Both the 1-dimensional line from Sieve of Eratosthenes and 2-dimensional trajectory from Riemann zeta function have Centroids as perfect Point symmetry

John Y. C. Ting[1]

[a]*Affiliated with University of Tasmania, Churchill Avenue, Hobart, 7005, Tasmania, Australia*

Abstract

From perspective of Number theory, Dirichlet eta function (proxy function for Riemann zeta function as generating function for all nontrivial zeros) and Sieve of Eratosthenes (as generating algorithm for all prime numbers) are essentially infinite series. We apply infinitesimals to their outputs. Riemann hypothesis asserts the complete set of all nontrivial zeros from Riemann zeta function is located on its critical line. It is proven to be true when usefully regarded as an Incompletely Predictable Problem. We ignore even prime number 2. The complete set with derived subsets of Odd Primes all contain arbitrarily large number of elements while satisfying Prime number theorem for Arithmetic Progressions, Generic Squeeze theorem and Theorem of Divergent-to-Convergent series conversion for Prime numbers. Having these theorems satisfied by all Odd Primes, Polignac's and Twin prime conjectures are separately proven to be true when usefully regarded as Incompletely Predictable Problems.

Keywords: Abel-Ruffini theorem, Brun's constants, Centroid of n-dimensional geometric object, Enhanced regulators, Law of continuity, Polignac's and Twin prime conjectures, Riemann hypothesis, Theory of Symmetry from Langlands program
2020 MSC: 11M26, 11A41, 81Q10

Contents

*Corresponding author Prof. Dr. John Yuk Ching Ting states there is no conflict of interest.
Email address: `jycting@utas.edu.au` (John Y. C. Ting)
URL: `ORCID 0000-0002-3083-5071. Homepage https://jycting.wordpress.com` (John Y. C. Ting)
[1]Dental and Medical Surgery, 12 Splendid Drive, Bridgeman Downs, 4035, Queensland, Australia

1. Introduction

The complex number $z = a + bi$. Its real part a and imaginary part b are real numbers. Its imaginary unit i satisfy power-series expansions $\sum_{n=0}^{\infty} i^n$ [as well as basic facts about powers of i] with given terms:

$$\begin{array}{llll} i^0 = 1, & i^1 = i, & i^2 = -1, & i^3 = -i, \\ i^4 = 1, & i^5 = i, & i^6 = -1, & i^7 = -i \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Using power-series definition, we prove Euler's formula for real values of x:

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \frac{(ix)^8}{8!} + \cdots \\ &= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \frac{x^8}{8!} + \cdots \\ &= 1\left(\frac{x^0}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) + i\left(\frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \\ &= \cos x + i \sin x. \text{ [Note that when } x = \pi,\ e^{i\pi} = -1 \text{ (Euler's identity).]} \end{aligned}$$

In the last step above we recognize $\frac{x^0}{0!} = 1$ and the two terms are Maclaurin series [**alternating power series** or, broadly, alternating infinite series] for $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ with the rearrangement of terms justified because each series is absolutely convergent. Recall that $\cos x$ & $\cosh x$ are even functions, so $\cos(-x) = \cos(x)$ & $\cosh(-x) = \cosh(x)$; and $\sin x$ & $\sinh x$ are odd functions, so $\sin(-x) = -\sin(x)$ & $\sinh(-x) = -\sinh(x)$.

$\sinh i = \frac{e^i - e^{-i}}{2} = i \sin 1$, $\cosh i = \frac{e^i + e^{-i}}{2} = \cos 1$ & $\tanh i = \frac{\sinh i}{\cosh i} = \frac{(e^i - e^{-i})}{(e^i + e^{-i})} = i \tan 1$.

$\sin i = i\frac{e^1 - e^{-1}}{2} = i \sinh 1$, $\cos i = \frac{e^1 + e^{-1}}{2} = \cosh 1$ & $\tan i = \frac{\sin i}{\cos i} = \frac{i(e^1 - e^{-1})}{(e^1 + e^{-1})} = i \tanh 1$.

$\cos i = \sum_{n=0}^{\infty} \frac{1}{(2n)!} = \frac{1}{0!} + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \frac{1}{8!} + \cdots$ & $\sin i = i \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} = i\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\right)$

[Note: For $n = 0$ to ∞, $(i)^{2n} = (i^2)^n = (-1)^n$]. Euler's formula produces following analytical identities for sine, cosine and tangent in terms of e and i: $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$, $\cos x = \frac{e^{ix} + e^{-ix}}{2}$ & $\tan x = \frac{\sin x}{\cos x} = \frac{(e^{ix} - e^{-ix})}{i(e^{ix} + e^{-ix})}$. The related or extended Lindemann-Weierstrass theorem, Gelfond-Schneider theorem, Baker's theorem, four exponentials conjecture or Schanuel's conjecture could be used to establish transcendence of a large class of numbers constituted from the (algebraic) irrational numbers, transcendental (irrational) numbers and rational numbers. Natural logarithm of any natural number other than 0 and 1 (more generally, of any positive algebraic number other than 1) e.g. $\ln 2$ and $\ln\sqrt{2} = \ln 2^{\frac{1}{2}} = \frac{1}{2}\ln 2$ are transcendental numbers by the Lindemann-Weierstrass theorem. By the Gelfond-Schneider theorem, e^{π} [Gelfond's constant], $2^{\sqrt{2}}$ [Gelfond-Schneider constant as an example of a^b where a is algebraic but not 0 or 1, and b is (algebraic) irrational number], $e^{-\frac{\pi}{2}} = i^i$, etc are all transcendental numbers.

As sum of infinite series, Euler's number $e=\sum_{n=0}^{\infty}\frac{1}{(n)!}=\sum_{n=1}^{\infty}\frac{1}{(n-1)!}=1+\frac{1}{1}+\frac{1}{1\cdot 2}+\frac{1}{1\cdot 2\cdot 3}+\cdots$ $\approxeq 2.71828$ is the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$. It can be characterized using integral $\int_1^e \frac{dx}{x} = 1$. As sum of infinite series, $\ln 2 = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = \frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots \approxeq 0.693147$. This infinite series can also be expressed using Riemann zeta function as $\sum_{n=1}^{\infty}\frac{1}{n}[\zeta(2n)-1] = \ln 2$. Some explicit formulas for $\ln 2$ as a result of integration include $\int_0^1 \frac{dx}{1+x} = \int_1^2 \frac{dx}{x} = \ln 2$, $\int_0^{\infty} e^{-x}\frac{1-e^{-x}}{x}\,dx = \ln 2$, $\int_0^{\infty} 2^{-x}dx = \frac{1}{\ln 2}$, $\int_0^{\frac{\pi}{3}} \tan x\,dx = 2\int_0^{\frac{\pi}{4}} \tan x\,dx = \ln 2$, $-\frac{1}{\pi i}\int_0^{\infty}\frac{\ln x \ln\ln x}{(x+1)^2}\,dx = \ln 2$. In the principal branch of logarithm, $\ln(-1) = i\pi$.

The analytic identity using natural logarithm $-\ln(1-i)$ is analogous to Euler's formula for chosen transcendental (real) number values as based on inverse functions $\ln i = \ln(e^{i\frac{\pi}{2}}) = 0+\frac{\pi}{2}i =$ $1.57079632679i$ & $e^i = \cos(1) + i\sin(1) = 0.540302306 + 0.841470985i$. It conforms to the *Langlands program's Theory of Symmetry* w.r.t. imaginary number (point) $i = \sqrt{-1} = 0 + i$ $= \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$; viz, $\ln(e^i) = i$ & $e^{(\ln i)} = i$ [c.f. Figure 8 manifesting (perfect) diagonal symmetry via $\ln(e^x) = x$ & $e^{(\ln x)} = x$]. Then $-\ln(1-i) = -\ln\sqrt{2}+i\frac{\pi}{4}$

$$= 0 + i + \frac{(i)^2}{2} + \frac{(i)^3}{3} + \frac{(i)^4}{4} + \frac{(i)^5}{5} + \frac{(i)^6}{6} + \frac{(i)^7}{7} + \frac{(i)^8}{8} + \cdots$$
$$= 0 + \frac{i}{1} - \frac{1}{2} - \frac{i}{3} + \frac{1}{4} + \frac{i}{5} - \frac{1}{6} - \frac{i}{7} + \frac{1}{8}\cdots$$
$$= 1\left(0 - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{8} - \cdots\right) + i\left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

Transcendental numbers $-\ln\sqrt{2} = -\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n} \approxeq -0.3465...$ & $\frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} \approxeq 0.7853...$ [*aka* Leibniz formula for π] as two **alternating harmonic series** [or, broadly, alternating infinite series] are recognizably related to each other as they represent the two terms in the last step above. As expected, our **additive identity 0** in $-\ln(1-i)$ is analogous to **multiplicative identity 1** [viz, $\frac{x^0}{0!}$] in Euler's formula e^{ix}.

A formal series is an infinite series (sum) that is considered independently from any notion of convergence, and is manipulated with usual algebraic operations on series such as addition, subtraction, multiplication, division, partial sums, etc. A power series defines a function by taking numerical values for the variable WITHIN a radius of convergence. In contrast with NO requirements of convergence, a *formal* power series is a special kind of formal series whose terms are of the form ax^n where x^n is the n^{th} power of a variable x (n is a non-negative integer), and a is called the coefficient. Hence, a formal power series can be viewed as a generalization of polynomials where the number of terms is allowed to be infinite.

Not actually regarded as a function *per se* with its "variable" remaining an indeterminate, a generating function (or series) is a representation of infinite sequences of numbers as coefficients of a formal power series. More generally, a formal power series includes series with any finite (or countable) number of variables, and with coefficients in an arbitrary ring. Rings of formal power series are complete local rings, and this allows using calculus-like methods in the purely algebraic framework of algebraic geometry and commutative algebra. They are analogous in many ways to p-adic integers which can be defined as formal series of the powers of p (see Page 22 – 23 of [6]). Various types of generating functions include ordinary generating functions, exponential generating functions, Lambert series, Bell series, and Dirichlet series. Sieve of Eratosthenes (as generating algorithm for all prime numbers) and Dirichlet eta function (*proxy* function for Riemann zeta function as generating function for all nontrivial zeros) are infinite series since they both encapsulate "infinite sequences of numbers". In this sense, *generating functions and generating algorithms are literally synonymous with infinite series*. By same token as discussed below, harmonic series formed by summing all positive [or alternating positive and negative] unit fractions, are infinite series and is also conveniently regarded as generating functions.

Remark 1. *An L-series is a Dirichlet series, usually convergent on a half-plane, that may give rise to an L-function via analytic continuation. L-functions as denoted by $\mathbb{L}$ [e.g. Dedekind zeta function, Riemann zeta function $\zeta(s)$, Dirichlet eta function $\eta(s)$, Dirichlet L-functions, Hecke L-functions, Artin L-functions, automorphic L-functions, elliptic functions, etc] are meromorphic functions on complex plane, associated to one out of several categories of mathematical objects [viz, anything that has been or could be formally defined, and with which one may do deductive reasoning and mathematical proofs] e.g. Dirichlet character, Hecke character, Artin representations of Galois group G, modular form, λ-ring, Hilbert space, dual vector space, elliptic curve E (abelian variety / group) defined over field K (which can be general field, finite fields, quadratic field $\mathbb{Q}\sqrt{d}$ with d a square-free integer, field of p-adic numbers $\mathbb{Q}_p$, rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$), etc.*

A 'general' Dirichlet series is an infinite series of the form $\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ where a_n, s are complex numbers and $\{\lambda_n\}$ is a strictly increasing sequence of nonnegative real numbers that tends to infinity. An 'ordinary' Dirichlet series $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ is obtained by substituting $\lambda_n = \ln n$ while a power series $\sum_{n=1}^{\infty} a_n (e^{-s})^n$ is obtained when $\lambda_n = n$. **Riemann zeta function $\zeta(s)$ as non-alternating harmonic series Eq. (1) is the **most basic** 'ordinary' Dirichlet series with complex sequence $a_n = 1$ for $n = 1$ to ∞**. Hurwitz zeta function is one of many zeta functions that is formally defined for complex variables s with $\mathrm{Re}(s) > 1$ and $a \neq 0, -1, -2, -3, ...$ by $\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}$. This series is absolutely convergent for given values of s & a, and can be extended to a meromorphic function defined for all $s \neq 1$. **Riemann zeta function is then $\zeta(s, 1)$**.

Dirichlet L-series is a function of the form $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ where χ is a Dirichlet character and s a complex variable with real part greater than 1. It is a special case of a Dirichlet series. By analytic continuation, it can be extended to a meromorphic function on whole complex plane, and is then called Dirichlet L-function and also denoted $L(s, \chi)$. Since Dirichlet character χ is completely multiplicative, its L-function can also be written as an Euler product in the half-plane of absolute convergence $L(s, \chi) = \prod_p (1 - \chi(p)p^{-s})^{-1}$ for $\mathrm{Re}(s) > 1$ where the product is over all prime numbers. Dirichlet L-functions may be written as a linear combination of Hurwitz zeta function at rational values. Fixing an integer $k \geq 1$, Dirichlet L-functions for characters modulo k are linear combinations, with constant coefficients, of $\zeta(s, a)$ where $a = \frac{r}{k}$ and $r = 1, 2, 3, ..., k$. This means Hurwitz zeta function for rational a has analytic properties that are closely related to Dirichlet L-functions. Specifically, let χ be a character modulo k. Then we can write its Dirichlet L-function as $L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \frac{1}{k^s} \sum_{r=1}^{k} \chi(r)\, \zeta\left(s, \frac{r}{k}\right)$.

Dirichlet L-functions satisfy a functional equation, which provides a way to analytically continue them throughout the complex plane. The functional equation relates the value of $L(s, \chi)$ to the value of $L(1 - s, \overline{\chi})$. Let χ be a primitive character modulo q, where $q > 1$. One way to express the functional equation is $L(s, \chi) = \varepsilon(\chi) 2^s \pi^{s-1} q^{1/2-s} \sin\left(\frac{\pi}{2}(s + a)\right) \Gamma(1 - s) L(1 - s, \overline{\chi})$. In this equation, Γ denotes the gamma function; a is 0 if $\chi(-1) = 1$, or 1 if $\chi(-1) = -1$; and $\varepsilon(\chi) = \frac{\tau(\chi)}{i^a \sqrt{q}}$ where $\tau(\chi)$ is a Gauss sum $\tau(\chi) = \sum_{n=1}^{q} \chi(n) \exp(2\pi i n/q)$. It is a property of Gauss sums that $|\tau(\chi)| = q^{\frac{1}{2}}$, so $|\varepsilon(\chi)| = 1$. Another way to state the functional equation is in terms of $\xi(s, \chi) = \left(\frac{q}{\pi}\right)^{(s+a)/2} \Gamma\left(\frac{s + a}{2}\right) L(s, \chi)$. The functional equation can be expressed as $\xi(s, \chi) = \varepsilon(\chi) \xi(1 - s, \overline{\chi})$. The functional equation implies that $L(s, \chi)$ and $\xi(s, \chi)$ are entire functions of s. Again, this assumes that χ is primitive character modulo q with $q > 1$. If $q = 1$, then $L(s, \chi) = \zeta(s)$ has a pole at s = 1.

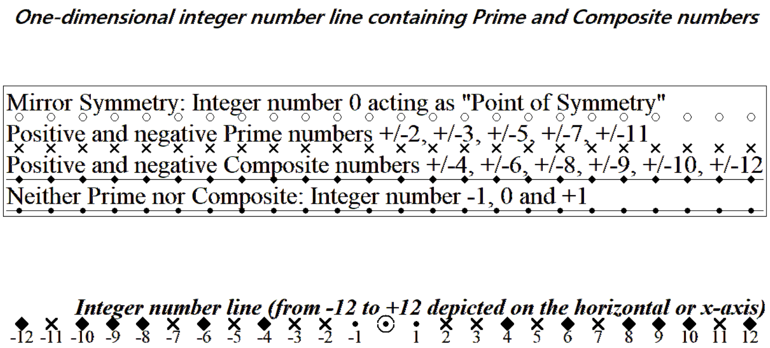
In mathematics and theoretical physics, techniques of zeta function regularization, dimensional regularization and analytic regularization are types of regularization or summability methods that assigns finite values to divergent sums or products. They are then used to define determinants and traces of some self-adjoint operators [which admit orthonormal eigenbasis with real eigenvalues]. Inspired by the Method of Smoothed asymptotics previously developed by Prof. Terence Tao in 2010, we broadly base some deductions in this paper on recent introduction in 2024 by Prof. Antonio Padilla & Prof. Robert Smith of a new ultra-violet regularization scheme for loop integrals in Quantum field theory dubbed η regularization. We outline in section 4 rich underlying connections between analytic number theory & perturbative quantum field theory.

Broadly viewed as vast "resource materials" that support the completed 2001 proofs on modularity theorem, we have bi-directional correspondences (bridges) existing between Number theory $\leftrightarrow$ Harmonic analysis forming "framework" for L-functions and modular forms database (LMFDB, launched on May 10, 2016)[2]: (i) {Elliptic curves $\leftrightarrow$ Modular forms}; (ii) {Counting problem $1 + p-$number of solutions mod p [in *finite series* Elliptic curves] $\leftrightarrow$ Coefficients of q^p [in *infinite series* Modular forms]} whereby nome $q = e^{\pi i \tau}$ & p = prime numbers from Modular forms act as (periodic) 'generating series or functions' having Group of symmetry = $SL_2(\mathbb{Z})$ [involving unit disk in complex plane], which is analogous to Group of symmetry = Group of integers $\mathbb{Z}$ [involving real number line present in general solutions such as $\sin(x + 2\pi n) = \sin(x)$ with $n = ... - 3, -2, -1, 0, 1, 2, 3...$]; viz, these properties conform to the *Langlands program "Theory of Symmetry"* [for Transformations of Rotation, Translation, Dilation and Reflection]; and (iii) {Representations of Galois groups $\leftrightarrow$ Automorphic forms} whereby the modular forms are classified as a specific type of these [more general] automorphic forms, which are ultimate objects in Harmonic analysis.

Useful relationship: integer numbers $\mathbb{Z} \subset$ rational numbers $\mathbb{Q}$ since $\mathbb{Q}$ = quotient or fraction $\frac{p}{q}$ of two $\mathbb{Z}$ and when denominator $q = 1$, $\mathbb{Q}$ is always an $\mathbb{Z}$ [whereby $\frac{0}{1} = 0$]. Diophantine equations are effectively various *"finite series"* polynomial equations that generally involve operation of adding finitely many terms e.g. Fermat's equation $x^n + y^n = z^n$ and elliptic curve $y^2 = x^3 + ax + b$. Proposed by Pierre de Fermat in 1637, Fermat's Last Theorem states that no three positive integers a, b and c can satisfy Fermat's equation for any integer value of n greater than 2. The modularity theorem asserts that every elliptic curve [as fundamental mathematical objects defined by cubic equations in two variables] over the rational numbers is modular, meaning that it can be associated with an *"infinite series"* modular form. In a nutshell, this was broadly a crucial step in proving Fermat's Last Theorem because it famously allowed Prof. Andrew Wiles to prove the theorem in 1994 by establishing a deep connection between [semistable] elliptic curves and modular forms. Sir Andrew Wiles was deservingly awarded the 2016 Abel Prize for this work. Finally, the [unsolved] Birch and Swinnerton-Dyer (BSD) conjecture asserts an elliptic curve has either an infinite number or a finite number of rational points (solutions) according to whether $\zeta(1) = 0$ or $\zeta(1) \neq 0$, respectively [when these rational points (solutions) are points of an abelian variety and $\zeta(1)$ is an associated zeta function $\zeta(s)$ near point $s = 1$]. Therefore this conjecture relates number of points on an elliptic curve mod p to the rank of the group of rational points.

We have infinities or infinitely large numbers as the unbounded and limitless quantities (∞) at the big end, and infinitesimals or infinitely small numbers as the extremely small but nonzero quantities ($\frac{1}{\infty}$) at the small end. Applying infinitesimals to their corresponding outputs in section 6 allow us to prove 1859-dated Riemann hypothesis [viz, the proposal that relevant outputs as infinitely many nontrivial zeros or Origin intercept points of Riemann zeta function are all located on its $\sigma = \frac{1}{2}$-critical line or $\sigma = \frac{1}{2}$-Origin point], and Polignac's and Twin prime conjectures [viz, the proposal that relevant outputs as subsets of Odd Primes derived from every even Prime gaps 2, 4, 6, 8, 10... all contain infinitely many unique elements]. Referring to even Prime gap 2, 1846-dated Twin prime conjecture is simply a subset of 1849-dated Polignac's conjecture [which refers to all even Prime gaps 2, 4, 6, 8, 10...]. Altered terminology on cardinality of Odd Primes being *arbitrarily large number* (ALN) instead of *infinitely many* was previously used to denote *Modified* Polignac's and Twin prime conjectures.

Our *generic mathematical approaches* for solving Riemann hypothesis, Polignac's and Twin prime conjectures is applicable to selected branches of science such as relativistic quantum mechanics, quantum gravity or string theory. When usefully construed as a self-sufficient research paper, correct and complete mathematical arguments condensed in this current paper are major (core) arguments from publications [4], [5] & [6] whereby Riemann zeta function [= function that faithfully generates output of all nontrivial zeros via its *proxy* Dirichlet eta function] and Sieve of Eratosthenes [= algorithm that faithfully generates output of all prime numbers] are treated as de novo or derived infinite series in order to prove their connected open problems in

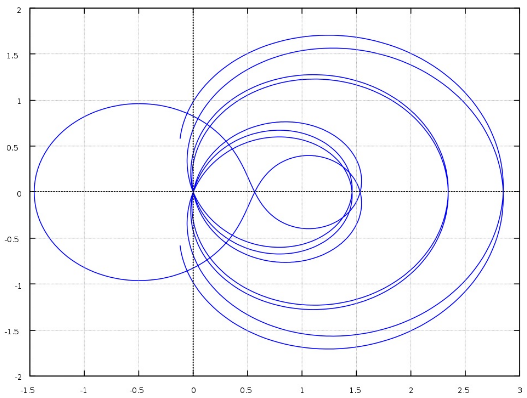


1: Narrow range of positive & negative prime and composite numbers plotted together on integer number line generated using Sieve-of-Eratosthenes and complement-Sieve-of-Eratossthenes. The combined [positive] image and [negative] mirror image will conceptually represent a one-dimensional line (state) having perfect Mirror symmetry with integer number 0 acting as the Point of symmetry.

Number theory. These infinite series are either convergent series or divergent series where partial sums of sequence from the former tends to a finite limit, while that from the later do not tend to a finite limit [viz, it tends to infinity]. Prime number theorem for Arithmetic Progressions [as Axiom 1], Generic Squeeze theorem [as Theorem 2] and Theorem of Divergent-to-Convergent series conversion for Prime numbers [as Theorem 3] are outlined (respectively) in section 2, section 3 and section 4. Lemma 4 and Lemma 5 in section 5 (respectively) introduce novel concept of Incompletely Predictable entities and innovatively classifying countably infinite sets into accelerating, linear or decelerating subtypes. To the extent that many associated minor (peripheral) arguments were not included in this paper, we advocate their absence do not adversely reflect the rigorous nature of our derived proofs but, rather, helps disseminate mathematical knowledge to the lay person and scientific community.

A function [sometimes loosely termed an operator or an equation] is usefully defined as the relation between a set of inputs (called domain) and a set of possible outputs (called codomain) where each input is related to EXACTLY one output. More precisely, the classical example of [linear] operator performed on [eligible] functions is differentiation. An algorithm is usefully defined as the finite sequence of rigorous instructions typically used to solve a class of specific problems or to perform a computation. Functions or algorithms as infinite-dimensional vectors: A function or algorithm defined on real numbers $\mathbb{R}$ can be represented by an uncountably infinite set of vectors (as a vector field) while a function or algorithm defined on natural numbers $\mathbb{N}$ [or any other countably infinite domain such as prime numbers and composite numbers] can be represented by a countably infinite set of vectors (as a vector field). One could also use the later countably infinite set of vectors involving [discrete] $\mathbb{N}$ {e.g. all nontrivial zeros of Riemann zeta function interpolated as "nearest" t-valued $\mathbb{N}$ 14, 21, 25, 30, 33, 38, 41, 43...} to approximate the former uncountably infinite set of vectors that "pseudo-represent" [continuous] $\mathbb{R}$ {for the same nontrivial zeros when precisely given as t-valued transcendental numbers} $\approxeq$ Law of continuity: If a quantity changes "continuously", then its value at any point between two given values can be determined by the process of interpolation.

Based on Figure 1 and Figure 2 that accommodate both positive (+*ve*) parts and negative



2: OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Polar graph of $\zeta(\frac{1}{2} + \imath t)$ depicted as a two-dimensional figure (state) plotted along critical line for real values of t between -30 and $+30$ [viz, for $s = \sigma \pm t$ range], horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Origin intercept points are present. There is manifestation of perfect Mirror symmetry about horizontal x-axis acting as the line of symmetry.

($-ve$) counterparts of prime numbers, composite numbers and nontrivial zeros, **we can represent eligible functions with complex vector space [having +ve and –ve complex vectors pointing in opposite directions] and eligible algorithms with real vector space [having +ve and –ve real vectors pointing in opposite directions]**: Recall that a row vector or a column vector is, respectively, a one-row matrix or a one-column matrix. Real numbers $\mathbb{R}$ [and natural numbers $\mathbb{N}$] are exactly one-dimensional vectors (on a line) and complex numbers $\mathbb{C}$ are exactly two-dimensional vectors (in a plane). A complex vector (or complex matrix) as Cartesian representation $z = x + iy$ or Polar representation $z = r(\cos\theta + i \cdot \sin\theta)$ is simply a vector (matrix) of the complex numbers. A two-dimensional real vector (or real matrix) in a plane is given by Cartesian representation as $v = x + y$ or Polar representation as $v = r(\cos\theta + \sin\theta)$. x & y are $\mathbb{R}$, modulus $r = |z \text{ or } v| = \sqrt{x^2 + y^2}$, multi-valued arg($z$ or v) or principal-valued Arg(z or v) = $\theta = \arctan(y/x)$, and imaginary unit $i = \sqrt{-1}$.

Integers {0, 1} are neither prime nor composite. Prime & composite numbers form distinct countably infinite sets of integers as two subsets in uncountably infinite set of real numbers. Both [algorithmic] inputs Sieve-of-Eratosthenes and Complement-Sieve-of-Eratosthenes in section 2 that faithfully generate outputs prime & composite numbers are visually represented by countably infinite set of real vectors. We recognize all real vector sub-spaces for even Prime gaps 2, 4, 6, 8, 10... with each unique sub-space constituted by its corresponding countably infinite set of real vectors, imply Polignac's and Twin prime conjectures are true.

Where σ, t, $Re\{\zeta(s)\}$, $Im\{\zeta(s)\}$, $Re\{\eta(s)\}$ and $Im\{\eta(s)\}$ are $\mathbb{R}$, (input) parameter $s = \sigma \pm it$ used in (output) functions from section 2 such as non-alternating Riemann zeta function Eq. 1 $\zeta(s) = Re\{\zeta(s)\} + i \cdot Im\{\zeta(s)\}$ and alternating Dirichlet eta function Eq. 2 $\eta(s) = Re\{\eta(s)\} + i \cdot Im\{\eta(s)\}$ are recognized to all be given in $z = x + iy$ format, thus allowing uncountably infinite set of complex vectors to visually represent them. Next consider two derived functions from section 2: simplified Dirichlet eta function or sim-$\eta(s)$ and Dirichlet Sigma-Power Law or DSPL [= $\int$ sim-$\eta(s)dn$ ≡ "signed area under a curve" for this Riemann integrable function] with their corresponding horizontal and vertical axes being perpendicular to each other or, equivalently, being $\frac{\pi}{2}$ out-of-phase with each other (as per Page 12 of [4]). Complex vectors representing sim-$\eta(s)$ and DSPL when combined together form an orthonormal set in the inner product space since all these vectors in the set are mutually orthogonal ("perpendicular") and can be depicted using their ("normalized") unit length. When equivalently expressed using countably infinite set of complex vectors; we recognize nontrivial zeros of $\zeta(s)$, $\eta(s)$, sim-$\eta(s)$ or DSPL that can only exist in unique $\sigma = \frac{1}{2}$ complex vector sub-space, must imply Riemann hypothesis is true.

Non-alternating power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$

Alternating power series $\sum_{n=0}^{\infty} (-1)^n a_n x^n = a_0 - a_1 x + a_2 x^2 - a_3 x^3 + \ldots$

$\boxed{\text{Non-alternating harmonic series } \sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots}$

$\boxed{\text{Alternating harmonic series } \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots}$

When $s = 1$ in Eq. 1 $\zeta(s)$ & Eq. 2 $\eta(s)$ with n = +ve integers, we (respectively) obtain the above **most basic $\boxed{\text{Non-alternating harmonic series}}$ and $\boxed{\text{Alternating harmonic series}}$**. An infinite series [listed above as various types of power series and harmonic series] (or a finite series) is sum of [≥ 1] infinite (or finite) sequence of terms constituted by numbers, scalars, or anything *e.g. functions, vectors, matrices*. As previously discussed, power series [with VARYING coefficients a_n] are infinite polynomials. Sieve-of-Eratosthenes & Complement-Sieve-of-Eratosthenes as well-defined infinite algorithms give rise to [infinite] n solutions of all primes & composites; viz, they are the "analogs" of power or harmonic series as well-defined infinite functions. With SAME coefficients a, the (non-alternating) geometric series $\sum_{n=0}^{\infty} ax^n = a + ax + ax^2 + ax^3 + \ldots$ having +ve common ratio x between successive terms, is simply a special case of (non-alternating) power series e.g. when $a = \frac{1}{2}$ & $\frac{1}{2}$ for +ve common ratio. **Cf** when $a = \frac{1}{2}$ & $-\frac{1}{2}$ for −ve common ratio in an "inverse" (alternating) geometric series, which is simply a special case of (alternating) power series (Page 56 of [6]):

$$\sum_{n=0}^{\infty} \frac{1}{2}(-\frac{1}{2})^n = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots = \frac{\frac{1}{2}}{1-(-\frac{1}{2})} = \frac{1}{3} \textbf{ cf } \sum_{n=0}^{\infty} \frac{1}{2}(\frac{1}{2})^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \frac{\frac{1}{2}}{1-(+\frac{1}{2})} = 1$$

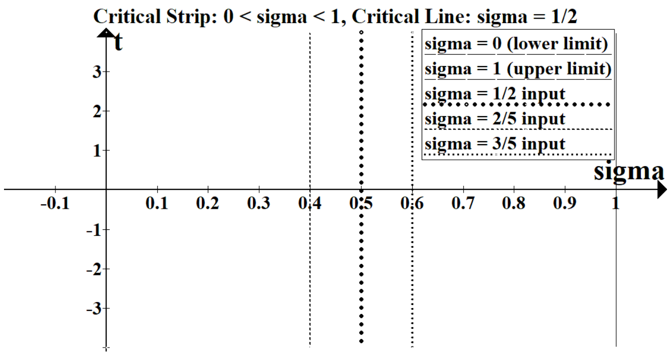
An expression is in closed form if it is formed with constants, variables and a finite set of basic functions connected by arithmetic operations (viz, +, −, ×, ÷, and integer powers) and function composition. The commonly allowed functions are [I] the algebraic functions [viz, defined as the root of an irreducible polynomial equation] e.g. nth root or raising to a fractional power and [II] the transcendental (non-algebraic) functions e.g. exponential function, logarithmic function, Γ function, trigonometric functions and their inverses. Algebraic and transcendental (non-algebraic) solutions form two subsets of closed-form expressions. Thus, a solution in radicals or algebraic solution is a closed-form expression, and more specifically a closed-form algebraic expression, that is the solution of a polynomial equation, and relies only on addition, subtraction, multiplication, division, raising to integer powers, and the extraction of nth roots (square roots, cube roots, and other integer roots). Following directly from Galois theory using polynomial $f(x) = x^5 - x - 1$ as one of the simplest examples of a non-solvable quintic polynomial, Abel-Ruffini theorem states that there is no solution in radicals to SOME general (finite) polynomial equations of degree five or higher with arbitrary coefficients. Here, *general* meant the coefficients of a polynomial equation are viewed and manipulated as indeterminates. We extrapolate: Any power series [e.g. e^x, $\sin x$, $\sinh x$, $\ln x$, etc] as general (infinite) polynomial equations having infinitely many coefficients should have no solution in radicals [viz, have transcendental solutions]. However some power series with coefficients involving (infinite) polynomials [e.g. geometric series, binomial series, etc] can have solutions expressible in terms of radicals, provided the series converges within the domain where such expressions are valid. Similar to, but not categorized as, power series are various hypergeometric series [as defined by the generalized hypergeometric function] that could have either transcendental solutions or solutions in radicals.

Eq. 1 $\zeta(s)$ & Eq. 2 $\eta(s)$ have complex variable $s = \sigma \pm it$. In $0 < \sigma < 1$ critical strip containing $\sigma = \frac{1}{2}$ critical line, $\eta(s)$ must act as proxy function for $\zeta(s)$ [with both ≡ infinite series]. Useful relationship: z as a complex number $\mathbb{C}$ is defined by $z = a+bi$ with i being the imaginary unit, and a & b being real numbers $\mathbb{R}$. Thus $\mathbb{R} \subset \mathbb{C}$ since when $b = 0$, $z = a + 0i = a$ will always be $\mathbb{R}$. Our "amalgated" *generic Fundamental Theorem of Algebra* heuristically $\Longrightarrow$ (eligible) general [finite or infinite or ALN] algorithms and functions (of degree n with real or complex coefficients) have exactly [finite or infinite or ALN] n roots or n solutions as real or complex numbers, counting *multiplicities* {e.g. $\sin(x + 2\pi n)$ with $n = ...-3, -2, -1, 0, 1, 2, 3...$; ±nontrivial zeros; ±Primes; ±Composites; etc}. Riemann hypothesis is true when nontrivial zeros as Origin point intercepts are the infinitely many n roots that only occur when parameter $\sigma = \frac{1}{2}$ resulting in [optimal] "formula symmetry" for $\eta(s)$ [as **infinite series**]. Polignac's and Twin prime conjectures are true when Sieve-of-Eratosthenes algorithm and its derived sub-algorithms [as **"infinite series"** via $\sum_{n=i}^{ALN} p_{n+1} = 3 + \sum_{i=2}^{n} g_i$] have ALN of n solutions represented by Set [≡ total] of Odd Primes and Subsets [≡ subtotals] of Odd Primes derived from all even Prime gaps.

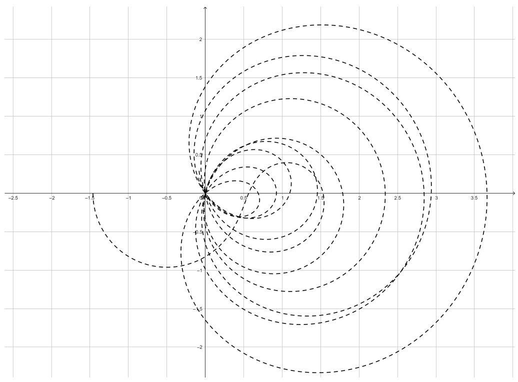
2. General notations including Prime number theorem for Arithmetic Progressions and creating *de novo* Infinite Series

Common abbreviations used in this paper: CP = Completely Predictable, IP = Incompletely Predictable, FL = Finite-Length, IL = Infinite-Length, CFS = countably finite set, CIS = countably infinite set, IM = infinitely-many, ALN = arbitrarily large number. We treat eligible algorithms and functions as *de novo* infinite series.

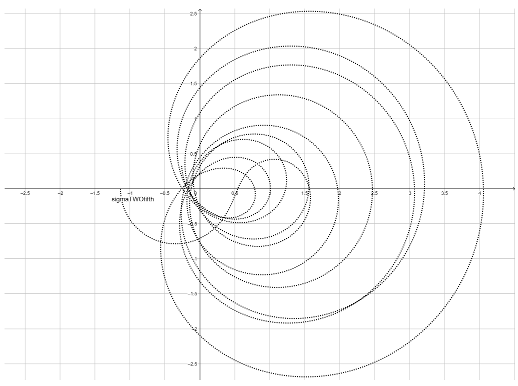
Critical strip≡$\{s \in \mathbb{C} : 0 < \text{Re}(s) < 1\}$ & Critical line≡$\{s \in \mathbb{C} : \text{Re}(s) = \frac{1}{2}\}$ in Figure 3. Phrase "*inside* the critical strip" refers to parameter s [$= \sigma \pm it$ with $0 < \sigma < 1$; viz, $0 < \text{Re}(s) < 1$] having complex number values defined for $\eta(s)$ as given by parameter t over ± real numbers. Phrase "*outside* the critical strip" refers to parameter s [$= \sigma \pm it$ with $\sigma > 1$; viz, $\text{Re}(s) > 1$] having complex number values defined for $\zeta(s)$ as given by parameter t over ± real numbers. When s is considered for (purely) real number values: $\zeta(-1) = -\frac{1}{12}$, $\zeta(0) = -\frac{1}{2}$, $\zeta(\frac{1}{2}) = -1.4603545...$, etc. Via Eq. (3) as its functional equation, $\zeta(s)$ has Completely Predictable infinitely many trivial zeros at each even negative integer $s = -2n$ for $n = 1, 2, 3, 4, 5....$ The point $s = 1$ in $\zeta(s)$ corresponds to a simple pole with complex residue 1. Even though $\zeta(1)$ is undefined as it diverges



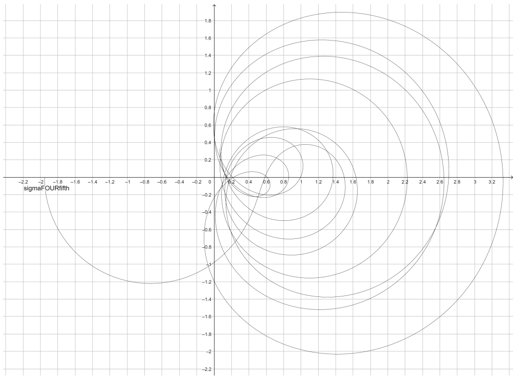
3: INPUT for $\sigma = \frac{1}{2}$ (for Figure 4), $\frac{2}{5}$ (for Figure 5), and $\frac{3}{5}$ (for Figure 6). Riemann zeta function $\zeta(s)$ has two countable infinite sets of firstly, Completely Predictable trivial zeros located at s = all negative even numbers and secondly, Incompletely Predictable nontrivial zeros located at $\sigma = \frac{1}{2}$ as various t-valued transcendental numbers.



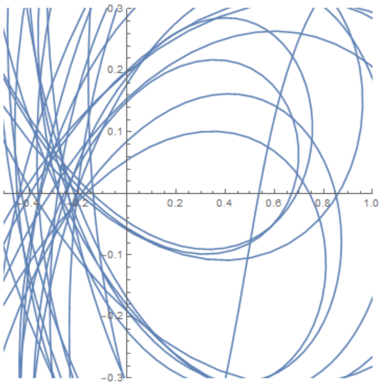
4: OUTPUT for $\sigma = \frac{1}{2}$ as Gram points. Polar graph of $\zeta(\frac{1}{2} + \imath t)$ plotted along critical line for real values of t running from 0 to 34. Horizontal axis: $Re\{\zeta(\frac{1}{2} + \imath t)\}$. Vertical axis: $Im\{\zeta(\frac{1}{2} + \imath t)\}$. Presence of Origin intercept points. **Nil-shift w.r.t. Origin point when** $\sigma = \frac{1}{2}$.



5: OUTPUT for $\sigma = \frac{2}{5}$ as virtual Gram points. Varying Loops are shifted to left of Origin with horizontal axis: $Re\{\zeta(\frac{2}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{2}{5} + \imath t)\}$. Nil Origin intercept points. **Left-shift w.r.t. Origin point when** $\sigma < \frac{1}{2}$**; viz,** $0 < \sigma < \frac{1}{2}$.



6: OUTPUT for $\sigma = \frac{3}{5}$ as virtual Gram points. Varying Loops are shifted to right of Origin with horizontal axis: $Re\{\zeta(\frac{3}{5} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{3}{5} + \imath t)\}$. Nil Origin intercept points. **Right-shift w.r.t. Origin point when $\sigma > \frac{1}{2}$; viz, $\frac{1}{2} < \sigma < 1$.**



7: Close-up view of virtual Origin points when $\sigma = \frac{1}{3}$. OUTPUT for $\sigma = \frac{1}{3}$ [$\sigma < \frac{1}{2}$ situation] as virtual Gram points. Polar graph of $\zeta(\frac{1}{3} + \imath t)$ plotted along non-critical line for real values of t running between 0 and 100, horizontal axis: $Re\{\zeta(\frac{1}{3} + \imath t)\}$, and vertical axis: $Im\{\zeta(\frac{1}{3} + \imath t)\}$. Total absence of all Origin intercept points at "static" Origin point. Total presence of all virtual Origin intercept points (as additional negative virtual Gram[y=0] points on x-axis) at "varying" [infinitely many] virtual Origin points. **With respect to $\sigma = \frac{1}{2}$-Origin point being analogically the $\sigma = \frac{1}{2}$-'Centroid', then the [depicted] "left-shifted" $\sigma = \frac{1}{3}$ as being $\frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$ and the [undepicted] "right-shifted" $\sigma = \frac{2}{3}$ as being $\frac{2}{3} - \frac{1}{2} = +\frac{1}{6}$ are BOTH equidistant from 'Centroid' [thus fully satisfying (Remark 3) Principle of Equidistant for Multiplicative Inverse – see last paragraph discussion in section 6 Conclusions.**

to ∞, its Cauchy principal value $\lim_{\varepsilon\to 0} \frac{\zeta(1+\varepsilon)+\zeta(1-\varepsilon)}{2}$ exists and is equal to Euler-Mascheroni constant $\gamma = 0.577218...$ [a transcendental number]. For small positive integer values of s: $\zeta(1) = \infty$, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(3) = 1.2020569032...$, $\zeta(4) = \frac{\pi^4}{90}$, $\zeta(5) = 1.0369277551...$, $\zeta(6) = \frac{\pi^6}{945}$, $\zeta(7) = 1.0083492774...$, $\zeta(8) = \frac{\pi^8}{9450}$, $\zeta(9) = 1.0020083928...$, $\zeta(10) = \frac{\pi^{10}}{93555}$, etc. When s = 2, 4, 6, 8, 10...; computed ζ(s) values all contain transcendental irrational number π. When s = 3, 5, 7, 9, 11...; computed ζ(s) values are "likely" all algebraic irrational numbers. In fact, only $\zeta(3)$ or Apery's constant is proven to be an irrational number but it is unknown whether it is also a transcendental number derived from (e.g.) π^3 or another unrelated transcendental number. Despite these unknowns, the computed ζ(s) solutions from substituting s = even numbers 2, 4, 6, 8, 10... versus s = odd numbers 3, 5, 7, 9, 11... should all be irrational numbers that are, crucially, *mutually exclusive* by being **mathematically, geometrically and topologically** different from each other. **List of abbreviations incorporating relevant definitions:**

·**CP entities**: These entities manifest CP *independent* properties.

·**IP entities**: These entities manifest IP *dependent* properties.

·**ζ(s)**: $f(n)$ Riemann zeta function [≡ **infinite (converging) series** for Re(s) > 1] in Eq. (1) below containing variable n, and parameters t and σ will generate [via its *proxy* Dirichlet eta function] Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.

·**η(s)**: $f(n)$ Dirichlet eta function [≡ **infinite (converging) series** for Re(s) > 0] in Eq. (2) below as the analytic continuation of ζ(s), containing variable n, and parameters t and σ will generate Zeroes when $\sigma = \frac{1}{2}$ and virtual Zeroes when $\sigma \neq \frac{1}{2}$.

·**sim-η(s)**: $f(n)$ simplified Dirichlet eta function [≡ **infinite (converging) series** for Re(s) > 0], derived by applying Euler formula to η(s), containing variable n, and parameters t and σ will generate Zeroes when $\sigma = \frac{1}{2}$ in Eq. (4) below and virtual Zeroes when $\sigma \neq \frac{1}{2}$ in Eq. (5) below.

·**DSPL**: $F(n)$ Dirichlet Sigma-Power Law [≡ "continuous" **infinite (converging) series** for Re(s) > 0] = $\int$ sim-$\eta(s)dn$ containing variable n, and parameters t and σ will generate Pseudo-zeroes when $\sigma = \frac{1}{2}$ in Eq. (6) below and virtual Pseudo-zeroes when $\sigma \neq \frac{1}{2}$ whereby the (virtual) Zeros = (virtual) Pseudo-zeros − $\frac{\pi}{2}$ relationship allows (virtual) Pseudo-zeros to (virtual) Zeros conversion and *vice versa*.

·**NTZ**: Nontrivial zeros located on the one-dimensional (mathematical) $\sigma = \frac{1}{2}$-critical line are precisely equivalent to **G[x=0,y=0]P**: Gram[x=0,y=0] points as Origin intercept points which are located at the zero-dimensional (geometrical) $\sigma = \frac{1}{2}$-Origin point [as per Figure 4]. These entities, mathematically defined by $\sum ReIm\{\eta(s)\} = Re\{\eta(s)\} + Im\{\eta(s)\} = 0$, are generated by equation G[x=0,y=0]P-η(s) containing exponent $\frac{1}{2}$ when $\sigma = \frac{1}{2}$.

·**GP or G[y=0]P**: 'usual' or 'traditional' Gram points = Gram[y=0] points = x-axis intercept points that are [multiple-positioned] located on one-dimensional x-axis line are generated by equation G[y=0]P-η(s) when $\sigma = \frac{1}{2}$. These entities are mathematically defined by $\sum$ReIm$\{\eta(s)\}$ = Re$\{\eta(s)\}$ + 0, or simply Im$\{\eta(s)\}$ = 0. Riemann hypothesis is usefully stated as none of the [additional] virtual G[x=0]P generated by equation G[x=0]P-η(s) when $\sigma \neq \frac{1}{2}$ – as demonstrated by Figure 7 for $\sigma = \frac{1}{3}$ – can be constituted by t transcendental number values that [incorrectly] coincide with t transcendental number values for NTZ when $\sigma = \frac{1}{2}$.

·**G[x=0]P**: Gram[x=0] points = y-axis intercept points that are [multiple-positioned] located on one-dimentional y-axis line are generated by equation G[x=0]P-η(s) when $\sigma = \frac{1}{2}$. These entities are mathematically defined by $\sum$ReIm$\{\eta(s)\}$ = 0 + Im$\{\eta(s)\}$, or simply Re$\{\eta(s)\}$ = 0.

·**virtual NTZ**: virtual nontrivial zeros or **virtual G[x=0,y=0]P**: virtual Gram[x=0,y=0] points.

These are virtual Origin intercept points located at the multiple-positioned virtual Origin points which are generated by equation virtual-G[x=0,y=0]P-η(s) containing exponent values $\neq \frac{1}{2}$ when $\sigma \neq \frac{1}{2}$. We note that each virtual NTZ when $\sigma < \frac{1}{2}$ in Figure 5 equates to an [additional] negative virtual G[y=0]P located at IP varying positions on horizontal axis, and each virtual NTZ when $\sigma > \frac{1}{2}$ in Figure 6 equates to an [additional] positive virtual G[y=0]P located at IP varying positions on horizontal axis. We observe overall less virtual G[x=0]P when $\sigma > \frac{1}{2}$, and overall more virtual G[x=0]P when $\sigma < \frac{1}{2}$.

·**Sieve-of-Eratosthenes (S-of-E)**: For i = 1, 2, 3, 4, 5... and with $p_1 = 2$ [≡ even prime number 2 forming a CFS with cardinality of 1] as the first term in S-of-E; the algorithm S-of-E as symbolically denoted by $p_{n+1} = 2 + \sum_{i=1}^{n} g_i$ with $g_n = p_{n+1} - p_n$ and its derived sub-algorithms faithfully generate the set of all prime numbers 2, 3, 5, 7, 11, 13... and subsets of Odd Primes derived from even Prime gaps 2, 4, 6, 8, 10.... We now ignore even prime number 2 by changing variable i to instead commence from 2^{nd} position. For i = 2, 3, 4, 5, 6... and with $p_2 = 3$ [≡ first Odd Prime 3] as the first term in Modified-S-of-E; the altered algorithm Modified-S-of-E as symbolically denoted by $p_{n+1} = 3 + \sum_{i=2}^{n} g_i$ with $g_n = p_{n+1} - p_n$ and its derived sub-algorithms will faithfully generate the set of all Odd Primes 3, 5, 7, 11, 13, 17... and subsets of Odd Primes derived from even Prime gaps 2, 4, 6, 8, 10.... By performing summation [viz, conducting repeated addition of sequence from ALN of prime gaps and prime numbers that are arranged in an unique order] on above two algorithms as $\sum_{n=i}^{ALN} p_{n+1} = 2 + \sum_{i=1}^{n} g_i$ and $\sum_{n=i}^{ALN} p_{n+1} = 3 + \sum_{i=2}^{n} g_i$, we obtain (de novo) infinite series. These infinite series are all **diverging series** for this two algorithms [and their derived sub-algorithms]. In contrast, Brun's constants as outlined in section 4 are **converging series**. The cardinality of CIS-ALN-decelerating is applicable for (i) set of all prime numbers, (ii) set of all Odd Primes, (iii) subsets of Odd Primes, and (iv) set of all even Prime gaps $\Longrightarrow$ Modified Polignac's and Twin prime conjectures are true.

·**Complement-Sieve-of-Eratosthenes**: For i = 1, 2, 3, 4, 5... and with $c_1 = 4$; this algorithm as symbolically denoted by $c_{n+1} = 4 + \sum_{i=1}^{n} c_i$ with $g_n = c_{n+1} - c_n$ and its derived sub-algorithms will faithfully generate all composite numbers. Parallel arguments to construct de novo infinite series as **diverging series** for (sub)sets of composite numbers are also possible.

In general, the infinite-length sequence of a given converging series or diverging series is theoretically constituted by either positive terms e.g. $\zeta(s)$ as non-alternating harmonic series Eq. (1) OR alternating positive and negative terms e.g. $\eta(s)$ as alternating harmonic series Eq. (2).

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad (1)\\ &= \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots\\ &= \Pi_{p\ prime} \frac{1}{(1-p^{-s})}\\ &= \frac{1}{(1-2^{-s})} \cdot \frac{1}{(1-3^{-s})} \cdot \frac{1}{(1-5^{-s})} \cdot \frac{1}{(1-7^{-s})} \cdot \frac{1}{(1-11^{-s})} \cdots \frac{1}{(1-p^{-s})} \cdots\end{aligned}$$

Eq. (1) non-alternating harmonic series Riemann zeta function $\zeta(s)$ is a function of complex variable s $(= \sigma \pm \imath t)$ that continues sum of infinite series $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$ for $\mathrm{Re}(s) > 1$, and its analytic continuation elsewhere for $0 < \mathrm{Re}(s) < 1$. Containing no nontrivial zeros, $\zeta(s)$ is defined only in $1 < \sigma < \infty$ region where it is absolutely convergent. The common convention is to write s as $\sigma + \imath t$ with $\imath = \sqrt{-1}$, and with σ and t real. Valid for $\sigma > 1$, we write $\zeta(s)$ as $Re\{\zeta(s)\}+\imath Im\{\zeta(s)\}$ and note that $\zeta(\sigma + \imath t)$ when $0 < t < +\infty$ is the complex conjugate of $\zeta(\sigma - \imath t)$ when $-\infty < t < 0$. In Eq. (1), the equivalent **Euler product formula** with product over all prime numbers implies the presence of Sieve of Eratosthenes. Also note that for all $s \in \mathbb{C}, s \neq 1$, the integral relation $\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + 2\int_0^{\infty} \frac{\sin(s \arctan t)}{(1+t^2)^{s/2}\,(e^{2\pi t} - 1)}\, \mathrm{d}t$ holds true, which may be used for a numerical evaluation of the zeta function.

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \cdots \tag{2}$$

Eq. (2) alternating harmonic series Dirichlet eta function $\eta(s)$ that faithfully generates all three types of Gram points as three dependent CIS-IM-linear Incompletely Predictable entities when $\sigma = \frac{1}{2}$ must represent and act as *proxy* function for Eq. (1) in $0 < \sigma < 1$-critical strip [viz, for $0 < \mathrm{Re}(s) < 1$] containing $\sigma = \frac{1}{2}$-critical line because $\zeta(s)$ only converges when $\sigma > 1$. They are related to each other as $\zeta(s) = \gamma \cdot \eta(s)$ or equivalently as $\eta(s) = \frac{1}{\gamma} \cdot \zeta(s)$ with proportionality factor $\gamma = \frac{1}{(1 - 2^{1-s})}$.

$$\zeta(s) = 2^s \pi^{s-1} \, \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s) \tag{3}$$

$\zeta(s)$ satisfies Eq. (3) as the reflection functional equation whereby Γ is the gamma function. [NOTE: Derived for complex numbers with a positive real part, Γ is defined via a convergent improper integral $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}\, \mathrm{d}t, \Re(z) > 0$. Γ is then defined as analytic continuation of this integral function to a meromorphic function that is holomorphic in whole complex plane except zero and negative integers, where the function has simple poles. The main motivation for its development is $\Gamma(x+1)$ interpolates factorial function $x! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot x$ to non-integer values.] As an equality of meromorphic functions valid on whole complex plane, Eq. (3) relates values of $\zeta(s)$ at points s and $1 - s$; in particular, it relates even positive integers with odd negative integers. Owing to zeros of sine function, the functional equation implies $\zeta(s)$ has a simple zero at each even negative integer $s = -2n = -2, -4, -6, -8, -10...$ known as trivial zeros of $\zeta(s)$. When s is an even positive integer, product $\sin(\frac{\pi s}{2})\Gamma(1-s)$ on right is non-zero because $\Gamma(1-s)$ has a simple pole, which cancels simple zero of sine factor. With perfect line symmetry at the vertical line $s = \frac{1}{2}$, we have a **symmetric version** of this functional equation applied to the Lambda-function given by $\Lambda(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$, which satisfies $\Lambda(s) = \Lambda(1-s)$ OR to the Riemann xi-function given by $\xi(s) = \frac{1}{2} \pi^{-\frac{s}{2}} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s)$, which satisfies $\xi(s) = \xi(1-s)$. $\Lambda(s)$ OR $\xi(s)$ is thus the 'completed zeta function' whereby $\pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) = \Gamma_{\mathbb{R}}(s)$ is "Gamma factor" as the local L-factor corresponding to the Archimedean place, with the other factors in the Euler product expansion being the local L-factors of the non-Archimedean places. The conductor of L-function is the positive integer N from $N^{\frac{s}{2}}$. For Riemann zeta function, its conductor N as derived from $\pi^{-\frac{s}{2}} = (\frac{1}{\pi} \cdot 1)^{\frac{s}{2}}$ is 1.

Remark 2. *Consider sequence* $\lambda_n = \dfrac{1}{(n-1)!} \dfrac{d^n}{ds^n} \left[s^{n-1} \log \xi(s) \right] \Big|_{s=1}$. *As a possible pathway to solve Riemann hypothesis, Li's criterion states this hypothesis is equivalent to statement* $\lambda_n > 0$ *for every positive integer n [viz, positivity of* λ_n*].*

The numbers λ_n (as the Additive invariants denoted by $\mathbb{L}^*$ and sometimes defined with a slightly different normalization) are called Keiper-Li coefficients or Li coefficients. They may also be expressed in terms of nontrivial zeros of Riemann zeta function $\lambda_n = \sum_\rho \left[1 - \left(1 - \frac{1}{\rho} \right)^n \right]$ where the sum extends over ρ, the nontrivial zeros of the zeta function. This conditionally convergent sum should be understood in the sense usually used in Number theory; namely, that $\sum_\rho = \lim_{N\to\infty} \sum_{|\operatorname{Im}(\rho)| \le N}$. [Re($s$) and Im($s$) denote real and imaginary parts of s].

At $\sigma = \frac{1}{2}$, sim-$\eta(s)$ =

$$\sum_{n=1}^{\infty} (2n)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n) + \frac{1}{4}\pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{1}{2}} 2^{\frac{1}{2}} \cos(t \ln(2n-1) + \frac{1}{4}\pi) \tag{4}$$

At $\sigma = \frac{2}{5}$, sim-$\eta(s)$ =

$$\sum_{n=1}^{\infty} (2n)^{-\frac{2}{5}} 2^{\frac{1}{2}} \cos(t \ln(2n) + \frac{1}{4}\pi) - \sum_{n=1}^{\infty} (2n-1)^{-\frac{2}{5}} 2^{\frac{1}{2}} \cos(t \ln(2n-1) + \frac{1}{4}\pi) \tag{5}$$

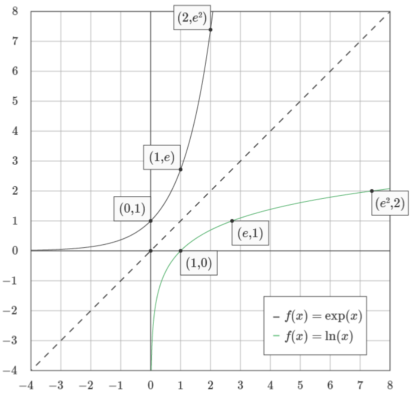
For any real number n, $e^{\imath n} = \cos n + \imath \cdot \sin n$ is Euler's formula where e [≊transcendental number 2.71828] is base of natural logarithm, $\imath = \sqrt{-1}$ is imaginary unit. When $n = \pi$ [≊transcendental number 3.14159], then $e^{\imath\pi} + 1 = 0$ or $e^{\imath\pi} = -1$, known as Euler's identity. Applying this formula to f(n) η(s) results in Eq. (4) f(n) simplified η(s) at $\sigma = \frac{1}{2}$ that incorporate all nontrivial zeros [as Zeroes]. There is total absence of (non-existent) virtual nontrivial zeros [as virtual Zeroes]. Eq. (5) f(n) simplified η(s) at $\sigma = \frac{2}{5}$ will incorporate all (non-existent) virtual nontrivial zeros [as virtual Zeroes]. There is total absence of nontrivial zeros [as Zeroes].

At $\sigma = \frac{1}{2}$, DSPL =

$$\frac{1}{2^{\frac{1}{2}}} \left(t^2 + \frac{1}{4} \right)^{\frac{1}{2}} \left[(2n)^{\frac{1}{2}} \cos(t \ln(2n) - \frac{1}{4}\pi) - (2n-1)^{\frac{1}{2}} \cos(t \ln(2n-1) - \frac{1}{4}\pi) + C \right]_1^{\infty} \tag{6}$$

$F(n)$ Dirichlet Sigma-Power Law, denoted by DSPL, refers to $\int$sim-η(s)dn. Eq. (6) is F(n) DSPL at $\sigma = \frac{1}{2}$ that will incorporate all nontrivial zeros [as Pseudo-zeroes to Zeroes conversion].

Remark 3. *Given* $\delta = \frac{1}{10}$*, the left-shifted* $\sigma = \frac{1}{2} - \delta = \frac{2}{5}$*-non-critical line (Figure 5) and right-shifted* $\sigma = \frac{1}{2} + \delta = \frac{3}{5}$*-non-critical line (Figure 6) are* ***equidistant*** *from nil-shifted* $\sigma = \frac{1}{2}$*-critical line (Figure 4). Let* $x = (2n)$ *or* $\dfrac{1}{(2n)}$ *or* $(2n-1)$ *or* $\dfrac{1}{(2n-1)}$*. With multiplicative*



8: The natural logarithm function $log_e x$ or $ln(x)$ and natural exponential function exp(x) or e^x. The graphs of $log_e x$ and its inverse e^x are symmetric with respect to line y = x thus geometrically denoting diagonal symmetry of these two functions; viz, $\ln(e^x) = x$ and $e^{(\ln x)} = x$.

*inverse operation of $x^{\delta} \cdot x^{-\delta} = 1$ or $\frac{1}{x^{\delta}} \cdot \frac{1}{x^{-\delta}} = 1$ that is applicable, this imply intrinsic presence of **Multiplicative Inverse** in sim-$\eta(s)$ or DSPL for all σ values with this function or law rigidly obeying relevant trigonometric identity. Then both $f(n)$ sim-$\eta(s)$ and $F(n)$ DSPL will manifest **Principle of Equidistant for Multiplicative Inverse** (as per Page 41 of [6]).*

The dissertation based on Figure 8 with inverse functions $ln(x)$ & $e(x)$ in Page 30 – 35 of [6] confirms the Asymptotic law of distribution for prime numbers as $\lim_{x\to\infty} \frac{\text{Prime-}\pi(x)}{\left[\frac{x}{\ln(x)}\right]} = 1$ and the Asymptotic law of distribution for composite numbers as $\lim_{x\to\infty} \frac{\text{Composite-}\pi(x)}{\left[\frac{x}{e(x)}\right]} = 1$. This fully supports Prime number theorem [viz, Prime-$\pi(x) \approx \frac{x}{ln(x)}$] & derived Composite number theorem [viz, Composite-$\pi(x) \approx \frac{x}{e(x)}$].

A number base, consisting of any whole number greater than 0, is number of digits or combination of digits that a number system uses to represent numbers e.g. decimal number system or base 10, binary number system or base 2, octal number system or base 8, hexa-decimal number system or base 16. Prime counting function, Prime-$\pi(x)$ = number of primes $\leq x$ and Composite counting function, Composite-$\pi(x)$ = number of composites $\leq x$. As $x \to \infty$, derived properties of Prime-$\pi(x)$ occur in, for instance, Prime number theorem for Arithmetic Progressions, Prime-$\pi(x; b, a)$ [= number of primes $\leq x$ with last digit of primes given by a in base b]. For any choice of digit a in base b with gcd(a,b) = 1: Prime-$\pi(x; b, a) \sim \frac{\text{Prime-}\pi(x)}{\phi(b)}$. Here, Euler's totient function $\phi(n)$ is defined as the number of positive integers $\leq n$ that are relatively prime to (i.e., do not contain any factor in common with) n, where 1 is counted as being relatively prime to all numbers. Then each of the last digit of primes given by digit a in base b as $x \to \infty$ is equally distributed between the permitted choices for digit a with this result being valid for, and is independent of, any chosen base b.

Numbers with their last digit ending in (i) 1, 3, 7 or 9 [which can be either primes or com-

posites] constitute ∼40% of all integers; and (ii) 0, 2, 4, 5, 6 or 8 [which must be composites] constitute ∼60% of all integers. We validly ignore the only single-digit even prime number 2 and odd prime number 5. We note ≥ 2-digit Odd Primes can only have their last digit ending in 1, 3, 7 or 9 but not in 0, 2, 4, 5, 6 or 8. These are given as the **complete List**:
The last digit of Odd Primes having their Prime gaps with last digit ending in 2 [viz, Gap 2, Gap 12, Gap 22, Gap 32...] can only be 1, 3 or 9 [but not (5) or 7] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 4 [viz, Gap 4, Gap 14, Gap 24, Gap 34...] can only be 1, 3 or 7 [but not (5) or 9] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 6 [viz, Gap 6, Gap 16, Gap 26, Gap 36...] can only be 3, 7 or 9 [but not (5) or 1] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 8 [viz, Gap 8, Gap 18, Gap 28, Gap 38...] can only be 1, 7 or 9 [but not (5) or 3] as three choices.
The last digit of Odd Primes having their Prime gaps with last digit ending in 0 [viz, Gap 10, Gap 20, Gap 30, Gap 40...] can only be 1, 3, 7 or 9 [but not (5)] as four choices.

Axiom 1. ***Applications of the Prime number theorem for Arithmetic Progressions will confirm Modified Polignac's and Twin prime conjectures to be true (as per Page 31 – 32 in [6]).***

Proof. We use decimal number system (base $b = 10$), and ignore the only single-digit even prime number 2 and odd prime number 5. For $i = 1, 2, 3, 4, 5...$; the last digit of all Gap $2i$-Odd Primes can only end in 1, 3, 7 or 9 that are each proportionally and equally distributed as ∼25% when $x \to \infty$, whereby this result is consistent with Prime number theorem for Arithmetic Progressions. The 100%-Set of, and its derived four unique 25%-Subsets of, Gap $2i$-Odd Primes based on their last digit being 1, 3, 7 or 9 must all be CIS-ALN-decelerating. *"Different Prime numbers literally equates to different Prime gaps"* is a well-known intrinsic property. Since the ALN of Gap $2i$ as fully represented by all Prime gaps with last digit ending in 0, 2, 4, 6 or 8 are associated with various permitted combinations of last digit in Gap $2i$-Odd Primes being 1, 3, 7 and/or 9 as three or four choices [outlined above in **List** from preceding paragraph]; then these ALN unique subsets of Prime gaps based on their last digit being 0, 2, 4, 6 or 8 together with their correspondingly derived ALN unique subsets constituted by Gap $2i$-Odd Primes having last digit 1, 3, 7 or 9 must also all be CIS-ALN-decelerating. The Probability (any Gap $2i$ abruptly terminating as $x \to \infty$) = Probability (any Gap $2i$-Odd Primes abruptly terminating as $x \to \infty$) = 0. Thus Modified Polignac's and Twin prime conjectures is confirmed to be true. With ordinary Riemann hypothesis being a special case, we additionally note the generalized Riemann hypothesis formulated for Dirichlet L-function holds once $x > b^2$, or base $b < x^{\frac{1}{2}}$ as $x \to \infty$.
The ["statistical" or "probabilistic"] proof is now complete for Axiom 1□.

3. Generic Squeeze theorem as a novel mathematical tool

We adopt abbreviations $\mathbb{P}$ = Prime numbers, $\mathbb{C}$ = Composite numbers, NTZ = nontrivial zeros, G[y=0]P = Gram[y=0] points (usual / traditional Gram points), and G[x=0]P = Gram[x=0] points. Gram's Law and Rosser's Rule for Riemann zeta function via its proxy Dirichlet eta function at $\sigma = \frac{1}{2}$ are perpetually associated with recurring violations (failures). Violations of Gram's Law equates to intermittently observing various geometric variants of two consecutive (positive first and then negative) G[y=0]P that is alternatingly followed by two consecutive NTZ. Violations of Rosser's Rule equates to intermittently observing various geometric variants of reduction in expected number of certain x-axis intercept points. Both types of violations give rise to intermittent or cyclical events of two missing G[y=0]P or, equivalently, two extra NTZ.

We hereby artificially and conveniently regard the G[y=0]P ≤ G[x=0]P ≤ NTZ inequality as being applicable for Theorem 2 below. Observe that this particular inequality has never been definitively confirmed to be true *over the large range of numbers*. With full analysis, one of the following alternative inequalities G[x=0]P ≤ G[y=0]P ≤ NTZ or NTZ ≤ G[y=0]P ≤ G[x=0]P or NTZ ≤ G[x=0]P ≤ G[y=0]P or G[x=0]P ≤ NTZ ≤ G[y=0]P or G[y=0]P ≤ NTZ ≤ G[x=0]P *over the large range of numbers* could instead be true. Even the equality G[y=0]P = G[x=0]P = NTZ *over the large range of numbers* could instead also be true. It may even be the case that all types of inequalities mentioned above could cyclically co-exist *over the large range of numbers*. In principle, Theorem 2 should intuitively be validly applicable to the correctly chosen inequality [or equality].

Theorem 2. *(Generic Squeeze theorem). Crucially applicable to all prime numbers, composite numbers and nontrivial zeros, our devised Theorem 2 is formally stated as follows (as per Page 51 – 53 in [6]).*

Let I be an interval containing point a. Let g, f, and h be algorithms or functions defined on I, except possibly at a itself. Suppose for every x in I not equal to a, we have $g(x) \leq f(x) \leq h(x)$ and also suppose $\lim\limits_{x\to a} g(x) = \lim\limits_{x\to a} h(x) = L$. Then $\lim\limits_{x\to a} f(x) = L$. The algorithms or functions g and h are said to be lower and upper bounds (respectively) of f. Here, a is not required to lie in the interior of I. Indeed, if a is an endpoint of I, then the above limits are left- or right-hand limits. A similar statement holds for infinite intervals e.g. applicable to the IM t-valued NTZ (as CIS-IM-linear) obtained from Riemann zeta function via its proxy Dirichlet eta function, and the ALN of $\mathbb{P}$ (as CIS-ALN-decelerating) obtained from Sieve-of-Eratosthenes and IM $\mathbb{C}$ (as CIS-IM-accelerating) obtained from Complement-Sieve-of-Eratosthenes. In particular, if $I = (0, \infty)$ or (0, ALN), then the conclusion holds, taking the limits as $x \to \infty$ or ALN.

Let a_n, c_n be two sequences converging to ℓ, and b_n a sequence. If $\forall n \geq N$, $N \in \mathbb{N}$ we have $a_n \leq b_n \leq c_n$, then b_n also converges to ℓ. From previous arguments, we logically notice Generic Squeeze theorem is valid for carefully selected sequences e.g. those precisely derived from algorithm Sieve-of-Eratosthenes generating set of all unique $\mathbb{P}$ 2, 3, 5, 7, 11, 13, 17, 19, 23, 29... with progressive "cummulative" cardinality $\equiv c_n$ and sub-algorithms from Complement-Sieve-of-Eratosthenes generating two subsets of all unique pre-prime-Gap 2-Even $\mathbb{C}$ 4, 6, 10, 12, 16, 18, 22, 28... with progressive "cummulative" cardinality $\equiv b_n$ and of all unique 1^{st} post-prime-Gap 1-Even $\mathbb{C}$ 8, 14, 20, 24, 32, 38, 44... with progressive "cummulative" cardinality $\equiv a_n$. We recognize even $\mathbb{P}$ 2 is not a pre-prime-Gap 2-Even $\mathbb{C}$, and 1^{st} $\mathbb{P}$ 3, 5, 11, 17, 29, 41, 59... from all twin prime pairings (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43), (59, 61)... are never associated with 1^{st} post-prime-Gap 1-Even $\mathbb{C}$ as these even numbers 4, 6, 12, 18, 30, 42, 60... [which must be ***eternally ubiquitous***, not least, to comply with Law of Continuity] are all pre-prime-Gap 2-Even $\mathbb{C}$. Incorporating mixtures of $\mathbb{P}$ & $\mathbb{C}$, our findings on twin prime pairings $\Longrightarrow$ {c_n representing progressive total of all $\mathbb{P}$} > {b_n representing progressive total of all pre-prime-Gap 2-Even $\mathbb{C}$} > {a_n representing progressive total of all 1^{st} post-prime-Gap 1-Even $\mathbb{C}$}. Since $\lim\limits_{n\to ALN} a_n = \lim\limits_{n\to ALN} c_n$ = CIS-ALN-decelerating, then $\lim\limits_{n\to ALN} b_n$ = CIS-ALN-decelerating. Stated in another insightful way: The perpetual recurrence of intermittent inevitable DISAPPEARANCE of 1^{st} post-prime-Gap 1-Even $\mathbb{C}$ is solely due to coinciding intermittent inevitable APPEARANCE of twin primes $\Longrightarrow$ Twin prime conjecture is true.

*The 1^{st} post-prime-Gap 1-Even $\mathbb{C}$ precisely forms OEIS sequence A014574 *Average of twin prime pairs* 4, 6, 12, 18, 30, 42, 60, 72, 102, 108, 138, 150, 180, 192, 198, 228, 240, 270, 282, 312, 348, 420, 432, 462, 522, 570, 600, 618... by R. K. Guy, N. J. A. Sloane & E. W. Weisstein

(June 11, 2011) https://oeis.org/A014574 whereby
(i) With an initial 1 added, these numbers form part of the complement of closure of {2} under the operations $a * b + 1$ and $a * b - 1$ within the set of all non-zero positive even numbers $U = \{2, 4, 6, 8, 10...\}$. For $a * b + 1$: $2 * 2 + 1 = 5$. For $a * b - 1$: $2 * 2 - 1 = 3$. Under both operations, we obtain the set $S = \{2, 3, 5\}$. Therefore the complement of S within U would be all even numbers except 2 [and 5 & 3]; viz, $S' = \{4, 6, 8, 10, 12, 14, 16...\}$.
(ii) These numbers are also the square root of the product of twin prime pairs + 1. Two consecutive odd numbers can be written as $2k+1$, $2k+3$. Then $(2k+1)(2k+3)+1 = 4(k^2+2k+1) = 4(k+1)^2$, a perfect square [where the countably infinite set of all perfect squares ≡ product of an integer multiplied by itself = 1, 4, 9, 16, 25, 36, 49, 64, 81, 100...]. Since twin prime pairs are two consecutive odd numbers, the statement is true for all CIS-ALN-decelerating twin prime pairs.
(iii) These numbers are single (or isolated) composites. Nonprimes k such that neither $k - 1$ nor $k + 1$ is nonprime.
(iv) These form the numbers n such that $\sigma(n - 1) = \phi(n + 1)$. This equation involves two arithmetic functions: the sum of divisors function σ [which calculates the sum of all positive divisors of n e.g. when $n = 30$: Prime factorization of $(n - 1) = 29$ is $29 = 29^1$, and $\sigma(29) = 1 + 29 = 30$] and Euler's totient function ϕ [which gives the count of positive integers less than n that are coprime to n e.g. Prime factorization of $(n + 1) = 31$ is $31 = 31^1$, and $\phi(31) = 31 - 1 = 30$].
(v) Aside from the first term 4 in the sequence, all remaining terms 6, 12, 18, 30, 42, 60, 72, 102, 108, 138, 150... have digital root 3, 6, or 9 e.g. the digital root of 138 is 3 since 138 = 1 + 3 + 8 = 12 and 1 + 2 = 3.
(vi) These form the numbers n such that $n^2 - 1$ is a semiprime [a natural number that is the product of two prime numbers].
(vii) Every term but the first term 4 is a multiple of 6 [and all the multiple of 6 clearly constitute a countably infinite set].

From above synopsis that is valid for [mixed] prime & composite numbers as $x \to$ALN, we conclude: Since there is an ALN of all prime numbers as (c_n) and also an ALN of all 1^{st} post-prime-Gap 1-Even composite numbers as (a_n), then by the Generic Squeeze theorem, there must also be an ALN of all Gap 2-Even composite numbers as (b_n). Thus ℓ must have the value of ALN. In theory, even if there are [incorrectly] only finitely many twin primes, the mathematical relationship of $a_n \leq b_n \leq c_n$ will still hold except that the Generic Squeeze theorem is no longer applicable as there will be inevitable "errors" present in computed a_n, b_n and c_n.

By applying Generic Squeeze theorem [only] to Odd $\mathbb{P}$, we now prove Polignac's and Twin prime conjectures are true: We ignore even $\mathbb{P}$ 2. Let algorithm Sieve-of-Eratosthenes that generate the set of all unique Total Odd $\mathbb{P}$ 3, 5, 7, 11, 13, 17, 19, 23, 29... with progressive "cummulative" cardinality ≡ c_n and sub-algorithms from Sieve-of-Eratosthenes that generate the two [randomly selected] subsets of all unique Gap 4-Odd $\mathbb{P}$ 7, 13, 19, 37, 43, 67... with progressive "cummulative" cardinality ≡ a_n and of all unique Gap 2, 6, 8, 10, 12...-Odd $\mathbb{P}$ 3, 5, 11, 17, 23, 23, 29, 31, 41, 47, 53, 59, 61... [viz, not including Gap 4-Odd $\mathbb{P}$] with progressive "cummulative" cardinality ≡ b_n. Instead of choosing b_n to be even Prime gap 4, one could choose any other eligible even Prime gap derived from the set of all even Prime gaps [which will inevitably also include Zhang's landmark result of an unknown even Prime gap $N < 70$ million]. Since $\lim_{n \to ALN} a_n = \lim_{n \to ALN} c_n$ = CIS-ALN-decelerating, then $\lim_{n \to ALN} b_n$ = CIS-ALN-decelerating. Stated in another insightful way: In order for our novel method Generic Squeeze theorem to be ubiquitously applicable for Odd $\mathbb{P}$, all even Prime gaps 2, 4, 6, 8, 10... must be associated with their corresponding ALN of Odd $\mathbb{P}$.

On 17 April 2013, Yitang Zhang announced an incredible proof that there are infinitely many pairs of prime numbers that differ by less than 70 million[7]; viz, there is an arbitrarily large number of Odd Primes with an unknown even Prime gap N of less than 70 million. By optimizing Zhang's bound, subsequent Polymath Project collaborative efforts using a new refinement of GPY sieve in 2014 lowered N to 246; and assuming Elliott-Halberstam conjecture and its generalized form further lower N to 12 and 6, respectively. Intuitively, N has more than one valid values such that the same condition holds for each N value. Using different methods, we can at most lower N to 2 and 4 in regards to Odd Primes having small even Prime gaps 2 & 4 with each uniquely generating CIS-ALN-decelerating Odd Primes. We anticipate there are all remaining even Prime gaps w.r.t. Odd Primes with large even Prime gaps ≥ 6 as denoted by corresponding $N \geq 6$ values whereby each large even Prime gap will generate its own unique CIS-ALN-decelerating Odd Primes.

We justify **"Zhang's optimized result ≥ 3 up to ALN even Prime gaps with each having ALN of elements"**: Always as finite [but NOT infinite] length, we observe as side note that two or more consecutive Odd Primes are validly and rarely constituted by [same] even Prime gap of 6 or multiples of 6. With just one or two existing even Prime gaps that have ALN of elements being simply "insufficient" in the large range of prime numbers, then the landmark result by Zhang on this unknown even Prime gap N of less than 70 million is usefully extrapolated as "There must be at least one subset of Odd Primes having ALN of elements". Hence there are aesthetically at least two, if not three, existing even Prime gaps that generate their corresponding CIS-ALN-decelerating Odd Primes. Modified Polignac's and Twin prime conjectures equates to all even Prime gaps 2, 4, 6, 8, 10... generating corresponding CIS-ALN-decelerating Odd Primes.

Near-identical arguments can be made for three types of Gram points located at $\sigma = \frac{1}{2}$-critical line of Riemann zeta function but we leave out the full exercise of applying Generic Squeeze theorem to NTZ as progressive "cummulative" cardinality $\equiv c_n$, G[x=0]P as progressive "cummulative" cardinality $\equiv b_n$ and G[y=0]P as progressive "cummulative" cardinality $\equiv a_n$. We immediately recognize the [trivial] conclusion: Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n$ = CIS-IM-linear, then $\lim_{n\to\infty} b_n$ = CIS-IM-linear.

Eq. (4) manifests *exact* Dimensional analysis homogeneity when $\sigma = \frac{1}{2}$ whereby Σ(all fractional exponents) = $2(-\sigma)$ = exact negative whole number -1 [c.f. Eq. (5) manifests *inexact* Dimensional analysis homogeneity when $\sigma = \frac{2}{5}$ whereby Σ(all fractional exponents) = $2(-\sigma)$ = inexact negative fractional number $-\frac{4}{5}$]. Only Dirichlet eta function having parameter $\sigma = \frac{1}{2}$ will mathematically depict [optimal] "formula symmetry" on Σ(all fractional exponents) as an exact negative whole number, whereby absolute values of all fractional exponents = $\frac{1}{2}$ when associated with constant 2 and variable $(2n)$ or $(2n-1)$. This formula symmetry is not equivalent to geometrical symmetry about X-axis, Y-axis, Diagonal, or Origin point that do not exist for any Dirichlet eta function when considered for either $-\infty < t < 0$ or $0 < t < +\infty$ from full range $-\infty < t < +\infty$; whereby we conventionally adopt the positive range. Simple observation of [optimal] "formula symmetry" implies only $\sigma = \frac{1}{2}$-Dirichlet eta function will perpetually & geometrically intercept $\sigma = \frac{1}{2}$-Origin point as Origin intercept points or Gram[x=0,y=0] points (i.e. will perpetually & mathematically lie on $\sigma = \frac{1}{2}$-critical line as nontrivial zeros) an infinite number of times.

Completely conforming to Langlands program "**Theory of Symmetry**", all IL (sub-)algorithms or IL (sub-)equations and all FL (sub-)algorithms or FL (sub-)equations will respectively generate infinitely-many and finitely-many entities. All FL (sub-)algorithms or FL (sub-)equations are CP but IL (sub-)algorithms or IL (sub-)equations can be either CP or IP. Here, we validly regard equation Dirichlet eta function (*proxy* for Riemann zeta function that generate nontrivial zeros when $\sigma = \frac{1}{2}$), and algorithms Sieve-of-Eratosthenes [for prime numbers] and Complement-Sieve-of-Eratosthenes [for composite numbers] as **non-overlapping** "IP IL number generators".

Remark 4. *Not least to maintain Dimensional analysis homogeneity and to conserve Total number of elements (cardinality), it is a crucial* ***sine qua non*** *Pre-requisite Mathematical Condition that a parent IP IL algorithm which is precisely constituted by its IP IL sub-algorithms or a parent IP IL equation which is precisely constituted by its IP IL sub-equations must generally all be wholly IP IL [and not be mixed IP IL and CP FL]. Useful self-explanatory analogy using CP IL (sub)algorithms or (sub)equations: Set "twin" even numbers 0, 2, 4, 6, 8, 10... with Even gap 2, Subset "cousin" even numbers 0, 4, 8, 12, 16, 20... with Even gap 4, Subset "sexy" even numbers 0, 6, 12, 18, 24, 30... with Even gap 6, etc must all be constituted by CP IL [and not mixed CP IL and IP IL] even numbers that are derived from, paradoxically,* ***overlapping*** *"CP IL number generators".*

Remark 5. *It was correctly asserted on Page 3 – 4 of [6] that any created Prime-tuplet or Prime-tuple is not able to be used to either prove or disprove Modified Polignac's and Twin prime conjectures. The main reason is Prime-tuplets or Prime-tuples are simply* ***"overlapping and incomplete" (Sub)Tuples Classification of consecutive primes****. In contrast, we can use* ***"non-overlapping and complete" (Sub)Sets Classification of grouped primes*** *to prove Modified Polignac's and Twin prime conjectures. Thus even Prime gap 2 = Prime 2-tuplets of diameter 2 and even Prime gaps 4, 6, 8, 10, 12... = Prime 2-tuples of diameter 4, 6, 8, 10, 12....*

4. Theorem of Divergent-to-Convergent series conversion for Prime numbers as a novel mathematical tool

Recall from section 2 the algorithms Sieve-of-Eratosthenes (S-of-E) and Modified-S-of-E. Both algorithms and their derived sub-algorithms faithfully generate set of all prime numbers 2, 3, 5, 7, 11, 13...; set of all Odd Primes 3, 5, 7, 11, 13, 17...; and subsets of Odd Primes derived from even Prime gaps 2, 4, 6, 8, 10.... By performing summation given by $\sum_{n=i}^{ALN} p_{n+1} = 2 + \sum_{i=1}^{n} g_i$ and $\sum_{n=i}^{ALN} p_{n+1} = 3 + \sum_{i=2}^{n} g_i$, we obtain (de novo) infinite series as diverging series for these two algorithms [and their derived sub-algorithms]. For Polignac's and Twin prime conjectures to be true, we deduce the cardinality for (i) set of all prime numbers, (ii) set of all Odd Primes, (iii) subsets of Odd Primes, and (iv) set of all even Prime gaps must all be CIS-ALN-decelerating. In contrast, we deduce below after Theorem 3 that all Brun's constants as (derived) infinite series are, in fact, converging series.

Helpful preliminary information about Theorem 3: Four basic arithmetic operations of addition [and complementary substraction] and multiplication [and complementary division] obey Axioms of Addition and Multiplication, and Axioms of Order. Division of one number by another is equivalent to multiplying first number by reciprocal (or multiplicative inverse) of second number, whereby division by 0 is always undefined. Subtraction of one number from another is equivalent to adding additive inverse of second number (viz, a negative number) to first number (viz, a positive number). Completely Predictable properties arising from (non-)alternating addition of any Even numbers ($\mathbb{E}$) 0, 2, 4, 6, 8, 10, 12... and any Odd numbers ($\mathbb{O}$)

1, 3, 5, 7, 9, 11, 13...:
(1) $\mathbb{E} + \mathbb{E} + \mathbb{E} + \mathbb{E}$... when involving any number of terms = $\mathbb{E}$.
(2) $\mathbb{O} + \mathbb{O} + \mathbb{O} + \mathbb{O}$... when involving an even number of terms = $\mathbb{E}$; and when involving an odd number of terms = $\mathbb{O}$.
The alternating sum $\mathbb{E} + \mathbb{O} + \mathbb{E} + \mathbb{O} + \mathbb{E} + \mathbb{O}$... when involving $(1 + n)$ terms for $n = 1, 2, 3, 4, 5...$ = repeating patterns of $\mathbb{O}$, $\mathbb{O}$, $\mathbb{E}$, $\mathbb{E}$, $\mathbb{O}$, $\mathbb{O}$,....

A convergent series (CS) as an infinite series having its partial sums of sequence that tends to a finite limit is validly represented by the [defined] value of this finite limit. A divergent series (DS) as an infinite series having its partial sums of sequence that tends to a infinite limit is validly represented by the [undefined] value of this infinite limit. As previously discussed in section 2, the infinite-length sequence of a given CS or DS can theoretically be constituted by either positive terms OR alternating positive and negative terms. Below are Completely Predictable properties arising from addition of any infinite series constituted by ≥ 1 CS and/or ≥ 1 DS:
I. DS + DS + DS +... when involving any number of DS terms = DS.
II. CS + CS +... + DS + DS +... when involving any number of CS terms and any number of DS terms = DS.
III. CS + CS + CS +... when involving a finite number of CS terms = CS.
IV. CS + CS + CS +... when involving an infinite number of CS terms or arbitrarily large number (ALN) of CS terms = DS.

Theorem 3. *(Theorem of Divergent-to-Convergent series conversion for Prime numbers) (as per Page 53 – 54 in [6]).*

We validly ignore even prime number 2. Theorem 3, *aka* **Smoothed asymptotics for Prime numbers with an enhanced regulator**, as given in next two paragraphs is further expanded below using three Brun's constants computed for twin primes, cousin primes and sexy primes.

For [eligible] homogenous entities of prime numbers with application of divergent series (DS) to convergent series (CS) conversion relationship, we obtain CS + CS + CS +... when involving arbitrarily large number (ALN) of CS terms [that faithfully "represent" all Subsets of Odd Primes] = DS [that faithfully "represent" the Set of all Odd Primes]. We recognize the ALN of computed CS terms will precisely correspond to Brun's constants. The correctly chosen **enhanced regulator for prime numbers** ≡ *sine qua non* condition [that must be fully complied with by all Odd Primes]: *Derived from the set of all Odd Primes, there must be an ALN of subsets of Odd Primes derived from even Prime gaps 2, 4, 6, 8, 10... with each subset of Odd Primes containing an ALN of unique elements.*

The elimination of a DS to CS under our novel *Divergent-to-Convergent series theorem for Prime numbers* fully supports Polignac's and Twin prime conjectures to be true. As alluded to in section 1, this procedure is reminiscent of invoking 'Method of Smooth asymptotics' and 'regularization of divergent series or integrals' to enable elimination of divergences in analytic number theory and preservation of gauge invariance at one loop in a wide class of non-abelian gauge theories coupled to Dirac fermions that preserves Ward identity for vacuum polarisation tensor [viz, a regularized quantum field theory]. This is achieved by Padilla and Smith via adopting suitable choices from their proposed families of enhanced regulators[3] with analytic continuation that converge to Riemann zeta function value $\zeta(-1) = -\frac{1}{12}$ of extra relevance to quantum gravity, string theory, etc.

Considering Euler products $\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots\right) = \prod_{p} \frac{1}{1 - p^{-1}}$ when taken over the set of all infinitely many primes, Leonhard Euler in 1737 showed the [harmonic] **infinite series** of all infinitely many primes (as sum of the reciprocals of all infinitely many primes) **diverges** very slowly; viz, $\sum_{p \text{ prime}} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{17} + \frac{1}{19} + \cdots = \infty$. If it were the case that this sum of the reciprocals of twin primes (Prime gap 2), cousin primes (Prime gap 4), sexy primes (Prime gap 6), etc all diverged; then that fact would imply that there are infinitely many of twin primes, cousin primes, sexy primes, etc. However twin primes are less frequent than all infinitely many prime numbers by nearly a logarithmic factor with this bound giving the intuition that the sum of the reciprocals of twin primes **converges** very slowly, or stated in other words, twin primes form a small set. The sum $\sum_{p\,:\,p+2\in\mathbb{P}} \left(\frac{1}{p} + \frac{1}{p+2}\right) = \left(\frac{1}{3} + \frac{1}{5}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{11} + \frac{1}{13}\right) + \left(\frac{1}{17} + \frac{1}{19}\right) + \cdots = 1.902160583104...$ in explicit terms either has finitely many terms or has infinitely many terms but is very slowly convergent with its value known as Brun's constant for (consecutive) twin primes. Similar deductive arguments can be developed for the sum of the reciprocals of cousin primes, sexy primes, etc that also **converges** very slowly with their associated Brun's constant for (consecutive) cousin primes [$\approx$ 1.19705479], (consecutive) sexy primes [$\approx$ 1.13583508], etc. All these heuristically computed Brun's constants are irrational (transcendental) numbers ONLY IF there are infinitely many twin primes, cousin primes, sexy primes, etc. Based on Zhang's result[7], there must be at least one computed Brun's constant that is irrational (transcendental) associated with infinitely many Odd Primes having an even Prime gap < 70 million. Ignore solitary even prime number 2. Use "Arbitrarily Large Number" to denote "infinitely many". As an absolutely indispensable condition, there are ALN of subsets of Odd Primes with each subset of Odd Primes containing ALN of elements – this is akin to choosing the correct "enhanced regulator". From above discussions, we heuristically deduce very slowly **diverging** sum (series) of the reciprocals of all ALN Odd Primes are fully constituted by very slowly **converging** sum (series) of the reciprocals of ALN Odd Primes derived from each and every subsets of Odd Primes.

Erdos primitive set conjecture, now proven as a theorem by Prof. Jared Lichtman[1], is the assertion that for any primitive set S with exactly k prime factors (with repetition), $\sum_{n \in S} \frac{1}{\text{n log n}} \le \sum_{p} \frac{1}{\text{p log p}} = \frac{1}{\text{2 log 2}} + \frac{1}{\text{3 log 3}} + \frac{1}{\text{5 log 5}} + \frac{1}{\text{7 log 7}} + \frac{1}{\text{11 log 11}} + ... = 1.6366...$ [as a very slowly converging sum when $k = 1$ over infinitely-many integers 1, 2, 3, 4, 5...] $\implies f_k$ is maximized by the prime sum $f_1 = \sum_{p} \frac{1}{\text{p log p}} = 1.6366...$ [representing the unique "largest" primitive set that ONLY contains all infinitely-many prime numbers 2, 3, 5, 7, 11, 13...]. As supporting Modified Polignac's and Twin prime conjectures to be true [with **all Odd Primes belonging to CIS-ALN-decelerating**]; one can calculate the equivalent $f_1 = \sum_{p} \frac{1}{\text{p log p}}$ [also as very slowly converging sums with values < 1.6366...] for individual subsets of Odd Primes obtained from even Prime gaps 2, 4, 6, 8, 10... and notice these [derived] **"infinite series"** calculations must all, in principle and in synchrony, incorporate corresponding **CIS-ALN-decelerating Odd Primes from each subset**. This last statement is heavily supported by Yitang Zhang's result[7] which is extrapolated as "There must be at least one subset of Odd Primes [obtained from an even Prime gap < 70 million] having infinitely many elements".

5. Three subtypes of Countably Infinite Sets with Incompletely Predictable entities from Riemann zeta function and Sieve of Eratosthenes

The sets of numbers generated using power (exponent) such as 2 or $\frac{1}{2}$, even numbers, odd numbers, etc are morphologically constituted by *Completely Predictable (CP) numbers* in the sense that these sets of numbers are actually ***not*** random and do not behave like one. The sets of nontrivial zeros, primes, composites, etc are morphologically constituted by *Incompletely Predictable (IP) numbers* [or *pseudo-random numbers*] in the sense that these sets of numbers are actually ***not*** random but behave like one; thus giving rise to "Mathematics for Incompletely Predictable Problems". The word *number* [singular noun] or *numbers* [plural noun] in reference to CP even and odd numbers, IP prime and composite numbers, IP Gram points and virtual Gram points can be interchanged with the word *entity* [singular noun] or *entities* [plural noun].

Lemma 4. *We can formally define the elements from (sub)sets and (sub)tuples as Completely Predictable or Incompletely Predictable entities (as per Page 18 in [6]). Please also see Remark 4 & Remark 5 above in section 3 indicating the important significances arising from Lemma 4.*

Proof. A set is a collection of zero (viz, the empty set) or more elements (viz, a finite set with a finite number of elements or an infinite set with an infinite number of elements). A singleton refers to a finite set with a single element. A set can be any kind of mathematical objects: numbers, symbols, points in space, lines, other geometrical shapes, variables, or even other sets whereby these [mutable] non-repeating elements are not arranged in an unique order. A subset can be a [smaller] finite set derived from its [larger] parent finite set or its [larger] parent infinite set. A subset can also be a [smaller] infinite set derived from its [larger] parent infinite set. A tuple, which can potentially be subdivided into subtuples, is a finite ordered list (sequence) of elements whereby these [immutable] non-repeating elements are arranged in an unique order. Thus a tuple or a subtuple is regarded as a special type of finite set with the extra imposed restriction. Shown using worked examples:
CP simple equation or algorithm generates CP numbers e.g. even numbers 0, 2, 4, 6, 8, 10... or odd numbers 1, 3, 5, 7, 9, 11.... Thus a generated CP number is **locationally defined** as a number whose ith position is *independently* determined by simple calculations without needing to know related positions of all preceding numbers – this is a **Universal Property**.
IP complex equation or algorithm generates IP numbers e.g. prime numbers 2, 3, 5, 7, 11, 13... or composite numbers 4, 6, 8, 9, 10, 12.... Thus a generated IP number is **locationally defined** as a number whose ith position is *dependently* determined by complex calculations with needing to know related positions of all preceding numbers – this is a **Universal Property**.
We clearly note the elements in (sub)sets and (sub)tuples when generated by equations or algorithms will precisely constitute relevant entities or numbers of interest.
The proof is now complete for Lemma 4□.

A formula for primes in Number theory is a formula generating all prime numbers 2, 3, 5, 7, 11, 13, 17, 19, 23... exactly and without exception. Computationally slow and inefficient formulas for calculating primes exist e.g. 1964 Willans formula $p_n = 1+\sum_{i=1}^{2^n}\left\lfloor\left(\frac{n}{\sum_{j=1}^{i}\left\lfloor\left(\cos\frac{(j-1)!+1}{j}\pi\right)^2\right\rfloor}\right)^{1/n}\right\rfloor$

which is based on Wilson's theorem $n + 1$ is prime *iff* $n! \equiv n \pmod{n+1}$. Then critics may ask the question "For $n = 1, 2, 3, 4, 5,...$; does Willans formula that faithfully compute corresponding n^{th} prime number p_n for all infinitely-many primes contradict Sieve-of-Eratosthenes algorithm as being an Infinite Length (IL) and Incompletely Predictable (IP) algorithm?" The answer is categorically 'no' based on carefully analyzing this formula using following arguments [which lend further support to Polignac's and Twin prime conjectures being true]: Willans formula has two sub-components $\left\lfloor \left(\cos \frac{(j-1)!+1}{j}\pi\right)^2 \right\rfloor = \begin{matrix} 1 \text{ if } j \text{ is prime or } 1 \\ 0 \text{ if } j \text{ is composite} \end{matrix}$ & $\sum_{j=1}^{i} \left\lfloor \left(\cos \frac{(j-1)!+1}{j}\pi\right)^2 \right\rfloor$ $= (\#\text{ primes} \le i) + 1$. We recognize this second sub-component stipulating $(\#\text{ primes} \le i) + 1$ meant the actual position of every n^{th} prime number will have to be fully and indirectly computed each time, thus confirming the infinitely-many prime numbers are IP and of IL. Note all [complementary] composite numbers 4, 6, 8, 9, 10, 12, 14, 15, 16, 18... are simply obtained by discarding all prime numbers from integers 2, 3, 4, 5, 6, 7, 8, 9, 10... whereby "special" integers 0 & 1 are neither prime nor composite. We ignore even prime number 2. Zhang's landmark result[7] states there are infinitely many Odd Primes having an even Prime gap < 70 million. One could extrapolate Zhang's result to: There must be at least two or three up to all even Prime gaps being each associated with infinitely many Odd Primes. All obtained consecutive Odd Primes p_n and p_{n+1} can have their calculated $p_{n+1} - p_n$ values grouped together as belonging to even Prime gaps 2, 4, 6, 8, 10... whereby when the Zhang's result is maximally extrapolated, Polignac's and Twin prime conjectures are supported to be true.

Lemma 5. *We can validly classify countably infinite sets as accelerating, linear or decelerating subtypes (as per Page 18 – 19 in [6]).*

Proof. We provide the following required mathematical arguments.
Cardinality: With increasing size, arbitrary Set [or Subset] **X** can be countably finite set (**CFS**), countably infinite set (**CIS**) or uncountably infinite set (**UIS**). Denoted as $\|\mathbf{X}\|$ in this paper, the cardinality of Set **X** measures *number of elements* in Set **X**. E.g., Set **negative Gram[y=0] point** as constituted by a [solitary] rational ($\mathbb{Q}$) t-value of 0 instead of a usual transcendental ($\mathbb{R} - \mathbb{A}$) t-value has CFS of negative Gram[y=0] point with this particular ||**negative Gram[y=0] point**|| = 1, Set even Prime number ($\mathbb{P}$) has CFS of solitary even $\mathbb{P}$ 2 with ||**even** $\mathbb{P}$|| = 1, Set Natural numbers ($\mathbb{N}$) has CIS of $\mathbb{N}$ with $\|\mathbb{N}\| = \aleph_0$, and Set Real numbers ($\mathbb{R}$) has UIS of $\mathbb{R}$ with $\|\mathbb{R}\| = \mathfrak{c}$ (cardinality of the continuum). Then with ||**CIS**|| = $\aleph_0$ = [countably] infinitely many elements; we provide a novel classification for CIS based on its number of elements (cardinality) manifesting linear, accelerating or decelerating property constituting three subtypes of CIS.
CIS-IM-accelerating: CIS with cardinality = ||**CIS-IM-accelerating**|| = $\aleph_0$-accelerating = [countably] infinitely many elements that (overall) acceleratingly reach an *infinity value*. Examples: CP integers 0, 1, 4, 9, 16... generated by simple equation $y = x^2$ for $x = 0, 1, 2, 3, 4...$ and CP values obtained from natural exponential function $y = e(x)$; and IP composite numbers 4, 6, 8, 9, 10... faithfully generated by complex Complement-Sieve-of-Eratosthenes algorithm [which is equivalent to simply discarding 0, 1, and all generated prime numbers via Sieve-of-Eratosthenes algorithm from the set of integers 0, 1, 2, 3, 4, 5...].
CIS-IM-linear: CIS with cardinality = ||**CIS-IM-linear**|| = $\aleph_0$-linear = [countably] infinitely many elements that (overall) linearly reach an *infinity value*. Examples: CP entities 0, 1, 2, 3, 4, 5... [representing all positive integer numbers] generated by simple equation $y = x$ for $x = 0, 1, 2, 3, 4...$; CP entities 0, 2, 4, 6, 8, 10... [representing all positive even numbers] generated by simple equation $y = 2x$ for $x = 0, 1, 2, 3, 4...$; CP entities 1, 3, 5, 7, 9, 11... [representing all positive odd

numbers] generated by simple equation $y = 2x - 1$ for $x = 1, 2, 3, 4, 5...$; and IP nontrivial zeros, Gram[y=0] points and Gram[x=0] points (all given as $\mathbb{R} - \mathbb{A}$ t-values) generated from complex equation Riemann zeta function via its *proxy* Dirichlet eta function. These IP entities will inevitably manifest IP perpetual repeating violations (failures) in Gram's Law and Rosser's Rule occuring infinitely many times. E.g., the former give rise to Set **negative Gram[y=0] points** whereby CIS negative Gram[y=0] points is constituted by $\mathbb{R} - \mathbb{A}$ t-values classified as having ||**negative Gram[y=0] points**|| = ||**CIS-IM-linear**|| = $\aleph_0$-linear.
CIS-ALN-decelerating: CIS with cardinality = ||**CIS-ALN-decelerating**|| = $\aleph_0$-decelerating = [countably] arbitrarily large number of elements that (overall) deceleratingly reach an *Arbitrarily Large Number* value. Examples: CP entities 0, 1, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$... generated by simple equation $y = \sqrt{x}$ for $x = 0, 1, 2, 3, 4, 5...$ and CP values obtained from natural logarithm function $y = ln(x)$; and IP prime numbers 2, 3, 5, 7, 11... faithfully generated by the complex Sieve-of-Eratosthenes algorithm. *The proof is now complete for Lemma 5*□.

6. Conclusions including applying infinitesimals to outputs from Sieve of Eratosthenes and Riemann zeta function

Figure 1 [depicting positive & negative prime numbers and composite numbers] and Figure 2 [depicting the Co-linear Riemann zeta function for positive & negative range] will manifest perfect Mirror symmetry and fully comply with Law of Continuity. Valid comments: Whereas the continuous-like equation Riemann zeta function $\zeta(s)$ Eq. (1) [via *proxy* Dirichlet eta function $\eta(s)$ Eq. (2)] for $s = \sigma \pm t$ range that generate mutually exclusive CIS-IM-linear σ-valued co-lines be mathematically regarded as smoothly *continuous everywhere* thus obeying Law of continuity; so must the discrete-like algorithms Sieve-of-Eratosthenes and Complement-Sieve-of-Eratosthenes that generate mutually exclusive Primes and Composites be conceptually regarded as jaggedly *continuous everywhere* thus also obeying Law of continuity. CIS-ALN-decelerating Primes and CIS-IM-accelerating Composites are dependent complementary entities. In $\zeta(s)$ Eq. (1), the equivalent Euler product formula with product over prime numbers [instead of summation over natural numbers] represents $\zeta(s) \implies$ all primes and, by default, [complementary] composites are intrinsically encoded in $\zeta(s)$. Since via analytic continuation, $\eta(s) = \frac{1}{\gamma} \cdot \zeta(s)$ [*proxy* function for $\zeta(s)$ in $0 < \sigma < 1$- critical strip]; then all primes and, by default, [complementary] composites are also intrinsically encoded in $\eta(s)$ Eq. (2).

For $i = 1, 2, 3, 4, 5,..., n$ (Page 14 of [6]): Recurring *Accelerating primes as Prime gap*$_{i+2}$ *– Prime gap*$_{i+1}$ > *Prime gap*$_{i+1}$ *– Prime gap*$_i$, *Decelerating primes as Prime gap*$_{i+2}$ *– Prime gap*$_{i+1}$ < *Prime gap*$_{i+1}$ *– Prime gap*$_i$ and *Steady primes as Prime gap*$_{i+2}$ *– Prime gap*$_{i+1}$ = *Prime gap*$_{i+1}$ *– Prime gap*$_i$ [≡***"Alternating Prime Gaps series"*** with Prime gaps alternatingly ↑ & ↓] are computed by (sub-)algorithms to obtain mutually exclusive (solitary) even prime number 2 with odd Prime gap 1; odd Twin primes, odd Cousin primes & odd Sexy primes with even Prime gaps 2, 4 & 6.

(a) For IP IL algorithm [Gap 2, 4, 6, 8, 10...]-Sieve of Eratosthenes $p_{n+1} = 3 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd $\mathbb{P}$ {3, 5, 7, 11, 13, 17, 19...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd $\mathbb{P} - (n)^{th}$ Odd $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n = 2, 2, 4, 2, 4, 2....$

(b) For CP FL sub-algorithm [Gap 1]-Sieve of Eratosthenes $p_{n+1} = 2 + \sum_{i=1}^{n} g_i$ [where $n = 1$ and not ALN] that faithfully generates the first and only Even $\mathbb{P}$ {2} ≡ first and only paired Even $\mathbb{P}$ {(2,3)} with cardinality CFS of 1, the solitary n^{th} odd prime gap between two successive primes is denoted by $g_n = (n+1)^{st}$ Odd $\mathbb{P}$ – $(n)^{th}$ Even $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n = 3 - 2 = 1$.

(c) For IP IL sub-algorithm [Gap 2]-Sieve of Eratosthenes $p_{n+1} = 3 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd twin $\mathbb{P}$ {3, 5, 11, 17, 29, 41, 59...} ≡ all paired Odd twin $\mathbb{P}$ {(3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61)...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd twin $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd twin $\mathbb{P}$ – $(n)^{th}$ Odd twin $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 2, 6, 6, 12, 12, 18....

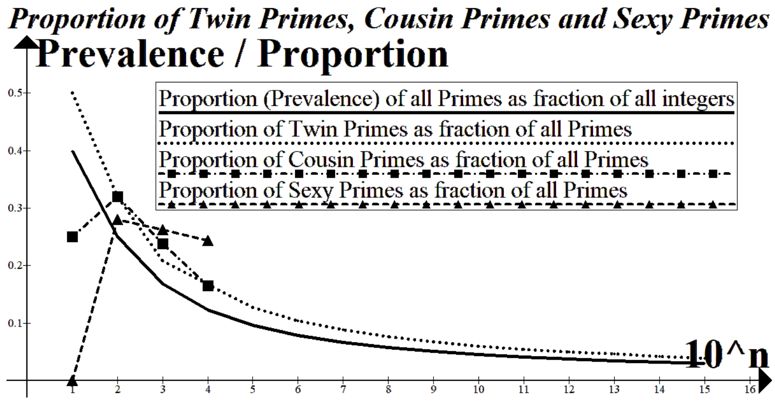
(d) For IP IL sub-algorithm [Gap 4]-Sieve of Eratosthenes $p_{n+1} = 7 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd cousin $\mathbb{P}$ {7, 13, 19, 37, 43, 67...} ≡ all paired Odd cousin $\mathbb{P}$ {(7,11), (13,17), (19,23), (37,41), (43,47), (67,71)...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd cousin $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd cousin $\mathbb{P}$ – $(n)^{th}$ Odd cousin $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 6, 6, 8, 6, 24....

(e) For IP IL sub-algorithm [Gap 6]-Sieve of Eratosthenes $p_{n+1} = 23 + \sum_{i=1}^{n} g_i$ [where n = ALN] that faithfully generates all Odd sexy $\mathbb{P}$ {23, 31, 47, 53, 61, 73, 83...} ≡ all paired Odd sexy $\mathbb{P}$ {(23,29), (31,37), (47,53), (53,59), (61,67), (73,79), (83,89)...} with cardinality $\aleph_0$-decelerating, the n^{th} even Prime gap between two successive Odd sexy $\mathbb{P}$ is denoted by $g_n = (n+1)^{st}$ Odd sexy $\mathbb{P}$ – $(n)^{th}$ Odd sexy $\mathbb{P}$, i.e. $g_n = p_{n+1} - p_n$ = 8, 16, 6, 8, 12, 10....

With n = ALN or, traditionally, ∞; rigorous algorithm-type proof for Modified Polignac's and Twin prime conjectures can be stated here as two statements. Statement 1: All known prime numbers = IP IL algorithm (a) + CP FL sub-algorithm (b). Statement 2: IP IL algorithm (a) = IP IL sub-algorithm (c) + IP IL sub-algorithm (d) + IP IL sub-algorithm (e) +... [that involves all even Prime gaps 2, 4, 6, 8, 10...].

As *proxy* function for Riemann zeta function in $0 < \sigma < 1$ critical strip, Dirichlet eta function when treated as equation and sub-equation at (unique) $\sigma = \frac{1}{2}$-critical line will faithfully generate all x-axis intercept points as *usual* Gram points or Gram[y=0] points, all y-axis intercept points as Gram[x=0] points, and all Origin intercept points as Gram[x=0,y=0] points or nontrivial zeros. Confirming Riemann hypothesis to be true, these entities that constitute three types of Gram points are mutually exclusive, dependent and endowed with t-valued irrational (transcendental) numbers except for initial Gram[y=0] point having a t-valued rational number:
(a) Considered for $t = 0$ to $+\infty$ at $\sigma = \frac{1}{2}$, Dirichlet eta function as IP IL equation will faithfully generate all above-mentioned three types of Gram points endowed with t-valued irrational (transcendental) numbers except for first Gram[y=0] point.
(b) Considered only for $t = 0$ at $\sigma = \frac{1}{2}$, Dirichlet eta function as CP FL sub-equation will faithfully generate the first and only Gram[y=0] point endowed with t-valued rational number 0.

We analyze the data of all CIS-IM-linear computed nontrivial zeros (NTZ) when extrapolated out over a wide range of t ≥ 0 real number values. Akin to Prime counting function Prime-π(x) = number of primes ≤ x, we can symbolically define nontrivial zeros counting function NTZ-π(t) = number of NTZ ≤ t with t assigned to having real number values which are conveniently designated by 10^n whereby n = 1, 2, 3, 4, 5.... The *cumulative* **Prevalence of nontrivial zeros**



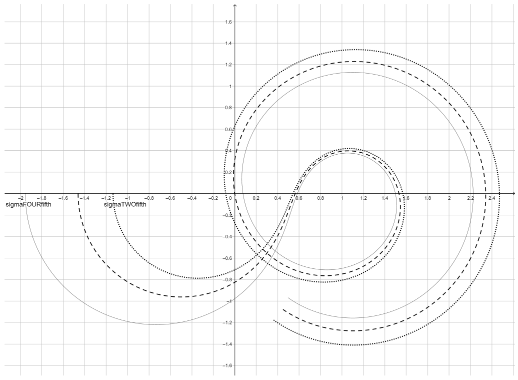
9: Proportion (Prevalence) of Twin primes, Cousin primes [as partial calculations] and Sexy Primes [as partial calculations] with Proportion (Prevalence) of all Primes included. These Proportions (Prevalences) are essentially *self-similar fractal objects*. The n = 1, 2, 3, 4, 5, 6, 7, 8... in 10^n that is denoted with horizontal x-axis $\Longrightarrow$ the scale of this axis is non-linearly depicted using increasing powers of 10.

= NTZ-π(t) / t = NTZ-π(t) / (10^n) when t = 0 to 10^n, whereby denominator t is [artificially] regarded as having integer number values. We conceptually define all consecutive NTZ gaps as i^{th} t-valued NTZ – (i-1)th t-valued NTZ. Thus there are CIS-IM-linear computed NTZ gaps. The numbers of NTZ between $10^0 - 10^1$ [interval = 9], $10^1 - 10^2$ [interval = 90], $10^2 - 10^3$ [interval = 900], $10^3 - 10^4$ [interval = 9000], $10^4 - 10^5$ [interval = 90000], $10^5 - 10^6$ [interval = 900000], $10^6 - 10^7$ [interval = 9000000], $10^7 - 10^8$ [interval = 90000000]... are 0, 29, 620, 9493, 127927, 1609077, 19388979, 226871900... with corresponding *rolling* **Prevalence of nontrivial zeros** = 0, 0.322, 0.689, 1.055, 1.421, 1.788, 2.154, 2.521... $\Longrightarrow$ *rolling* **Prevalence of nontrivial zeros** seems to overall fluctuatingly increase by around 0.366 in a "linear" manner. This limited observation alone suggests Cardinality of nontrivial zeros = ||**CIS-IM-linear**|| = $\aleph_0$-linear.

In comparison, we further notice here the numbers of NTZ between $10^0 - 10^1$ [interval = 9], $10^0 - 10^2$ [interval = 99], $10^0 - 10^3$ [interval = 999], $10^0 - 10^4$ [interval = 9999], $10^0 - 10^5$ [interval = 99999], $10^0 - 10^6$ [interval = 999999], $10^0 - 10^7$ [interval = 9999999], $10^0 - 10^8$ [interval = 99999999]... are 0, 29, 649, 10142, 138069, 1747146, 21136125, 248008025... with corresponding *cumulative* **Prevalence of nontrivial zeros** = 0, 0.293, 0.650, 1.014, 1.381, 1.747, 2.114, 2.480...

On the overall objective to rigorously derive Algorithm-type proofs for Modified Polignac's and Twin prime conjectures [as based on Figure 9] and Equation-type proof for Riemann hypothesis [as based on Figure 10], we apply infinitesimal numbers $\frac{1}{\infty}$ at two places using the following colloquially-stated propositions with their formal proofs given in Page 44 – 45 of [6].

Proposition 6. ***In the limit of never reaching a [nonexisting] zero*** *hereby conceptually visualized as Prevalences of both even Prime gaps and the associated [positive and negative] Odd Primes never becoming zero whereby arbitrarily large number of different even Prime gaps that uniquely accompany all Odd Primes in totality will never stop recurring. Foundation Figure 9 is roughly and analogically based on cohomology as an algebraic tool in topology allowing Geometrical-Mathematical interpretation for positive Odd Primes. We note these Prevalences can only have* $\frac{1}{\infty}$ *values above zero in the large range of prime numbers [but must never have*



10: Simulated dynamic trajectories showing Origin intercept points when $\sigma = \frac{1}{2}$ and virtual Origin intercept points when $\sigma = \frac{2}{5}$ and $\sigma = \frac{4}{5}$. Horizontal axis: $Re\{\zeta(\sigma + \imath t)\}$, and vertical axis: $Im\{\zeta(\sigma + \imath t)\}$. Total presence of all Origin intercept points at the [static] Origin point. Total presence of all virtual Origin intercept points as additional negative virtual Gram[y=0] points on the x-axis (e.g. when using $\sigma = \frac{2}{5}$ value) at the [infinitely many varying] virtual Origin points; viz, these negative virtual Gram[y=0] points on the x-axis cannot exist at the solitary Origin point since two trajectories form two colinear lines (or co-lines) [two parallel lines that never cross over].

zero values].

Proposition 7. ***In the limit of reaching an [existing] zero*** *hereby conceptually visualized as the entire $-\infty < t < +\infty$ trajectory of Dirichlet eta function, proxy for Riemann zeta function, touching (symbolic) zero-dimensional $\sigma = \frac{1}{2}$-Origin point only when parameter $\sigma = \frac{1}{2}$ whereby all nontrivial zeros [mathematically] located on (symbolic) one-dimensional $\sigma = \frac{1}{2}$-critical line will [geometrically] declare themselves in totality as corresponding Origin intercept points. Foundation Figure 10 is roughly and analogically based on cohomology as an algebraic tool in topology allowing Geometrical-Mathematical interpretation for $0 < t < +\infty$ range. Our* ***Corollary*** *is: Any $\sigma \neq \frac{1}{2}$ co-lines that are $\dfrac{1}{\infty}$ above or below the zero-dimensional $\sigma = \frac{1}{2}$-Origin point must never be classified as having nontrivial zeros. Then the* ***Proposition*** *must be: Only one unique $\sigma = \frac{1}{2}$ co-line that [repeatedly] touch the zero-dimensional $\sigma = \frac{1}{2}$-Origin point must always be classified as having [infinitely-many] nontrivial zeros.*

$(0 < \sigma < 1) \equiv (0 < \sigma < \frac{1}{2}) + (\sigma = \frac{1}{2}) + (\frac{1}{2} < \sigma < 1)$. Usefully regarded as *variants* of infinite series are various power series and harmonic series [e.g. (with $s = \sigma \pm it$) Riemann zeta function $\zeta(s)$ via Dirichlet eta function $\eta(s)$ generating infinitely-many $0 < \sigma < 1$-associated trajectories that are all of $-\infty < t < +\infty$ infinite length such as depicted by Figure 2 when $\sigma = \frac{1}{2}$], and various (sub)algorithms [e.g. Sieve of Eratosthenes generating Set of (±) prime numbers in its entirety and Subsets of (±) Odd Primes from even Prime gaps 2, 4, 6, 8, 10... that all have cardinality of ALN]. **Note that each $0 < \sigma < 1$-associated trajectory represents a unique infinite series that is, crucially, *mutually exclusive* by being **mathematically, geometrically and topologically** different from other infinite series**. Analogous to term 'centroid' referring to fixed invariant (0-dimensional) point with PERFECT Point Symmetry representing *center of a geometric object in (n-dimensional) Euclidean space*; there must be: (i) [being valid for entire range +ve & –ve integers] the easily deduced integer number 0 in (1-dimensional) Figure 1 as **Centroid point** and (ii) [being valid for entire range $-\infty < t < +\infty$] Origin point in (2-dimensional) $\sigma = \frac{1}{2}$ Figure 4 when combined together with (2-dimensional) $0 < \sigma < \frac{1}{2}$ Figure 5 and (2-dimensional)

$\frac{1}{2} < \sigma < 1$ Figure 6 [while fully satisfying (Remark 3) Principle of Equidistant for Multiplicative Inverse as previously discussed in Figure 7 with ONLY $\sigma = \frac{1}{2}$ containing the **most frequently & infinitely-often traversed or visited Centroid (Origin) point**]. Our unique Centroid (Origin) point for $\eta(s)$ is conceptually the Point Symmetry with ASSIGNED Central value as $\eta(\dfrac{1}{2} \pm it)$ = 0.0 + 0.0i = 0 at intersection of horizontal real axis & vertical imaginary axis [and having two Line Symmetry of horizontal real axis as depicted by Figure 2 and vertical line $\sigma = \frac{1}{2}$ as depicted by Figure 3]. In comparison, COMPUTED Central value for $\zeta(s)$ via its functional equation having Line Symmetry of vertical line $s = \frac{1}{2}$ [that intersect horizontal real axis] is $\zeta(\dfrac{1}{2})$ $\approxeq -1.4603545 + 0.0i \approxeq -1.4603545$. As overall summary, we insightfully conclude mutually exclusive (sub)sets arising from prime numbers, composite numbers, Gram points and virtual Gram points MUST all conceptually comply in full with *Theory of Symmetry from Langlands program* and *Inclusion-Exclusion Principle* when "extended to the infinite (sub)sets".

Acknowledgements and Declarations

The author is grateful to Editors and Reviewers for feedbacks. He contributes to all the work in this research paper whereby relevant generated data are included. This paper is dedicated to his daughter Jelena born 13 weeks early on May 14, 2012 with Very Low Birth Weight of 1010 grams. AUS $5,000 research grant was provided by Mrs. Connie Hayes and Mr. Colin Webb on January 20, 2020. He discloses receiving an additional AUS $3,250 reimbursement from Q-Pharm for participating in the EyeGene Shingles trial commencing on March 10, 2020. From Doctor of Philosophy (PhD) viewpoint on Ageing, Dementia, Sleep, Learning, Memory and Number theory; he possesses average level of working, short-term and long-term memory, and *Concrete Mathematics* ability. While conducting research that requires advance *Abstract Mathematics*, he routinely practices behavioral augmentation on personal Stage 3 Deep Sleep which contributes to insightful thinking, creativity and memory, and Stage 4 REM Sleep which is essential to cognitive functions memory, learning and creativity.

References

[1] Lichtman J. (2023). A proof of the Erdos primitive set conjecture. *Forum of Mathematics, Pi, 11*, E18, pp. 1 – 21. http://dx.doi:10.1017/fmp.2023.16

[2] LMFDB Collaboration. (2024). *The L-functions and modular forms database*. https://www.lmfdb.org [Online accessed on 5 May 2024]

[3] Padilla A. & Smith R.G.C. (2024). Smoothed asymptotics: from number theory to QFT. *arXiv:2401.10981v1*, pp. 1–28. https://doi.org/10.48550/arXiv.2401.10981

[4] Ting J.Y.C. (2020). Mathematical Modelling of COVID-19 and Solving Riemann Hypothesis, Polignac's and Twin Prime Conjectures Using Novel Fic-Fac Ratio With Manifestations of Chaos-Fractal Phenomena. *J. Math. Res., 12*(6) pp. 1 – 49. https://doi.org/10.5539/jmr.v12n6p1

[5] Ting J.Y.C. (2023). Origin Point Must Represent Critical Line as Location for Nontrivial Zeros of Riemann Zeta Function, and Set Prime Gaps With Subsets Odd Primes Are Arbitrarily Large in Number. *J. Math. Res., 15*(4) pp. 1 – 55. https://doi.org/10.5539/jmr.v15n4p1

[6] Ting J.Y.C. (2024). On the Universal Presence of Mathematics for Incompletely Predictable Problems in Rigorous Proofs for Riemann Hypothesis, Modified Polignac's and Twin Prime Conjectures. *J. Math. Res., 16*(2) pp. 1 – 61. https://doi.org/10.5539/jmr.v16n2p1

[7] Zhang Y. (2014). Bounded gaps between primes. *Ann. of Math., 179,* pp. 1121 – 1174. http://dx.doi.org/10.4007/annals.2014.179.3.7