then one may study the functor of Hamiltonian mechanics, which we just call *Hamiltonism.* It's target category is a observable set, such as

$$x \xrightarrow{V_H} y$$

$$x, y \in \text{6-d phase space}, \quad V_H \in \text{Hamiltonian vector flow}$$

Note that the use of category is different from the category of symplectic manifolds whose objects to be symplectic manifolds and morphisms to be symplectomorphisms.

4.4 *Einsteinism*

We can see special relativity in the sense of *Einsteinism.* Einsteinism is a physics from Idea to spacetime. Classically, we say that a theory is relativistic, if one coordinate system is Lorentz transformation of the other then there is invariant such as the spacetime interval for both reference frame. Then we translate it to the physics functor to be universal for any reference frame that is Lorentz transformation of given spacetime. For spacetime canvas X where x is observer and y is object and morphism to be observable. There is another canvas X' where an observer is a. Then (a,F) is universal such that below maps exists and commute,

$$\begin{tikzcd}
x \arrow[r, "f"] & y \arrow[r, Rightarrow, "F"] & F(x) \arrow[r, "F(f)"] & F(y) \\
& & & a \arrow[ul, "T_{Lor}"'] \arrow[u, "g"]
\end{tikzcd}$$

$$x, y \in \text{spacetime}, \quad T_{Lor} \in \text{Lorentz transformation}$$

This formality results Einsteinian category is universal for any observer a. Physically, it means that any observable g is defined upto the reference frame, which is differed by Lorentz transformation. Simply, g is determined by $F(f) \circ T_{Lor}$. By considering T_{Lor} as base change of spacetime vector, the observables for both observer x and a is defined universally. More details and physical explanations will be revisited again. One thing we want to emphasize is that the universality is one of main principles to define first quantization, and construction of quantum field theories.

In this language, we defined relativistic theory as spacetime canvas with universality. Note that in the above classical non-relativistic canvas, this universality may exist as a coordinates transformations. This shows the flexibility of the definition of physics as a functor. However, the Universality we mean physically contains invariant. In Einsteinism, there is invariant such that $|F(f)| = |g|$ or $ds^2 = ds'^2$. We also study the existence of invariant in first quantization.

Remark 1. The term universality is used from this formality and the context of statistical mechanics. When we made a choice of the terminology such as universal relativity, we have both meanings in mind. For a while, terminology is only about this formality until we define formal version of first quantization.

5 Quntum example

5.1 Schrödingerism

So far, our formality is covering only classical mechanics. Schrödinger's perspective on quantum mechanics is such that particle is represented by wave function and operator acts on it. Observables are eigenvalues of Hermitian operators and wave functions provide probability amplitudes. Heisenberg's perspective is to convert physical operators by matrices. And then their canonical commutation relation defines quantum behavior of physical operators between position and momentum. There is equivalence between both perspectives and both are used in quantum mechanics.

From Schrödinger's perspective, let us consider objects of canvas to be L^2-functions which forms Hilbert space. And the morphisms to be certain operators on them.

$$x \xrightarrow{\hat{O}} y$$

$$x, y \in \text{Hilbert space}, \quad \hat{O} \in \text{Operators on Hilb.}$$

There are certain types of operators in quantum mechanics. For example, any single particle's time translation can be described as follows

$$\Psi_0 \xrightarrow{\hat{H}_t} \Psi_t \Longrightarrow i\hbar \tfrac{d}{dt}\Psi = \hat{H}\Psi$$

This can be read as Schrödinger equation in quantum mechanics.

Any operator O_x and according observable x can be described as LHS of below diagram. And it's physical notation is by inner product of Hilbert space in quantum mechanics.

$$\begin{tikzcd}
 & x_0 & & \langle x_0| \hat{O}_x |\Psi\rangle \\
\Psi \arrow[ur, "\hat{O}_{x_0}"] \arrow[r, "\hat{O}_{x_1}"] \arrow[dr, "\hat{O}_{x_2}"'] & x_1 \arrow[rr, Rightarrow] & & \langle x_1| \hat{O}_x |\Psi\rangle \\
 & x_2 & & \langle x_2| \hat{O}_x |\Psi\rangle
\end{tikzcd}$$

So according to this language, any physical observable has a diagram chase as follows and object is always one of the end point of such diagram. The observable is read by the morphisms to it.

$$\begin{tikzcd}
 & & x_0 \\
\Psi_0 \arrow[r, "\hat{H}_t"] & \Psi_t \arrow[ur, "\hat{O}_{x_0}"] \arrow[r, "\hat{O}_{x_1}"] \arrow[dr, "\hat{O}_{x_2}"'] & x_1 \\
 & \vdots & x_2
\end{tikzcd}$$

6 Condensed observable set

We saw categorical representation of each style of physics, schematically. The problem we want to focus is to build a bridge between classical and quantum examples. In this way, we want to define first quantization as a natural process. For this purpose, we first extend the category of observable set to the category of condensed observable set. As it is shown in the diagram, one of differences in quantum canvas is that one input can have many arrows toward outputs with weighted morphisms. In classical case, any given input has one arrow to the output object. We need to cover this observation in the formality of condensed set.

One mathematical definition of condensed set is

Definition 5. *condensed* set is a functor from category of profinite sets to set.

$$\underline{\mathrm{X}} : \{Profinite\ sets\}^{op} \to X$$

In the original definition, the input category can be varied to category of weakly compact Hausdorff space or extremally disconnected set. Alternatives are there for convenience in dealing with algebraic geometry and topology. Also for physical use, we want to manipulate the input category for practical use, for example approximation of calculations. So for practical purpose, we generalized the definition of condensed set and gave alternative terminologies to define canvas.[2]

Schematically, we are going to use the category of condensed set as our observable category where the below syntactic representation is possible.

$$\underline{x} \xrightarrow{F} \underline{y}$$

We describe Idea by the observable category whose object $\underline{x}$ is a category of condensed set and morphisms are functors of them. With more elaboration of such category, one can see the canvas to be as below

$$\begin{tikzcd}
\varepsilon_2 \arrow[d] \arrow[dr] & \vdots & & \epsilon_2 \arrow[d] \arrow[dr] & \vdots \\
\varepsilon_1 \arrow[r] & x \arrow[r, Rightarrow, "F"] & \epsilon_1 \arrow[rr] & & y \\
\varepsilon_0 \arrow[ur] \arrow[from=u] & & & \epsilon_0 \arrow[ur] \arrow[from=u] &
\end{tikzcd}$$

More physical and analytic explanation of above abstract structure will be followed. So this is how we introduce simplified formal picture of canvas theory.

[2]We tried to invent new terminologies here and there. We believe that it requires physics-only terminology to defy confusions and amplify efficiency.

7 Objectives

The problem of quantum physics is that we know formal definition of each quantum and classical physics, but there is no formal bridge between both. There is physical bridge such as geometric quantization by starting from symplectic manifold and define prequantum line bundle then polarize the space then take metaplectic correction to have wave function like object on polarized symplectic manifold. But neither it is guaranteed as legitimate mathematical transformation nor there is clear physical motivation of this procedure. Also there is deformation quantization where one start from Poisson manifold then deform the Poisson bracket to get non-commutative algebra of observables and so on. This formality too is known to be non-functorial. And the construction doesn't say about physical interpretations.

We want to define first quantization by extending the classical canvas to condensed canvas. Or more clearly, we want to see classical mechanics as a trivial canvas by forgetting deeper categorical structures. We want to answer the question of existence of QFT and measurement problem by studying this specific canvas theory. Note that unlike the canvas theory of classical mechanics any object has multiple different morphisms to it. We want to describe this quantumness by multiple maps toward an object. Assume that the Idea actually contains the multiple morphisms to an object. But classically, the description is only represented by a single map as below forgetting other morphisms.

$$\begin{tikzcd}
\cdot \arrow[d] \arrow[ddr] & & & & \\
\cdot \arrow[d] \arrow[dr] & \vdots & & & \\
\cdot \arrow[r] & \cdot \arrow[r, Rightarrow, "classical"] & {} & \cdot \arrow[r, "f"] & \cdot
\end{tikzcd}$$

Then this physics functor can describe the Idea partially. For describing more details of Idea we have to consider a canvas with more morphisms in it.

$$\begin{tikzcd}
\cdot \arrow[d] \arrow[ddr] & & & \cdot \arrow[d] \arrow[ddr] & \\
\cdot \arrow[d] \arrow[dr] & \vdots & & \cdot \arrow[d] \arrow[dr] & \vdots \\
\cdot \arrow[r] & \cdot \arrow[r, Rightarrow, "quantum"] & {} & \cdot \arrow[r, "f"] & \cdot
\end{tikzcd}$$

For this we use the condensed observable set as our canvas to draw pictures of quantum mechanics. First quantization will be approached from classical Hamiltonism or Lagrangeism by extending the canvas to condensed version of it. We try to explain the measurement problem by universality of condensed observable set.

The goal we want to achieve is to define first quantization by condensification of classical observable set of both Hamiltonism and Lagrangeism. Shortly,

$$\begin{array}{ccc} \text{Hamiltonism} & \Longleftarrow & \text{condensed Hamiltonism + universality} \\ & & \Downarrow \\ & & \text{Schrödingerism} \Longleftrightarrow \text{QM} \end{array}$$

We see classical Hamiltonian mechanics as a trivial condensed Hamiltonism. And we define canonical quantization by a condensed canvas with universality. Similarly, path integral quantization is defined by condensed Lagrangeism with universality.

$$\begin{array}{ccc} \text{Lagrangeism} & \Longleftarrow & \text{condensed Lagrangeism + universality} \\ & & \Downarrow \\ & & \text{Feynmanism} \Longleftrightarrow \text{QM} \end{array}$$

Once again universality is for explaining measurement problem. After this construction of first quantization in Feynmanism, second quantization by including fields and interactions are followed. We focus on the existence problem of such theoretical constructions.

$$\begin{array}{ccc} \text{Lagrangeism } (\Phi, A, \Psi) & \Longleftarrow & \text{condensed Lag } (\Phi, A, \Psi) + \text{universality} \\ & & \Downarrow \\ & & \text{Feynmanism } (\Phi, A, \Psi) \Longleftrightarrow \text{QFT} \end{array}$$

One of goals of constructive QFT is about to show the existence of path integral formalism satisfying certain axioms. For example, Euclidean path integral formalism satisfying axioms of Osterwalder-Schrader also satisfies Wightman axioms. So the existence of QFT is done by checking (OS) to be hold in measure space of quantum field theory over Euclidean spacetime. We approach this problem via canvas theory by using condensed analytic geometry to show the existence of scalar field theory satisfying categorical axioms of (OS).

Next, we are going to study physical aspect of canvas theory. We give physical meaning to the abstract definition of canvas theory and universality. Then we analyze examples one by one to shape what we want to do with this framework for constructing QFT in general.

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