on the possible zero value of the spatial curvature constant

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Abstracts

The value of curvature k that appears in the Friedman's equation of the FLRW metric is the subject of controversy. Of its three possible values (+1, -1, 0), determining whether it is zero or not is an important problem in physics. The experimental results existing today do not allow us to resolve it. In this report we have studied this problem by carrying out a theoretical calculation of the parameters, curvature density Ω_k and matter density Ω_m. To do this we have obtained an equation that relates the spatial curvature constant to the energy density and through it and the Friedman's equation we have calculated Ω_k and Ω_m. The ratio between the two will determine whether the curvature k is zero or non-zero. The result obtained in this report leads us to think that the curvature constant that appears in the Friedman's equation is zero.

Keywords: Spatial curvature constant, Friedman´s equation, General Relativity, Cosmic spacetime

1. - The cosmic spacetime

We are going to study a uniform and isotropic spacetime from a physical point of view, this is equivalent from a geometric point of view to being invariant under translations and rotations.

According to Professor Fulvio Meliá in reference [1], we define "cosmic spacetime" as the set of points (t, r, Ø Φ) that satisfy the FLRW metric, that is, that satisfy the equation:

$$ds^2 = c^2dt^2 - a(t)^2(\frac{dr^2}{1-kr^2} + r^2d\Omega^2)$$

We define each of the "3D hypersurfaces" of cosmic spacetime as the set of points that have the same temporal coordinate. Thus, cosmic spacetime will have a different hypersurface for each time t. As we have defined them, these hypersurfaces do not intersect, that is, they have no common points and the set of all of them constitutes cosmic spacetime.

It is in these 3D hypersurfaces where we are going to calculate the spatial curvature constant that constitute the object of this report

2. - Calculating the spatial curvature constant in the 3D hypersurfaces of cosmic spacetime

First we are going to calculate the curvature scalar of a 3D hypersurface of our homogeneous and isotropic cosmic spacetime with an matter density ρ_m .

2.1- Birkhoff–Jebsen theorem

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We make a brief comment on this theorem of mathematics applied to the theory of generalized relativity. First, we summarize Professor Fulvio Melia in reference [2] to explain it.

"If we have a spherical universe of mass-energy density ρ and radius r and within it a concentric sphere of radius r_s smaller than r, it is true that the acceleration due to gravity at any point on the surface of the sphere of relative radius r_s to an observer at its origin, depends solely on the mass-energy relation contained within this sphere".

Thus, according to this, to calculate the curvature of the gravitational field of a point located at a distance "r_s" from the geometric center that we are considering in our continuous universe, it is only necessary to consider its interaction with the points that are at a radius smaller than "r_s", therefore, the mass "m" to be considered will only be that contained in the sphere of radius "r_s".

In general relativity, Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be statically and asymptotically flat. This means that the outer solution (that is, the spacetime outside a gravitational, non-rotating, spherical body) must be given by the Schwarzschild metric.

2.2- Calculating the spatial curvature constant

Let's consider our 3D hypersurface and a sphere of radius r inside, the Birkhoff–Jebsen theorem assures us that if we want to calculate the curvature at a point on its surface, we must consider only the interaction with the gravitational mass found inside, the gravitational mass inside for the sphere external point that we are considering behaves as a point mass of equal magnitude to that of the mass of the sphere and located at its central point. In this case we are already in the Schwarzschild model, and we can use its equations to calculate the corresponding curvature.

For all this, we can treat the problem of calculating the curvature scalar in each of the 3D hypersurfaces of our cosmic spacetime as a problem to be solved by the Schwarzschild model and calculate the curvature scalar from that model. In this model, spacetime is reduced to a 2D surface and so Gaussian curvatures are easily calculated; the scalar curvature in this case is twice the Gaussian curvature.

According to Annex I, we have found an equation that relates the Gaussian curvature K of the spacetime of the Schwarzschild model, with the cosmological parameters mass M and universal gravitation constant G. We are going to use this equation to solve our problem. This equation is the following:

$$K = -GM/c^2r^3$$

Since in our case it is a sphere, its mass will be given by

$$M = 4\pi r^3\rho_m/3$$

$$K = -4\pi G\,\rho_m/3c^2$$

The curvature scalar R in bidimensional spaces, 2D surfaces, will be given by twice the Gaussian curvature K, thus:

$$\mathbf{R/\rho_m = -8\pi G/3c^2}$$

R curvature scalar, spatial curvature constant (m^{-2}) and ρ_m is the matter density (Kg/m^3)

The curvature scalar in our case is the spatial curvature constant we are looking for.

Thus, the spatial curvature constant each point of the hypersurface is the same and is proportional to the density of matter.

2.3- Studying the spatial curvature constant

Applying the Friedmann equation and our equation that relates spatial curvature constant to energy density in 3D hypersurfaces, we study the ratio between the parameter Ω_K and the matter density parameter Ω_m will give us a value that can allow us to solve the question of whether the universe (spatially) is flat or not. We study this question here.

$\Omega_m = \rho_m/\rho_c$

$R/\rho_m = -8\pi G/3c^2$

Dividing the two terms of the fraction by ρ_c, we get:

$(R/\rho_c) / \Omega_m = 8\pi G/3c^2$

Defining:

$\Omega_k = (R/\rho_c))$

Result:

$$\mathbf{\Omega_k/\Omega_m = 8\pi G/3c^2 = 6.10^{-27}}$$

Friedmann´s equation:

$$H^2 = (a'/a)^2 = 8\pi G\rho/3 - kc^2/a^2$$

being H the Hubble constant, “a” the scale factor and “ρ” the energy density.

In a universe dominated by matter, such as ours:

$$\rho = \rho_m + \rho_\Lambda$$

ρ_m is the matter density,

ρ_Λ is the vacuum energy density.

Friedmann's equation can be written like this:

$$\mathbf{1 = \Omega_m + \Omega_\Delta + \Omega_k}$$

$\Omega_m = 8\pi G\rho_m/3H^2$

$\Omega_\Delta = 8\pi G\rho_\Lambda/3H^2$

$\Omega_k = 6.10^{-27}\Omega_m$

According to these calculations, the Friedmann equation can be written as:

$$\mathbf{1 = \Omega_m + \Omega_\Delta}$$

Therefore, according to our calculations, the value of the curvature parameter k appearing in the Friedman equation is zero, k=0.

2.4- Experimental data Ω_k

According to the reference [3]

$\Omega_k = 0{,}001 \pm 0{,}002$

Our value for the parameter Ω_k is within the possible experimental data. Our result is consistent with the experimental data.

3. - Conclusions

By studying the relationship of the cosmological parameters Ω_k and Ω_m we can know whether the spatial curvature constant is zero or not. If the result of our calculation is very close to zero, it is expected that this constant will take a value of zero, if the result of our calculation is greater, it is expected that this constant will not be zero. The current experimental results are not precise enough to determine this.

Performing a calculation as detailed in this report we have obtained a value very close to zero for the relationship between these two parameters Ω_k and Ω_m. Applying this result to the Friedman equation we have obtained a zero value for k, the curvature constant that appears in the equation without any doubt regarding another possible value. Thus, we consider that the problem we proposed at the beginning of this report has been resolved.

Opinion among current astrophysicists is divided, with those who think that this curvature is zero being very relevant. Our calculations also seem to indicate this.

Annex I

In this annex we obtain an equation that relates the Gaussian curvature of the Schwarzschild spacetime with several physical parameters.

The Flamm paraboloid, J. Droste's spacetime solution to the problem studied by Schwarzschild, [4], is a 2D surface inserted in an R^3 space. Its geometry allows us to parameterize the paraboloid as a function of the observer's distance from the point mass "r" and the azimuth angle "φ". The problem admits a mathematical treatment of differential geometry of surfaces [5], and with it we are going to calculate the Gaussian Curvature. (Rs = Schwarzschild radius)

Surface parameters (r, φ)

$0 \leq r < \infty, \;\; 0 \leq \varphi < 2\pi$

which has this parametric equation:

$x = r \cos\varphi$

$y = r \,\mathrm{sen}\varphi$

$z = 2(Rs\,(r - Rs))^{1/2}$

Vector Equation of the Surface

$f\,(x,y,z) = (r \cos\varphi,\; r\,\mathrm{sen}\varphi,\; 2(Rs(r - Rs))^{1/2})$

Determination of velocity, acceleration, and normal vectors to the surface

$\eth f/\eth\varphi = (-r\ \text{sen}\varphi, r\cos\varphi, 0)$

$\eth^2 f/\eth\varphi^2 = (-r\cos\varphi, -r\ \text{sen}\varphi, 0)$

$\eth f/\eth r = (\cos\varphi, \text{sen}\varphi, (r/Rs - 1)^{-1/2})$

$\eth^2 f/\eth r^2 = (0, 0, (-1/2R_s). (r/Rs - 1)^{-3/2})$

$\eth f/\eth\varphi\eth r = (-\text{sen}\varphi, \cos\varphi, 0)$

$$n = \frac{(\eth f/\eth\varphi \times \eth f/\eth r)}{[\frac{\eth f}{\eth\varphi} x \frac{\eth f}{\eth r}]}$$

$(\eth f/\eth\varphi \times \eth f/\eth r) = (r\cos\varphi/(r/Rs - 1)^{1/2},\ \ r\,\text{sen}\varphi/(r/Rs - 1)^{1/2},\ \ -r)$

$[\frac{\eth f}{\eth\varphi} \text{X} \frac{\eth f}{\eth r}] = r\,((1/(r/Rs - 1)) + 1)^{1/2}$

Curvature and curvature parameters

$$\text{Gauss curvature} \quad K = LN\text{-}M^2/EG\text{-}F^2$$

$L = \eth^2 f/\eth\varphi^2.\ n = -r(r/Rs)^{-1/2}$

$N = \eth^2 f/\eth r^2.\ n = (1/2Rs)\ (r/Rs)^{-1/2}\ (r/Rs - 1)^{-1}$

$M = (\eth f/\eth\varphi\eth r).\ n = 0$

$G = \eth f/\eth r.\ \eth f/\eth r = 1 + (1/\ (r/Rs - 1))$

$E = \eth f/\eth\varphi.\ \ \eth f/\eth\varphi = r^2$

$F = \eth f/\eth\varphi.\ \eth f/\eth r = 0$

$$K = -Rs/2r^3 = -\ GM/c^2 r^3$$

for Schwarzschild radius, $Rs = 2GM/c^2$

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