

**MACLOF-Chaining**  
(Introduction to the Elementary Mathematics and Designs of GOD)

© 2024 Claude Michael Cassano

I have used a variety of  $4 \times 4$  matrix factor pairs while using Helmholtzian operator factorizations in analyzing fermion architecture and interactions; some commutative, others not.

The following is a generalization to a set of  $4 \times 4$  associative-commutative matrix factor pairs of general linear/differential operators to establish as a baseline foundation to begin operating from.

Because the operators are associative-commutative, the factors lend themselves to chaining - and thus, to chaining of mesons, hadrons, and to chemical compound chaining - amino acid chains, proteins, nucleic acids, and so on

Much is known, but MACLOF chains may provide a mathematical foundation to chemical compound chaining

Note also, that since MACLOF factoring may be extended to higher dimensions, so too MACLOF chaining, and resulting expansion of understanding.

The Helmholtzian operator factorization is:

$$\begin{pmatrix} D_0 & D_3^{\leftrightarrow} & -D_2^{\leftrightarrow} & D_1 \\ -D_3^{\leftrightarrow} & D_0 & D_1^{\leftrightarrow} & D_2 \\ D_2^{\leftrightarrow} & -D_1^{\leftrightarrow} & D_0 & D_3 \\ D_1^{\updownarrow} & D_2^{\updownarrow} & D_3^{\updownarrow} & -D_0^{\updownarrow} \end{pmatrix} \begin{pmatrix} D_0^{\updownarrow} & -D_3^{\leftrightarrow} & D_2^{\leftrightarrow} & D_1 \\ D_3^{\leftrightarrow} & D_0^{\updownarrow} & -D_1^{\leftrightarrow} & D_2 \\ -D_2^{\leftrightarrow} & D_1^{\leftrightarrow} & D_0^{\updownarrow} & D_3 \\ D_1^{\updownarrow} & D_2^{\updownarrow} & D_3^{\updownarrow} & -D_0 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix} = ((\square - |m|^2)) \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix}$$

where:

$$D_{ij}^+ \equiv (\partial_i + m_i) \quad , \quad D_{ij}^- \equiv (\partial_i - m_i) \quad (2)$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\updownarrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\updownarrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix} \quad (3)$$

Note, firstly:

$$f \leq \begin{pmatrix} f_+^1 \\ f_-^1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} D_0 & D_3^{\leftrightarrow} & -D_2^{\leftrightarrow} & D_1 \\ -D_3^{\leftrightarrow} & D_0 & D_1^{\leftrightarrow} & D_2 \\ D_2^{\leftrightarrow} & -D_1^{\leftrightarrow} & D_0 & D_3 \\ D_1^{\updownarrow} & D_2^{\updownarrow} & D_3^{\updownarrow} & -D_0^{\updownarrow} \end{pmatrix} \begin{pmatrix} D_0^{\updownarrow} & -D_3^{\leftrightarrow} & D_2^{\leftrightarrow} & D_1 \\ D_3^{\leftrightarrow} & D_0^{\updownarrow} & -D_1^{\leftrightarrow} & D_2 \\ -D_2^{\leftrightarrow} & D_1^{\leftrightarrow} & D_0^{\updownarrow} & D_3 \\ D_1^{\updownarrow} & D_2^{\updownarrow} & D_3^{\updownarrow} & -D_0 \end{pmatrix} f = ((\square - |m|^2)) [f_+ + f_-]$$

where:

$$f_+ \equiv \begin{pmatrix} \begin{pmatrix} f_+^1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} f_+^2 \\ 0 \end{pmatrix} \\ \begin{pmatrix} f_+^3 \\ 0 \end{pmatrix} \\ \begin{pmatrix} f_+^0 \\ 0 \end{pmatrix} \end{pmatrix} \quad , \quad f_- \equiv \begin{pmatrix} \begin{pmatrix} 0 \\ f_-^1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ f_-^2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ f_-^3 \end{pmatrix} \\ \begin{pmatrix} 0 \\ f_-^0 \end{pmatrix} \end{pmatrix} \quad , \quad f \equiv \begin{pmatrix} \begin{pmatrix} f_+^1 \\ f_-^1 \end{pmatrix} \\ \begin{pmatrix} f_+^2 \\ f_-^2 \end{pmatrix} \\ \begin{pmatrix} f_+^3 \\ f_-^3 \end{pmatrix} \\ \begin{pmatrix} f_+^0 \\ f_-^0 \end{pmatrix} \end{pmatrix} = f_+ + f_-$$

Thus, this all may be considered a or a result of binary interaction.  
(of a matter-anti-matter pair)

Note, firstly, the factorization operation is right-to-left

NOTE:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

form a four-dimensional vector space closed under ordinary matrix multiplication.

(similar to the Pauli matrices without the  $i$  (but the first two are already equivalent to:  $(1, i)$ )

And:

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} = \begin{pmatrix} (L_{ij} + \ell_{ij}) & 0 \\ 0 & (L_{ij} - \ell_{ij}) \end{pmatrix} = L_{ij} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \ell_{ij} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} = \begin{pmatrix} (L_{ij} - \ell_{ij}) & 0 \\ 0 & (L_{ij} + \ell_{ij}) \end{pmatrix} = L_{ij} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \ell_{ij} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} = \begin{pmatrix} 0 & (L_{ij} - \ell_{ij}) \\ (L_{ij} + \ell_{ij}) & 0 \end{pmatrix} = L_{ij} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \ell_{ij} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Dirac factoring uses the 2x2 Pauli matrices:  $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,  $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ ,  $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

along with the 2x2 identity matrix:  $\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2$

So:

$$D_{ij} = L_{ij}\sigma^0 + \ell_{ij}\sigma^3 \quad D_{ij}^{\uparrow} = L_{ij}\sigma^0 - \ell_{ij}\sigma^3 \quad D_{ij}^{\leftrightarrow} = L_{ij}\sigma^1 - \ell_{ij}\sigma^2$$

More generally,  $\partial_i$  &  $m_i$  may be replaced with:

**Multidimensional Associative-Commutative Linear Operator Factors:**

$L_{ij}$  &  $\ell_{ij}$  with similar results, yet far reaching.

(In this case  $M = 2$  for dimension  $2^2 = 4 \Rightarrow$  MACLOFactorization)

**definiton 1:** Given: commutative-associative linear operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

$$D_{Bm} \equiv \begin{pmatrix} -D_{0m} & D_{3m}^{\leftrightarrow} & -D_{2m}^{\leftrightarrow} & -D_{1m} \\ -D_{3m}^{\leftrightarrow} & -D_{0m} & D_{1m}^{\leftrightarrow} & -D_{2m} \\ D_{2m}^{\leftrightarrow} & -D_{1m}^{\leftrightarrow} & -D_{0m} & -D_{3m} \\ -D_{1m}^{\uparrow} & -D_{2m}^{\uparrow} & -D_{3m}^{\uparrow} & D_{0m}^{\uparrow} \end{pmatrix} \quad \& \quad D_{An} \equiv \begin{pmatrix} -D_{0n}^{\uparrow} & -D_{3n}^{\leftrightarrow} & D_{2n}^{\leftrightarrow} & -D_{1n} \\ D_{3n}^{\leftrightarrow} & -D_{0n}^{\uparrow} & -D_{1n}^{\leftrightarrow} & -D_{2n} \\ -D_{2n}^{\leftrightarrow} & D_{1n}^{\leftrightarrow} & -D_{0n}^{\uparrow} & -D_{3n} \\ -D_{1n}^{\uparrow} & -D_{2n}^{\uparrow} & -D_{3n}^{\uparrow} & D_{0n} \end{pmatrix}$$

$D_{Bm}$  &  $D_{An}$  are called **B-factors** & **A-factors**, under indices  $m$  &  $n$ , respectively.

**Theorem I.1:** For commutative-associative linear operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

$$D_{Bm} \equiv \begin{pmatrix} -D_{0m} & D_{3m}^{\leftrightarrow} & -D_{2m}^{\leftrightarrow} & -D_{1m} \\ -D_{3m}^{\leftrightarrow} & -D_{0m} & D_{1m}^{\leftrightarrow} & -D_{2m} \\ D_{2m}^{\leftrightarrow} & -D_{1m}^{\leftrightarrow} & -D_{0m} & -D_{3m} \\ -D_{1m}^{\uparrow} & -D_{2m}^{\uparrow} & -D_{3m}^{\uparrow} & D_{0m}^{\uparrow} \end{pmatrix} \quad \& \quad D_{An} \equiv \begin{pmatrix} -D_{0n}^{\uparrow} & -D_{3n}^{\leftrightarrow} & D_{2n}^{\leftrightarrow} & -D_{1n} \\ D_{3n}^{\leftrightarrow} & -D_{0n}^{\uparrow} & -D_{1n}^{\leftrightarrow} & -D_{2n} \\ -D_{2n}^{\leftrightarrow} & D_{1n}^{\leftrightarrow} & -D_{0n}^{\uparrow} & -D_{3n} \\ -D_{1n}^{\uparrow} & -D_{2n}^{\uparrow} & -D_{3n}^{\uparrow} & D_{0n} \end{pmatrix}$$

then:

$$D_{Bm}D_{An} = \begin{pmatrix} (D_{0m}D_{0n}^{\uparrow} + D_{3m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{2m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{1m}D_{1n}^{\uparrow}) & (D_{0m}D_{3n}^{\leftrightarrow} - D_{3m}^{\leftrightarrow}D_{0n}^{\uparrow} - D_{2m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{1m}D_{2n}^{\uparrow}) & (-D_{0m}D_{2n}^{\leftrightarrow} - D_{3m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{2m}^{\leftrightarrow}D_{0n}^{\uparrow} + D_{1m}D_{3n}^{\uparrow}) \\ (D_{3m}^{\leftrightarrow}D_{0n}^{\uparrow} - D_{0m}D_{3n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{2m}D_{1n}^{\uparrow}) & (D_{3m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{0m}D_{0n}^{\uparrow} + D_{1m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{2m}D_{2n}^{\uparrow}) & (-D_{3m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{0m}D_{1n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{0n}^{\uparrow} + D_{2m}D_{3n}^{\uparrow}) \\ (-D_{2m}^{\leftrightarrow}D_{0n}^{\uparrow} - D_{1m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{0m}D_{2n}^{\leftrightarrow} + D_{3m}D_{1n}^{\uparrow}) & (-D_{2m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{1m}^{\leftrightarrow}D_{0n}^{\uparrow} - D_{0m}D_{1n}^{\leftrightarrow} + D_{3m}D_{2n}^{\uparrow}) & (D_{2m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{1m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{0m}D_{0n}^{\uparrow} + D_{3m}D_{3n}^{\uparrow}) \\ (D_{1m}^{\uparrow}D_{0n}^{\uparrow} - D_{2m}^{\uparrow}D_{3n}^{\leftrightarrow} + D_{3m}^{\uparrow}D_{2n}^{\leftrightarrow} - D_{0m}^{\uparrow}D_{1n}^{\uparrow}) & (D_{1m}^{\uparrow}D_{3n}^{\leftrightarrow} + D_{2m}^{\uparrow}D_{0n}^{\uparrow} - D_{3m}^{\uparrow}D_{1n}^{\leftrightarrow} - D_{0m}^{\uparrow}D_{2n}^{\uparrow}) & (-D_{1m}^{\uparrow}D_{2n}^{\leftrightarrow} + D_{2m}^{\uparrow}D_{1n}^{\leftrightarrow} + D_{3m}^{\uparrow}D_{0n}^{\uparrow} - D_{0m}^{\uparrow}D_{3n}^{\uparrow}) \end{pmatrix}$$

$$D_{An}D_{Bm} = \begin{pmatrix} (D_{0n}^{\uparrow}D_{0m} + D_{3n}^{\leftrightarrow}D_{3m}^{\leftrightarrow} + D_{2n}^{\leftrightarrow}D_{2m}^{\leftrightarrow} + D_{1n}D_{1m}^{\uparrow}) & (-D_{0n}^{\uparrow}D_{3m}^{\leftrightarrow} + D_{3n}^{\leftrightarrow}D_{0m} - D_{2n}^{\leftrightarrow}D_{1m}^{\leftrightarrow} + D_{1n}D_{2m}^{\uparrow}) & (D_{0n}^{\uparrow}D_{2m}^{\leftrightarrow} - D_{3n}^{\leftrightarrow}D_{1m}^{\leftrightarrow} - D_{2n}^{\leftrightarrow}D_{0m} + D_{1n}D_{3m}^{\uparrow}) \\ (-D_{3n}^{\leftrightarrow}D_{0m} + D_{0n}^{\uparrow}D_{3m}^{\leftrightarrow} - D_{1n}^{\leftrightarrow}D_{2m}^{\leftrightarrow} + D_{2n}D_{1m}^{\uparrow}) & (D_{3n}^{\leftrightarrow}D_{3m}^{\leftrightarrow} + D_{0n}^{\uparrow}D_{0m} + D_{1n}^{\leftrightarrow}D_{1m}^{\leftrightarrow} + D_{2n}D_{2m}^{\uparrow}) & (-D_{3n}^{\leftrightarrow}D_{2m}^{\leftrightarrow} - D_{0n}^{\uparrow}D_{1m}^{\leftrightarrow} + D_{1n}^{\leftrightarrow}D_{0m} + D_{2n}D_{3m}^{\uparrow}) \\ (D_{2n}^{\leftrightarrow}D_{0m} - D_{1n}^{\leftrightarrow}D_{3m}^{\leftrightarrow} - D_{0n}^{\uparrow}D_{2m}^{\leftrightarrow} + D_{3n}D_{1m}^{\uparrow}) & (-D_{2n}^{\leftrightarrow}D_{3m}^{\leftrightarrow} - D_{1n}^{\leftrightarrow}D_{0m} + D_{0n}^{\uparrow}D_{1m}^{\leftrightarrow} + D_{3n}D_{2m}^{\uparrow}) & (D_{2n}^{\leftrightarrow}D_{2m}^{\leftrightarrow} + D_{1n}^{\leftrightarrow}D_{1m}^{\leftrightarrow} + D_{0n}^{\uparrow}D_{0m} + D_{3n}D_{3m}^{\uparrow}) \\ (D_{1n}^{\uparrow}D_{0m} + D_{2n}^{\uparrow}D_{3m}^{\leftrightarrow} - D_{3n}^{\uparrow}D_{1m}^{\leftrightarrow} - D_{0n}^{\uparrow}D_{1m}^{\uparrow}) & (-D_{1n}^{\uparrow}D_{3m}^{\leftrightarrow} + D_{2n}^{\uparrow}D_{0m} + D_{3n}^{\uparrow}D_{1m}^{\leftrightarrow} - D_{0n}^{\uparrow}D_{2m}^{\uparrow}) & (-D_{1n}^{\uparrow}D_{2m}^{\leftrightarrow} + D_{2n}^{\uparrow}D_{1m}^{\leftrightarrow} + D_{3n}^{\uparrow}D_{0m} - D_{0n}^{\uparrow}D_{3m}^{\uparrow}) \end{pmatrix}$$

**Proof:**

$$D_{Bm}D_{An} = \begin{pmatrix} -D_{0m} & D_{3m}^{\leftrightarrow} & -D_{2m}^{\leftrightarrow} & -D_{1m} \\ -D_{3m}^{\leftrightarrow} & -D_{0m} & D_{1m}^{\leftrightarrow} & -D_{2m} \\ D_{2m}^{\leftrightarrow} & -D_{1m}^{\leftrightarrow} & -D_{0m} & -D_{3m} \\ -D_{1m}^{\uparrow} & -D_{2m}^{\uparrow} & -D_{3m}^{\uparrow} & D_{0m}^{\uparrow} \end{pmatrix} \begin{pmatrix} -D_{0n}^{\uparrow} & -D_{3n}^{\leftrightarrow} & D_{2n}^{\leftrightarrow} & -D_{1n} \\ D_{3n}^{\leftrightarrow} & -D_{0n}^{\uparrow} & -D_{1n}^{\leftrightarrow} & -D_{2n} \\ -D_{2n}^{\leftrightarrow} & D_{1n}^{\leftrightarrow} & -D_{0n}^{\uparrow} & -D_{3n} \\ -D_{1n}^{\uparrow} & -D_{2n}^{\uparrow} & -D_{3n}^{\uparrow} & D_{0n} \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} (D_{0m}D_{0n}^\dagger + D_{3m}^\leftarrow D_{3n}^\leftarrow + D_{2m}^\leftarrow D_{2n}^\leftarrow + D_{1m}D_{1n}^\dagger) & (D_{0m}D_{3n}^\leftarrow - D_{3m}^\leftarrow D_{0n}^\dagger - D_{2m}^\leftarrow D_{1n}^\leftarrow + D_{1m}D_{2n}^\dagger) & (-D_{0m}D_{2n}^\leftarrow - D_{3m}^\leftarrow D_{1n}^\leftarrow + D_{2m}^\leftarrow D_{0n}^\dagger + D_{1m}D_{3n}^\dagger) & (D_{0m}D_{1n}^\dagger) \\ (D_{3m}^\leftarrow D_{0n}^\dagger - D_{0m}D_{3n}^\leftarrow - D_{1m}^\leftarrow D_{2n}^\leftarrow + D_{2m}D_{1n}^\dagger) & (D_{3m}^\leftarrow D_{3n}^\leftarrow + D_{0m}D_{0n}^\dagger + D_{1m}^\leftarrow D_{1n}^\leftarrow + D_{2m}D_{2n}^\dagger) & (-D_{3m}^\leftarrow D_{2n}^\leftarrow + D_{0m}D_{1n}^\leftarrow - D_{1m}^\leftarrow D_{0n}^\dagger + D_{2m}D_{3n}^\dagger) & (D_{3m}^\leftarrow D_{1n}^\dagger) \\ (-D_{2m}^\leftarrow D_{0n}^\dagger - D_{1m}^\leftarrow D_{3n}^\leftarrow + D_{0m}D_{2n}^\leftarrow + D_{3m}D_{1n}^\dagger) & (-D_{2m}^\leftarrow D_{3n}^\leftarrow + D_{1m}^\leftarrow D_{0n}^\dagger - D_{0m}D_{1n}^\leftarrow + D_{3m}D_{2n}^\dagger) & (D_{2m}^\leftarrow D_{2n}^\leftarrow + D_{1m}^\leftarrow D_{1n}^\leftarrow + D_{0m}D_{0n}^\dagger + D_{3m}D_{3n}^\dagger) & (-D_{2m}^\leftarrow D_{1n}^\dagger) \\ (D_{1m}^\leftarrow D_{0n}^\dagger - D_{2m}^\leftarrow D_{3n}^\leftarrow + D_{3m}^\leftarrow D_{2n}^\leftarrow - D_{0m}D_{1n}^\dagger) & (D_{1m}^\leftarrow D_{3n}^\leftarrow + D_{2m}^\leftarrow D_{0n}^\dagger - D_{3m}^\leftarrow D_{1n}^\leftarrow - D_{0m}D_{2n}^\dagger) & (-D_{1m}^\leftarrow D_{2n}^\leftarrow + D_{2m}^\leftarrow D_{1n}^\leftarrow + D_{3m}^\leftarrow D_{0n}^\dagger - D_{0m}D_{3n}^\dagger) & (D_{1m}^\leftarrow D_{1n}^\dagger) \end{pmatrix} \\
D_{An}D_{Bm} &= \begin{pmatrix} -D_{0n}^\dagger & -D_{3n}^\leftarrow & D_{2n}^\leftarrow & -D_{1n}^\leftarrow \\ D_{3n}^\leftarrow & -D_{0n}^\dagger & -D_{1n}^\leftarrow & -D_{2n}^\leftarrow \\ -D_{2n}^\leftarrow & D_{1n}^\leftarrow & -D_{0n}^\dagger & -D_{3n}^\leftarrow \\ -D_{1n}^\dagger & -D_{2n}^\dagger & -D_{3n}^\dagger & D_{0n} \end{pmatrix} \begin{pmatrix} -D_{0m} & D_{3m}^\leftarrow & -D_{2m}^\leftarrow & -D_{1m}^\leftarrow \\ -D_{3m}^\leftarrow & -D_{0m} & D_{1m}^\leftarrow & -D_{2m}^\leftarrow \\ D_{2m}^\leftarrow & -D_{1m}^\leftarrow & -D_{0m} & -D_{3m}^\leftarrow \\ -D_{1m}^\dagger & -D_{2m}^\dagger & -D_{3m}^\dagger & D_{0m} \end{pmatrix} \\
&= \begin{pmatrix} (D_{0n}^\dagger D_{0m} + D_{3n}^\leftarrow D_{3m}^\leftarrow + D_{2n}^\leftarrow D_{2m}^\leftarrow + D_{1n}D_{1m}^\dagger) & (-D_{0n}^\dagger D_{3m}^\leftarrow + D_{3n}^\leftarrow D_{0m} - D_{2n}^\leftarrow D_{1m}^\leftarrow + D_{1n}D_{2m}^\dagger) & (D_{0n}^\dagger D_{2m}^\leftarrow - D_{3n}^\leftarrow D_{1m}^\leftarrow - D_{2n}^\leftarrow D_{0m} + D_{1n}D_{3m}^\dagger) & (D_{0n}^\dagger D_{1m}^\dagger) \\ (-D_{3n}^\leftarrow D_{0m} + D_{0n}^\dagger D_{3m}^\leftarrow - D_{1n}^\leftarrow D_{2m}^\leftarrow + D_{2n}D_{1m}^\dagger) & (D_{3n}^\leftarrow D_{3m}^\leftarrow + D_{0n}^\dagger D_{0m} + D_{1n}^\leftarrow D_{1m}^\leftarrow + D_{2n}D_{2m}^\dagger) & (-D_{3n}^\leftarrow D_{2m}^\leftarrow - D_{0n}^\dagger D_{1m}^\leftarrow + D_{1n}^\leftarrow D_{0m} + D_{2n}D_{3m}^\dagger) & (-D_{3n}^\leftarrow D_{1m}^\dagger) \\ (D_{2n}^\leftarrow D_{0m} - D_{1n}^\leftarrow D_{3m}^\leftarrow - D_{0n}^\dagger D_{2m}^\leftarrow + D_{3n}D_{1m}^\dagger) & (-D_{2n}^\leftarrow D_{3m}^\leftarrow - D_{1n}^\leftarrow D_{0m} + D_{0n}^\dagger D_{1m}^\leftarrow + D_{3n}D_{2m}^\dagger) & (D_{2n}^\leftarrow D_{2m}^\leftarrow + D_{1n}^\leftarrow D_{1m}^\leftarrow + D_{0n}^\dagger D_{0m} + D_{3n}D_{3m}^\dagger) & (D_{2n}^\leftarrow D_{1m}^\dagger) \\ (D_{1n}^\dagger D_{0m} + D_{2n}^\dagger D_{3m}^\leftarrow - D_{3n}^\dagger D_{2m}^\leftarrow - D_{0n}D_{1m}^\dagger) & (-D_{1n}^\dagger D_{3m}^\leftarrow + D_{2n}^\dagger D_{0m} - D_{3n}^\dagger D_{1m}^\leftarrow - D_{0n}D_{2m}^\dagger) & (-D_{1n}^\dagger D_{2m}^\leftarrow + D_{2n}^\dagger D_{1m}^\leftarrow + D_{3n}^\dagger D_{0m} - D_{0n}D_{3m}^\dagger) & (D_{1n}^\dagger D_{1m}^\dagger) \end{pmatrix} \\
&\square
\end{aligned}$$

**definiton 2:** Given: commutative-associative linear operators  $L_{ij}$  &  $\ell_{ij}$   
and: **B-factors & A-factors:**  $D_{Bm}$  &  $D_{An}$ , under indices  $m$  &  $n$ , respectively:  
 $D_{Bm}D_{An}$  &  $D_{An}D_{Bm}$  are called **BA-factorizations & AB-factorizations**, respectively.

**Corollary I.1.1:** For commutative-associative linear operators  $L_{ij}$  &  $\ell_{ij}$   
and:

$$\begin{aligned}
D_{ij}^+ &\equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij}) \\
D_{ij} &\equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^\leftarrow \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix} \\
D_{B\rho} &\equiv \begin{pmatrix} -D_{0\rho} & D_{3\rho}^\leftarrow & -D_{2\rho}^\leftarrow & -D_{1\rho}^\leftarrow \\ -D_{3\rho}^\leftarrow & -D_{0\rho} & D_{1\rho}^\leftarrow & -D_{2\rho}^\leftarrow \\ D_{2\rho}^\leftarrow & -D_{1\rho}^\leftarrow & -D_{0\rho} & -D_{3\rho}^\leftarrow \\ -D_{1\rho}^\dagger & -D_{2\rho}^\dagger & -D_{3\rho}^\dagger & D_{0\rho} \end{pmatrix} \quad \& \quad D_{A\sigma} \equiv \begin{pmatrix} -D_{0\sigma}^\dagger & -D_{3\sigma}^\leftarrow & D_{2\sigma}^\leftarrow & -D_{1\sigma}^\leftarrow \\ D_{3\sigma}^\leftarrow & -D_{0\sigma}^\dagger & -D_{1\sigma}^\leftarrow & -D_{2\sigma}^\leftarrow \\ -D_{2\sigma}^\leftarrow & D_{1\sigma}^\leftarrow & -D_{0\sigma}^\dagger & -D_{3\sigma}^\leftarrow \\ -D_{1\sigma}^\dagger & -D_{2\sigma}^\dagger & -D_{3\sigma}^\dagger & D_{0\sigma} \end{pmatrix}
\end{aligned}$$

then:

$$\begin{aligned}
D_{B\rho}D_{A\sigma} &= \begin{pmatrix} (D_{0\rho}D_{0\sigma}^\dagger + D_{3\rho}^\leftarrow D_{3\sigma}^\leftarrow + D_{2\rho}^\leftarrow D_{2\sigma}^\leftarrow + D_{1\rho}D_{1\sigma}^\dagger) & (D_{0\rho}D_{3\sigma}^\leftarrow - D_{3\rho}^\leftarrow D_{0\sigma}^\dagger - D_{2\rho}^\leftarrow D_{1\sigma}^\leftarrow + D_{1\rho}D_{2\sigma}^\dagger) & (-D_{0\rho}D_{2\sigma}^\leftarrow - D_{3\rho}^\leftarrow D_{1\sigma}^\leftarrow + D_{2\rho}^\leftarrow D_{0\sigma}^\dagger + D_{1\rho}D_{3\sigma}^\dagger) & (D_{0\rho}D_{1\sigma}^\dagger) \\ (D_{3\rho}^\leftarrow D_{0\sigma}^\dagger - D_{0\rho}D_{3\sigma}^\leftarrow - D_{1\rho}^\leftarrow D_{2\sigma}^\leftarrow + D_{2\rho}D_{1\sigma}^\dagger) & (D_{3\rho}^\leftarrow D_{3\sigma}^\leftarrow + D_{0\rho}D_{0\sigma}^\dagger + D_{1\rho}^\leftarrow D_{1\sigma}^\leftarrow + D_{2\rho}D_{2\sigma}^\dagger) & (-D_{3\rho}^\leftarrow D_{2\sigma}^\leftarrow + D_{0\rho}D_{1\sigma}^\leftarrow - D_{1\rho}^\leftarrow D_{0\sigma}^\dagger + D_{2\rho}D_{3\sigma}^\dagger) & (D_{3\rho}^\leftarrow D_{1\sigma}^\dagger) \\ (-D_{2\rho}^\leftarrow D_{0\sigma}^\dagger - D_{1\rho}^\leftarrow D_{3\sigma}^\leftarrow + D_{0\rho}D_{2\sigma}^\leftarrow + D_{3\rho}D_{1\sigma}^\dagger) & (-D_{2\rho}^\leftarrow D_{3\sigma}^\leftarrow + D_{1\rho}^\leftarrow D_{0\sigma}^\dagger - D_{0\rho}D_{1\sigma}^\leftarrow + D_{3\rho}D_{2\sigma}^\dagger) & (D_{2\rho}^\leftarrow D_{2\sigma}^\leftarrow + D_{1\rho}^\leftarrow D_{1\sigma}^\leftarrow + D_{0\rho}D_{0\sigma}^\dagger + D_{3\rho}D_{3\sigma}^\dagger) & (-D_{2\rho}^\leftarrow D_{1\sigma}^\dagger) \\ (D_{1\rho}^\dagger D_{0\sigma}^\dagger - D_{2\rho}^\dagger D_{3\sigma}^\leftarrow + D_{3\rho}^\dagger D_{2\sigma}^\leftarrow - D_{0\rho}D_{1\sigma}^\dagger) & (D_{1\rho}^\dagger D_{3\sigma}^\leftarrow + D_{2\rho}^\dagger D_{0\sigma}^\dagger - D_{3\rho}^\dagger D_{1\sigma}^\leftarrow - D_{0\rho}D_{2\sigma}^\dagger) & (-D_{1\rho}^\dagger D_{2\sigma}^\leftarrow + D_{2\rho}^\dagger D_{1\sigma}^\leftarrow + D_{3\rho}^\dagger D_{0\sigma}^\dagger - D_{0\rho}D_{3\sigma}^\dagger) & (D_{1\rho}^\dagger D_{1\sigma}^\dagger) \end{pmatrix} \\
D_{A\sigma}D_{B\rho} &= \begin{pmatrix} (D_{0\sigma}^\dagger D_{0\rho} + D_{3\sigma}^\leftarrow D_{3\rho}^\leftarrow + D_{2\sigma}^\leftarrow D_{2\rho}^\leftarrow + D_{1\sigma}D_{1\rho}^\dagger) & (-D_{0\sigma}^\dagger D_{3\rho}^\leftarrow + D_{3\sigma}^\leftarrow D_{0\rho} - D_{2\sigma}^\leftarrow D_{1\rho}^\leftarrow + D_{1\sigma}D_{2\rho}^\dagger) & (D_{0\sigma}^\dagger D_{2\rho}^\leftarrow - D_{3\sigma}^\leftarrow D_{1\rho}^\leftarrow - D_{2\sigma}^\leftarrow D_{0\rho} + D_{1\sigma}D_{3\rho}^\dagger) & (D_{0\sigma}^\dagger D_{1\rho}^\dagger) \\ (-D_{3\sigma}^\leftarrow D_{0\rho} + D_{0\sigma}^\dagger D_{3\rho}^\leftarrow - D_{1\sigma}^\leftarrow D_{2\rho}^\leftarrow + D_{2\sigma}D_{1\rho}^\dagger) & (D_{3\sigma}^\leftarrow D_{3\rho}^\leftarrow + D_{0\sigma}^\dagger D_{0\rho} + D_{1\sigma}^\leftarrow D_{1\rho}^\leftarrow + D_{2\sigma}D_{2\rho}^\dagger) & (-D_{3\sigma}^\leftarrow D_{2\rho}^\leftarrow - D_{0\sigma}^\dagger D_{1\rho}^\leftarrow + D_{1\sigma}^\leftarrow D_{0\rho} + D_{2\sigma}D_{3\rho}^\dagger) & (-D_{3\sigma}^\leftarrow D_{1\rho}^\dagger) \\ (D_{2\sigma}^\leftarrow D_{0\rho} - D_{1\sigma}^\leftarrow D_{3\rho}^\leftarrow - D_{0\sigma}^\dagger D_{2\rho}^\leftarrow + D_{3\sigma}D_{1\rho}^\dagger) & (-D_{2\sigma}^\leftarrow D_{3\rho}^\leftarrow - D_{1\sigma}^\leftarrow D_{0\rho} + D_{0\sigma}^\dagger D_{1\rho}^\leftarrow + D_{3\sigma}D_{2\rho}^\dagger) & (D_{2\sigma}^\leftarrow D_{2\rho}^\leftarrow + D_{1\sigma}^\leftarrow D_{1\rho}^\leftarrow + D_{0\sigma}^\dagger D_{0\rho} + D_{3\sigma}D_{3\rho}^\dagger) & (D_{2\sigma}^\leftarrow D_{1\rho}^\dagger) \\ (D_{1\sigma}^\dagger D_{0\rho} + D_{2\sigma}^\dagger D_{3\rho}^\leftarrow - D_{3\sigma}^\dagger D_{2\rho}^\leftarrow - D_{0\sigma}D_{1\rho}^\dagger) & (-D_{1\sigma}^\dagger D_{3\rho}^\leftarrow + D_{2\sigma}^\dagger D_{0\rho} - D_{3\sigma}^\dagger D_{1\rho}^\leftarrow - D_{0\sigma}D_{2\rho}^\dagger) & (-D_{1\sigma}^\dagger D_{2\rho}^\leftarrow + D_{2\sigma}^\dagger D_{1\rho}^\leftarrow + D_{3\sigma}^\dagger D_{0\rho} - D_{0\sigma}D_{3\rho}^\dagger) & (D_{1\sigma}^\dagger D_{1\rho}^\dagger) \end{pmatrix}
\end{aligned}$$

**Proof:**

This is merely a restatement of theorem I.1 with alternative subscripts (for convenience).

□

**Corollary I.1.2:** For commutative-associative linear operators  $L_{ij}$  &  $\ell_{ij}$   
and:

$$\begin{aligned}
D_{ij}^+ &\equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij}) \\
D_{ij} &\equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^\leftarrow \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix} \\
D_{Bm} &\equiv \begin{pmatrix} -D_{0m} & D_{3m}^\leftarrow & -D_{2m}^\leftarrow & -D_{1m}^\leftarrow \\ -D_{3m}^\leftarrow & -D_{0m} & D_{1m}^\leftarrow & -D_{2m}^\leftarrow \\ D_{2m}^\leftarrow & -D_{1m}^\leftarrow & -D_{0m} & -D_{3m}^\leftarrow \\ -D_{1m}^\dagger & -D_{2m}^\dagger & -D_{3m}^\dagger & D_{0m} \end{pmatrix} \quad \& \quad D_{An} \equiv \begin{pmatrix} -D_{0n}^\dagger & -D_{3n}^\leftarrow & D_{2n}^\leftarrow & -D_{1n}^\leftarrow \\ D_{3n}^\leftarrow & -D_{0n}^\dagger & -D_{1n}^\leftarrow & -D_{2n}^\leftarrow \\ -D_{2n}^\leftarrow & D_{1n}^\leftarrow & -D_{0n}^\dagger & -D_{3n}^\leftarrow \\ -D_{1n}^\dagger & -D_{2n}^\dagger & -D_{3n}^\dagger & D_{0n} \end{pmatrix}
\end{aligned}$$

$$\Rightarrow D_{Bm}D_{An} = \begin{pmatrix} (D_{0m}D_{0n}^\dagger + D_{3m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{2m}^{\leftarrow}D_{2n}^{\leftarrow} + D_{1m}D_{1n}^\dagger) & (D_{0m}D_{3n}^{\leftarrow} - D_{3m}^{\leftarrow}D_{0n}^\dagger - D_{2m}^{\leftarrow}D_{1n}^{\leftarrow} + D_{1m}D_{2n}^\dagger) & (-D_{0m}D_{2n}^{\leftarrow} - D_{3m}^{\leftarrow}D_{1n}^{\leftarrow} + D_{2m}^{\leftarrow}D_{0n}^\dagger + D_{1m}D_{3n}^\dagger) \\ (D_{3m}^{\leftarrow}D_{0n}^\dagger - D_{0m}D_{3n}^{\leftarrow} - D_{1m}^{\leftarrow}D_{2n}^{\leftarrow} + D_{2m}D_{1n}^\dagger) & (D_{3m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{0m}D_{0n}^\dagger + D_{1m}^{\leftarrow}D_{1n}^{\leftarrow} + D_{2m}D_{2n}^\dagger) & (-D_{3m}^{\leftarrow}D_{2n}^{\leftarrow} + D_{0m}D_{1n}^{\leftarrow} - D_{1m}^{\leftarrow}D_{0n}^\dagger + D_{2m}D_{3n}^\dagger) \\ (-D_{2m}^{\leftarrow}D_{0n}^\dagger - D_{1m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{0m}D_{2n}^{\leftarrow} + D_{3m}D_{1n}^\dagger) & (-D_{2m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{1m}^{\leftarrow}D_{0n}^\dagger - D_{0m}D_{1n}^{\leftarrow} + D_{3m}D_{2n}^\dagger) & (D_{2m}^{\leftarrow}D_{2n}^{\leftarrow} + D_{1m}^{\leftarrow}D_{1n}^{\leftarrow} + D_{0m}D_{0n}^\dagger + D_{3m}D_{3n}^\dagger) \\ (D_{1m}^{\leftarrow}D_{0n}^\dagger - D_{2m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{3m}^{\leftarrow}D_{2n}^{\leftarrow} - D_{0m}^{\leftarrow}D_{1n}^\dagger) & (D_{1m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{2m}^{\leftarrow}D_{0n}^\dagger - D_{3m}^{\leftarrow}D_{1n}^{\leftarrow} - D_{0m}^{\leftarrow}D_{2n}^\dagger) & (-D_{1m}^{\leftarrow}D_{2n}^{\leftarrow} + D_{2m}^{\leftarrow}D_{1n}^{\leftarrow} + D_{3m}^{\leftarrow}D_{0n}^\dagger - D_{0m}^{\leftarrow}D_{3n}^\dagger) \end{pmatrix}$$

$$\Rightarrow D_{Bn}D_{Am} = \begin{pmatrix} (D_{0n}D_{0m}^\dagger + D_{3n}^{\leftarrow}D_{3m}^{\leftarrow} + D_{2n}^{\leftarrow}D_{2m}^{\leftarrow} + D_{1n}D_{1m}^\dagger) & (D_{0n}D_{3m}^{\leftarrow} - D_{3n}^{\leftarrow}D_{0m}^\dagger - D_{2n}^{\leftarrow}D_{1m}^{\leftarrow} + D_{1n}D_{2m}^\dagger) & (-D_{0n}D_{2m}^{\leftarrow} - D_{3n}^{\leftarrow}D_{1m}^{\leftarrow} + D_{2n}^{\leftarrow}D_{0m}^\dagger + D_{1n}D_{3m}^\dagger) \\ (D_{3n}^{\leftarrow}D_{0m}^\dagger - D_{0n}D_{3m}^{\leftarrow} - D_{1n}^{\leftarrow}D_{2m}^{\leftarrow} + D_{2n}D_{1m}^\dagger) & (D_{3n}^{\leftarrow}D_{3m}^{\leftarrow} + D_{0n}D_{0m}^\dagger + D_{1n}^{\leftarrow}D_{1m}^{\leftarrow} + D_{2n}D_{2m}^\dagger) & (-D_{3n}^{\leftarrow}D_{2m}^{\leftarrow} + D_{0n}D_{1m}^{\leftarrow} - D_{1n}^{\leftarrow}D_{0m}^\dagger + D_{2n}D_{3m}^\dagger) \\ (-D_{2n}^{\leftarrow}D_{0m}^\dagger - D_{1n}^{\leftarrow}D_{3m}^{\leftarrow} + D_{0n}D_{2m}^{\leftarrow} + D_{3n}D_{1m}^\dagger) & (-D_{2n}^{\leftarrow}D_{3m}^{\leftarrow} + D_{1n}^{\leftarrow}D_{0m}^\dagger - D_{0n}D_{1m}^{\leftarrow} + D_{3n}D_{2m}^\dagger) & (D_{2n}^{\leftarrow}D_{2m}^{\leftarrow} + D_{1n}^{\leftarrow}D_{1m}^{\leftarrow} + D_{0n}D_{0m}^\dagger + D_{3n}D_{3m}^\dagger) \\ (D_{1n}^{\leftarrow}D_{0m}^\dagger - D_{2n}^{\leftarrow}D_{3m}^{\leftarrow} + D_{3n}^{\leftarrow}D_{2m}^{\leftarrow} - D_{0n}^{\leftarrow}D_{1m}^\dagger) & (D_{1n}^{\leftarrow}D_{3m}^{\leftarrow} + D_{2n}^{\leftarrow}D_{0m}^\dagger - D_{3n}^{\leftarrow}D_{1m}^{\leftarrow} - D_{0n}^{\leftarrow}D_{2m}^\dagger) & (-D_{1n}^{\leftarrow}D_{2m}^{\leftarrow} + D_{2n}^{\leftarrow}D_{1m}^{\leftarrow} + D_{3n}^{\leftarrow}D_{0m}^\dagger - D_{0n}^{\leftarrow}D_{3m}^\dagger) \end{pmatrix}$$

**Proof:**

Immediate from corollary I.1.1:  $\rho = n$  &  $\sigma = m$

□

Thus,  $D_A, D_B$  MACLOF chains may be written:

$D_{An}D_{Bm}D_{Ap}D_{B\sigma}$  or  $D_{Bm}D_{An}D_{B\sigma}D_{Ap}$  ;

generally, pairwise:

$D_{An_1}D_{Bn_1}D_{An_2}D_{Bn_2} \cdots D_{An_M}D_{Bn_M}$  or  $D_{Bm_1}D_{Am_1}D_{Bm_2}D_{Am_2} \cdots D_{Bm_M}D_{Am_M}$

and may link end-to-end; such as figured here:

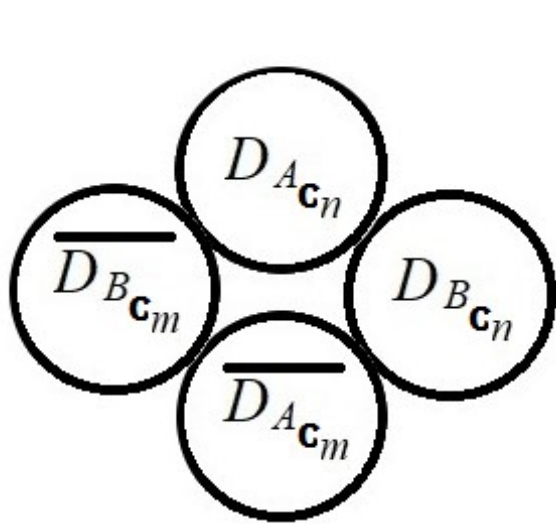


Fig 1

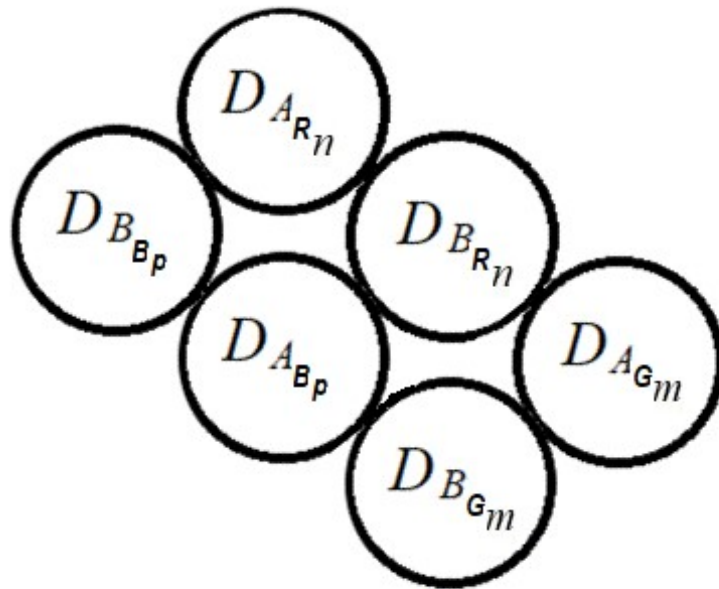


Fig 2

So, the following lemmas I.2 & 1.3 describe properties of the MACLOFs :

**Lemma I.2.1.1-(Bac1r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{0m}D_{0n}^\dagger + D_{3m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{2m}^{\leftarrow}D_{2n}^{\leftarrow} + D_{1m}D_{1n}^\dagger) = \begin{pmatrix} (L_{0m}L_{0n} + L_{3m}L_{3n} + L_{2m}L_{2n} + L_{1m}L_{1n}) - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - L_{3m}\ell_{3n} + \ell_{3m}L_{3n} - L_{2m}\ell_{2n} + \ell_{2m}L_{2n} - L_{1m}\ell_{1n} + \ell_{1m}L_{1n} & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) - L_{0n}\ell_{0n} + \ell_{0m}L_{0n} - L_{3n}\ell_{3n} + \ell_{3m}L_{3n} - L_{2n}\ell_{2n} + \ell_{2m}L_{2n} - L_{1n}\ell_{1n} + \ell_{1m}L_{1n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + (-\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{2n} + \ell_{2m})L_{2n} + (-\ell_{1n} + \ell_{1m})L_{1n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) - (\ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n} + \ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n}) & 0 \\ 0 & (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0m}L_{0n} + L_{3m}L_{3n} + L_{2m}L_{2n} + L_{1m}L_{1n}) - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - L_{3m}\ell_{3n} + \ell_{3m}L_{3n} - L_{2m}\ell_{2n} + \ell_{2m}L_{2n} - L_{1m}\ell_{1n} + \ell_{1m}L_{1n} & 0 \\ 0 & 0 \end{pmatrix}$$

**Proof:**

$$(D_{0m}D_{0n}^\dagger + D_{3m}^{\leftarrow}D_{3n}^{\leftarrow} + D_{2m}^{\leftarrow}D_{2n}^{\leftarrow} + D_{1m}D_{1n}^\dagger) = \left( \begin{pmatrix} (L_{0m} + \ell_{0m}) & 0 \\ 0 & (L_{0m} - \ell_{0m}) \end{pmatrix} \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \right) + \begin{pmatrix} 0 & (L_{3m} - \ell_{3m}) \\ (L_{3m} + \ell_{3m}) & 0 \end{pmatrix}$$

$$\begin{aligned}
&= \begin{pmatrix} (L_{0m} + \ell_{0m})(L_{0n} - \ell_{0n}) + (L_{3m} + \ell_{3m})(L_{3n} - \ell_{3n}) + (L_{2m} + \ell_{2m})(L_{2n} - \ell_{2n}) + (L_{1m} + \ell_{1m})(L_{1n} - \ell_{1n}) & (L_{0m} - \ell_{0m}) \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} (L_{0m}L_{0n} + L_{3m}L_{3n} + L_{2m}L_{2n} + L_{1m}L_{1n}) - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - L_{3m}\ell_{3n} + \ell_{3m}L_{3n} - L_{2m}\ell_{2n} + \ell_{2m}L_{2n} - L_{1m}\ell_{1n} + \ell_{1m}L_{1n} & 0 \end{pmatrix} \\
L_{jm} = L_{jn} : & \\
&= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) - L_{0n}\ell_{0n} + \ell_{0m}L_{0n} - L_{3n}\ell_{3n} + \ell_{3m}L_{3n} - L_{2n}\ell_{2n} + \ell_{2m}L_{2n} - L_{1n}\ell_{1n} + \ell_{1m}L_{1n} & 0 \end{pmatrix} \\
L_{jm} = L_{jn} \ \& \ \ell_{ij} \text{ are constants:} \\
&= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + (-\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{2n} + \ell_{2m})L_{2n} + (-\ell_{1n} + \ell_{1m})L_{1n} & 0 \end{pmatrix} \\
L_{jm} = L_{jn} \ \& \ \ell_{ij} \text{ are constants} \ \& \ \ell_{jm} = \ell_{jn} : \\
&= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) - (\ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n} + \ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n}) & (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) \\ 0 & 0 \end{pmatrix} \\
L_{jn} \text{ are functions or differential operators} \ \& \ \ell_{ij} \text{ are non-constant functions:} \\
&= \begin{pmatrix} (L_{0m}L_{0n} + L_{3m}L_{3n} + L_{2m}L_{2n} + L_{1m}L_{1n}) + (\ell_{0m}L_{0n} + \ell_{3m}L_{3n} + \ell_{2m}L_{2n} + \ell_{1m}L_{1n}) - (L_{0m}\ell_{0n} + L_{3m}\ell_{3n} + L_{2m}\ell_{2n} + L_{1m}\ell_{1n}) & 0 \end{pmatrix} \\
\square
\end{aligned}$$

**Lemma I.2.1.2-(BAC1r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \ , \ D_{ij}^- \equiv (L_{ij} - \ell_{ij}) \\
D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \ , \ D_{ij}^\uparrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{3m}^{\leftrightarrow}D_{0n}^\uparrow - D_{0m}D_{3n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{2m}D_{1n}^\uparrow) = \begin{pmatrix} (-L_{1m}L_{2n} + L_{2m}L_{1n}) - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} - L_{2m}\ell_{1n} + \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{3m}L_{0n} - L_{0m}L_{3n}) + L_{3m}\ell_{0n} - \ell_{3m}L_{0n} \\ (+L_{3m}L_{0n} - L_{0m}L_{3n}) - L_{3m}\ell_{0n} + \ell_{3m}L_{0n} - L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) + L_{1m}\ell_{2n} - \ell_{1m}L_{2n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} -L_{1n}\ell_{2n} + \ell_{1m}L_{2n} - L_{2n}\ell_{1n} + \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & +L_{3n}\ell_{0n} - \ell_{3m}L_{0n} + L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ -L_{3n}\ell_{0n} + \ell_{3m}L_{0n} - L_{0n}\ell_{3n} + \ell_{0m}L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & +L_{1n}\ell_{2n} - \ell_{1m}L_{2n} + L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (-\ell_{2n} + \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (\ell_{0n} - \ell_{0m})L_{3n} + (\ell_{3n} - \ell_{3m})L_{0n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ (-\ell_{0n} + \ell_{0m})L_{3n} + (\ell_{3m} - \ell_{3n})L_{0n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (\ell_{2n} - \ell_{2m})L_{1n} + (-\ell_{1m} + \ell_{1n})L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1m}L_{2n} + L_{2m}L_{1n}) + \ell_{2m}L_{1n} + \ell_{1m}L_{2n} + (-[L_{1m}\ell_{2n} + L_{2m}\ell_{1n}] + [+ \ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}]) & (L_{3m}L_{0n} - L_{0m}L_{3n}) - L_{3m}\ell_{0n} + \ell_{3m}L_{0n} \\ (+L_{3m}L_{0n} - L_{0m}L_{3n}) + \ell_{3m}L_{0n} + \ell_{0m}L_{3n} + (-[L_{3m}\ell_{0n} + L_{0m}\ell_{3n}] + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n})) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} \end{pmatrix}$$

**Proof:**

$$\begin{aligned}
(D_{3m}^{\leftrightarrow}D_{0n}^\uparrow - D_{0m}D_{3n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{2m}D_{1n}^\uparrow) &= \left( \begin{pmatrix} 0 & (L_{3m} - \ell_{3m}) \\ (L_{3m} + \ell_{3m}) & 0 \end{pmatrix} \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \right) - \begin{pmatrix} (L_{0m} + \ell_{0m}) & 0 \\ 0 & (L_{0m} - \ell_{0m}) \end{pmatrix} \\
&= \begin{pmatrix} -(L_{1m} - \ell_{1m})(L_{2n} + \ell_{2n}) + (L_{2m} + \ell_{2m})(L_{1n} - \ell_{1n}) & (L_{3m} - \ell_{3m})(L_{0n} + \ell_{0n}) - (L_{0m} + \ell_{0m})(L_{3n} - \ell_{3n}) \\ (L_{3m} + \ell_{3m})(L_{0n} - \ell_{0n}) - (L_{0m} - \ell_{0m})(L_{3n} + \ell_{3n}) & -(L_{1m} + \ell_{1m})(L_{2n} - \ell_{2n}) + (L_{2m} - \ell_{2m})(L_{1n} + \ell_{1n}) \end{pmatrix} \\
&= \begin{pmatrix} -L_{1m}L_{2n} - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} + \ell_{1m}\ell_{2n} + L_{2m}L_{1n} - L_{2m}\ell_{1n} + \ell_{2m}L_{1n} - \ell_{2m}\ell_{1n} & L_{3m}L_{0n} + L_{3m}\ell_{0n} - \ell_{3m}L_{0n} - \ell_{3m}\ell_{0n} \\ L_{3m}L_{0n} - L_{3m}\ell_{0n} + \ell_{3m}L_{0n} - \ell_{3m}\ell_{0n} - L_{0m}L_{3n} - L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + \ell_{0m}\ell_{3n} & -L_{1m}L_{2n} + L_{1m}\ell_{2n} - \ell_{1m}L_{2n} + \ell_{1m}\ell_{2n} \end{pmatrix} \\
&= \begin{pmatrix} (-L_{1m}L_{2n} + L_{2m}L_{1n}) - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} - L_{2m}\ell_{1n} + \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{3m}L_{0n} - L_{0m}L_{3n}) + L_{3m}\ell_{0n} - \ell_{3m}L_{0n} \\ (+L_{3m}L_{0n} - L_{0m}L_{3n}) - L_{3m}\ell_{0n} + \ell_{3m}L_{0n} - L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) + L_{1m}\ell_{2n} - \ell_{1m}L_{2n} \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  :

$$\begin{aligned}
&= \begin{pmatrix} (-L_{1n}L_{2n} + L_{2n}L_{1n}) - L_{1n}\ell_{2n} + \ell_{1m}L_{2n} - L_{2n}\ell_{1n} + \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{3n}L_{0n} - L_{0n}L_{3n}) + L_{3n}\ell_{0n} - \ell_{3n}L_{0n} \\ (L_{3n}L_{0n} - L_{0n}L_{3n}) - L_{3n}\ell_{0n} + \ell_{3m}L_{0n} - L_{0n}\ell_{3n} + \ell_{0m}L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + L_{1n}\ell_{2n} - \ell_{1n}L_{2n} \end{pmatrix} \\
&= \begin{pmatrix} -L_{1n}\ell_{2n} + \ell_{1m}L_{2n} - L_{2n}\ell_{1n} + \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & +L_{3n}\ell_{0n} - \ell_{3m}L_{0n} + L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ -L_{3n}\ell_{0n} + \ell_{3m}L_{0n} - L_{0n}\ell_{3n} + \ell_{0m}L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & +L_{1n}\ell_{2n} - \ell_{1m}L_{2n} + L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$\begin{aligned}
&= \begin{pmatrix} (-L_{1n}L_{2n} + L_{2n}L_{1n}) - \ell_{2n}L_{1n} + \ell_{2m}L_{1n} + \ell_{1m}L_{2n} - \ell_{1n}L_{2n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{3n}L_{0n} - L_{0n}L_{3n}) + \ell_{0n}L_{3n} - \ell_{3n}L_{0n} \\ (L_{3n}L_{0n} - L_{0n}L_{3n}) - \ell_{0n}L_{3n} + \ell_{0m}L_{3n} + \ell_{3m}L_{0n} - \ell_{3n}L_{0n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + \ell_{2n}L_{1n} - \ell_{2m}L_{1n} \end{pmatrix} \\
&= \begin{pmatrix} (-\ell_{2n} + \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (\ell_{0n} - \ell_{0m})L_{3n} + (\ell_{3n} - \ell_{3m})L_{0n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ (-\ell_{0n} + \ell_{0m})L_{3n} + (\ell_{3m} - \ell_{3n})L_{0n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (\ell_{2n} - \ell_{2m})L_{1n} + (-\ell_{1m} + \ell_{1n})L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1m}L_{2n} + L_{2m}L_{1n}) + \ell_{2m}L_{1n} + \ell_{1m}L_{2n} + (-[L_{1m}\ell_{2n} + L_{2m}\ell_{1n}] + [+ \ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}]) & (L_{3m}L_{0n} - L_{0m}L_{3n}) - \\ (+L_{3m}L_{0n} - L_{0m}L_{3n}) + \ell_{3m}L_{0n} + \ell_{0m}L_{3n} + (-[L_{3m}\ell_{0n} + L_{0m}\ell_{3n}] + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n})) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) - \end{pmatrix}$$

□

**Lemma I.2.1.3-(BAC1r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\hat{}} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\hat{}} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left(-D_{2m}^{\leftrightarrow}D_{0n}^{\hat{}} - D_{1m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{0m}D_{2n}^{\leftrightarrow} + D_{3m}D_{1n}^{\hat{}}\right) = \begin{pmatrix} (-L_{1m}L_{3n} + L_{3m}L_{1n}) - L_{1m}\ell_{3n} + \ell_{1m}L_{3n} - L_{3m}\ell_{1n} + \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (-L_{2m}L_{0n} + L_{0m}L_{2n}) - L_{2m}\ell_{0n} + \ell_{2m}L_{0n} - L_{0m}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1m}L_{3n} + L_{3m}L_{1n}) + L_{1m}\ell_{3n} - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \\ (-L_{2m}L_{0n} + L_{0m}L_{2n}) + L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + L_{0m}\ell_{2n} - \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1m}L_{3n} + L_{3m}L_{1n}) + L_{1m}\ell_{3n} - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} & (-L_{1m}L_{3n} + L_{3m}L_{1n}) + L_{1m}\ell_{3n} - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} -L_{1n}\ell_{3n} + \ell_{1m}L_{3n} - L_{3n}\ell_{1n} + \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & -L_{2n}\ell_{0n} + \ell_{2m}L_{0n} - L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ +L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & -L_{2n}\ell_{0n} + \ell_{2m}L_{0n} - L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1m}L_{3n} + L_{3m}L_{1n}) + \ell_{1m}L_{3n} + \ell_{3m}L_{1n} + (-L_{1m}\ell_{3n} - L_{3m}\ell_{1n} + \ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (-L_{2m}L_{0n} + L_{0m}L_{2n}) + \ell_{2m}\ell_{0n} - \ell_{2m}L_{0n} - \ell_{0m}L_{2n} + (+L_{2m}\ell_{0n} + L_{0m}\ell_{2n} + \ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1m}L_{3n} + L_{3m}L_{1n}) - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \\ (-L_{2m}L_{0n} + L_{0m}L_{2n}) - \ell_{2m}L_{0n} - \ell_{0m}L_{2n} + (+L_{2m}\ell_{0n} + L_{0m}\ell_{2n} + \ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1m}L_{3n} + L_{3m}L_{1n}) - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} & (-L_{1m}L_{3n} + L_{3m}L_{1n}) - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} \left(-D_{2m}^{\leftrightarrow}D_{0n}^{\hat{}} - D_{1m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{0m}D_{2n}^{\leftrightarrow} + D_{3m}D_{1n}^{\hat{}}\right) &= - \begin{pmatrix} 0 & (L_{2m} - \ell_{2m}) \\ (L_{2m} + \ell_{2m}) & 0 \end{pmatrix} \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} - \begin{pmatrix} 0 & (L_{1m} - \ell_{1m}) \\ (L_{1m} + \ell_{1m}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} -(L_{1m} - \ell_{1m})(L_{3n} + \ell_{3n}) + (L_{3m} + \ell_{3m})(L_{1n} - \ell_{1n}) & -(L_{2m} - \ell_{2m})(L_{0n} + \ell_{0n}) + (L_{0m} + \ell_{0m})(L_{2n} - \ell_{2n}) \\ -(L_{2m} + \ell_{2m})(L_{0n} - \ell_{0n}) + (L_{0m} - \ell_{0m})(L_{2n} + \ell_{2n}) & -(L_{1m} + \ell_{1m})(L_{3n} - \ell_{3n}) + (L_{3m} - \ell_{3m})(L_{1n} + \ell_{1n}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{1m}(L_{3n} + \ell_{3n}) + \ell_{1m}(L_{3n} + \ell_{3n}) + L_{3m}(L_{1n} - \ell_{1n}) + \ell_{3m}(L_{1n} - \ell_{1n}) & -L_{2m}(L_{0n} + \ell_{0n}) + \ell_{2m}(L_{0n} + \ell_{0n}) + L_{0m}(L_{2n} - \ell_{2n}) - \ell_{0m}(L_{2n} - \ell_{2n}) \\ -L_{2m}(L_{0n} - \ell_{0n}) - \ell_{2m}(L_{0n} - \ell_{0n}) + L_{0m}(L_{2n} + \ell_{2n}) - \ell_{0m}(L_{2n} + \ell_{2n}) & -L_{1m}(L_{3n} - \ell_{3n}) - \ell_{1m}(L_{3n} - \ell_{3n}) + L_{3m}(L_{1n} + \ell_{1n}) - \ell_{3m}(L_{1n} + \ell_{1n}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{1m}L_{3n} - L_{1m}\ell_{3n} + \ell_{1m}L_{3n} + \ell_{1m}\ell_{3n} + L_{3m}L_{1n} - L_{3m}\ell_{1n} + \ell_{3m}L_{1n} - \ell_{3m}\ell_{1n} & -L_{2m}L_{0n} - L_{2m}\ell_{0n} + \ell_{2m}L_{0n} + \ell_{2m}\ell_{0n} \\ -L_{2m}L_{0n} + L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + \ell_{2m}\ell_{0n} + L_{0m}L_{2n} + L_{0m}\ell_{2n} - \ell_{0m}L_{2n} - \ell_{0m}\ell_{2n} & -L_{1m}L_{3n} + L_{1m}\ell_{3n} - \ell_{1m}L_{3n} + \ell_{1m}\ell_{3n} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{1m}L_{3n} + L_{3m}L_{1n}) - L_{1m}\ell_{3n} + \ell_{1m}L_{3n} - L_{3m}\ell_{1n} + \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (-L_{2m}L_{0n} + L_{0m}L_{2n}) - L_{2m}\ell_{0n} + \ell_{2m}L_{0n} - L_{0m}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ (-L_{2m}L_{0n} + L_{0m}L_{2n}) + L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + L_{0m}\ell_{2n} - \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1m}L_{3n} + L_{3m}L_{1n}) + L_{1m}\ell_{3n} - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$\begin{aligned} &= \begin{pmatrix} (-L_{1n}L_{3n} + L_{3n}L_{1n}) - L_{1n}\ell_{3n} + \ell_{1m}L_{3n} - L_{3n}\ell_{1n} + \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (-L_{2n}L_{0n} + L_{0n}L_{2n}) - L_{2n}\ell_{0n} + \ell_{2m}L_{0n} - L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ (-L_{2n}L_{0n} + L_{0n}L_{2n}) + L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1n}L_{3n} + L_{3n}L_{1n}) + L_{1n}\ell_{3n} - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \end{pmatrix} \\ &= \begin{pmatrix} -L_{1n}\ell_{3n} + \ell_{1m}L_{3n} - L_{3n}\ell_{1n} + \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & -L_{2n}\ell_{0n} + \ell_{2m}L_{0n} - L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ +L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & -L_{2n}\ell_{0n} + \ell_{2m}L_{0n} - L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1m}L_{3n} + L_{3m}L_{1n}) + \ell_{1m}L_{3n} + \ell_{3m}L_{1n} + (-L_{1m}\ell_{3n} - L_{3m}\ell_{1n} + \ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (-L_{2m}L_{0n} + L_{0m}L_{2n}) + \ell_{2m}\ell_{0n} - \ell_{2m}L_{0n} - \ell_{0m}L_{2n} + (+L_{2m}\ell_{0n} + L_{0m}\ell_{2n} + \ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1m}L_{3n} + L_{3m}L_{1n}) - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \\ (-L_{2m}L_{0n} + L_{0m}L_{2n}) - \ell_{2m}L_{0n} - \ell_{0m}L_{2n} + (+L_{2m}\ell_{0n} + L_{0m}\ell_{2n} + \ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (-L_{1m}L_{3n} + L_{3m}L_{1n}) - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} & (-L_{1m}L_{3n} + L_{3m}L_{1n}) - \ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} \end{pmatrix}$$

□

**Lemma I.2.1.4-(BAC1r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( D_{1m}^\dagger D_{0n}^\dagger - D_{2m}^\dagger D_{3n}^\dagger + D_{3m}^\dagger D_{2n}^\dagger - D_{0m}^\dagger D_{1n}^\dagger \right) = \begin{pmatrix} (L_{1m}L_{0n} - L_{0m}L_{1n}) - L_{1m}\ell_{0n} - \ell_{1m}L_{0n} + L_{0m}\ell_{1n} + \ell_{0m}L_{1n} + (+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) & (-L_{2m}L_{3n} + L_{3m}L_{2n}) + L_{1m}\ell_{3n} - \ell_{1m}L_{3n} \\ (-L_{2m}L_{3n} + L_{3m}L_{2n}) - L_{2m}\ell_{3n} - \ell_{2m}L_{3n} + L_{3m}\ell_{2n} + \ell_{3m}L_{2n} + (-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) & (L_{1m}L_{0n} - L_{0m}L_{1n}) + L_{1m}\ell_{0n} - \ell_{1m}L_{0n} \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} -L_{1n}\ell_{0n} - \ell_{1n}L_{0n} + L_{0n}\ell_{1n} + \ell_{0n}L_{1n} + (+\ell_{1n}\ell_{0n} - \ell_{0n}\ell_{1n}) & +L_{2n}\ell_{3n} + \ell_{2n}L_{3n} - L_{3n}\ell_{2n} - \ell_{3n}L_{2n} + (-\ell_{2n}\ell_{3n} + \ell_{3n}\ell_{2n}) \\ -L_{2n}\ell_{3n} - \ell_{2n}L_{3n} + L_{3n}\ell_{2n} + \ell_{3n}L_{2n} + (-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) & +L_{1n}\ell_{0n} + \ell_{1n}L_{0n} - L_{0n}\ell_{1n} - \ell_{0n}L_{1n} + (+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{1n} - \ell_{1m})L_{0n} + (+\ell_{0m} - \ell_{0n})L_{1n} + (+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) & (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) \\ (+\ell_{3m} - \ell_{3n})L_{2n} + (+\ell_{2n} - \ell_{2m})L_{3n} + (-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) & (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1m}L_{0n} - L_{0m}L_{1n}) - \ell_{1m}L_{0n} + \ell_{0m}L_{1n} + (-L_{1m}\ell_{0n} + L_{0m}\ell_{1n} + \ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) & (-L_{2m}L_{3n} + L_{3m}L_{2n}) + L_{1m}\ell_{3n} - \ell_{1m}L_{3n} \\ (-L_{2m}L_{3n} + L_{3m}L_{2n}) - \ell_{2m}L_{3n} + \ell_{3m}L_{2n} + (-L_{2m}\ell_{3n} + L_{3m}\ell_{2n} - \ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) & (L_{1m}L_{0n} - L_{0m}L_{1n}) + L_{1m}\ell_{0n} - \ell_{1m}L_{0n} \end{pmatrix}$$

**Proof:**

$$\left( D_{1m}^\dagger D_{0n}^\dagger - D_{2m}^\dagger D_{3n}^\dagger + D_{3m}^\dagger D_{2n}^\dagger - D_{0m}^\dagger D_{1n}^\dagger \right) = \left( \begin{pmatrix} D_{1m} & 0 \\ 0 & D_{1m}^+ \end{pmatrix} \begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} D_{2m}^- & 0 \\ 0 & D_{2m}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{3m}^- & 0 \\ 0 & D_{3m}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} D_{1m}D_{0n}^- - D_{0m}^-D_{1n}^+ & -D_{2m}^-D_{3n}^+ + D_{3m}^-D_{2n}^+ \\ -D_{2m}^+D_{3n}^- + D_{3m}^+D_{2n}^- & D_{1m}^+D_{0n}^- - D_{0m}^+D_{1n}^- \end{pmatrix}$$

$$= \begin{pmatrix} (L_{1m} - \ell_{1m})(L_{0n}^- - \ell_{0n}^-) - (L_{0m} - \ell_{0m})(L_{1n}^- - \ell_{1n}^-) & -(L_{2m} - \ell_{2m})(L_{3n}^- - \ell_{3n}^-) + (L_{3m} - \ell_{3m})(L_{2n}^- - \ell_{2n}^-) \\ -(L_{2m} + \ell_{2m})(L_{3n}^- + \ell_{3n}^-) + (L_{3m} + \ell_{3m})(L_{2n}^- + \ell_{2n}^-) & (L_{1m} + \ell_{1m})(L_{0n}^- + \ell_{0n}^-) - (L_{0m} + \ell_{0m})(L_{1n}^- + \ell_{1n}^-) \end{pmatrix}$$

$$= \begin{pmatrix} (L_{1m}L_{0n}^- - L_{0m}L_{1n}^-) - L_{1m}\ell_{0n}^- - \ell_{1m}L_{0n}^- + L_{0m}\ell_{1n}^- + \ell_{0m}L_{1n}^- + (+\ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-) & (-L_{2m}L_{3n}^- + L_{3m}L_{2n}^-) + L_{1m}\ell_{3n}^- - \ell_{1m}L_{3n}^- \\ (-L_{2m}L_{3n}^- + L_{3m}L_{2n}^-) - L_{2m}\ell_{3n}^- - \ell_{2m}L_{3n}^- + L_{3m}\ell_{2n}^- + \ell_{3m}L_{2n}^- + (-\ell_{2m}\ell_{3n}^- + \ell_{3m}\ell_{2n}^-) & (L_{1m}L_{0n}^- - L_{0m}L_{1n}^-) + L_{1m}\ell_{0n}^- - \ell_{1m}L_{0n}^- \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} -L_{1n}\ell_{0n}^- - \ell_{1n}L_{0n}^- + L_{0n}\ell_{1n}^- + \ell_{0n}L_{1n}^- + (+\ell_{1n}\ell_{0n}^- - \ell_{0n}\ell_{1n}^-) & +L_{2n}\ell_{3n}^- + \ell_{2n}L_{3n}^- - L_{3n}\ell_{2n}^- - \ell_{3n}L_{2n}^- + (-\ell_{2n}\ell_{3n}^- + \ell_{3n}\ell_{2n}^-) \\ -L_{2n}\ell_{3n}^- - \ell_{2n}L_{3n}^- + L_{3n}\ell_{2n}^- + \ell_{3n}L_{2n}^- + (-\ell_{2m}\ell_{3n}^- + \ell_{3m}\ell_{2n}^-) & +L_{1n}\ell_{0n}^- + \ell_{1n}L_{0n}^- - L_{0n}\ell_{1n}^- - \ell_{0n}L_{1n}^- + (+\ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{1n} - \ell_{1m})L_{0n}^- + (+\ell_{0m} - \ell_{0n}^-)L_{1n}^- + (+\ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-) & (+\ell_{3n}^- - \ell_{3m}^-)L_{2n}^- + (\ell_{2m}^- - \ell_{2n}^-)L_{3n}^- + (-\ell_{2m}^-\ell_{3n}^- + \ell_{3m}^-\ell_{2n}^-) \\ (+\ell_{3m}^- - \ell_{3n}^-)L_{2n}^- + (+\ell_{2n}^- - \ell_{2m}^-)L_{3n}^- + (-\ell_{2m}^-\ell_{3n}^- + \ell_{3m}^-\ell_{2n}^-) & (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1m}L_{0n}^- - L_{0m}L_{1n}^-) - \ell_{1m}L_{0n}^- + \ell_{0m}L_{1n}^- + (-L_{1m}\ell_{0n}^- + L_{0m}\ell_{1n}^- + \ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-) & (-L_{2m}L_{3n}^- + L_{3m}L_{2n}^-) + L_{1m}\ell_{3n}^- - \ell_{1m}L_{3n}^- \\ (-L_{2m}L_{3n}^- + L_{3m}L_{2n}^-) - \ell_{2m}L_{3n}^- + \ell_{3m}L_{2n}^- + (-L_{2m}\ell_{3n}^- + L_{3m}\ell_{2n}^- - \ell_{2m}\ell_{3n}^- + \ell_{3m}\ell_{2n}^-) & (L_{1m}L_{0n}^- - L_{0m}L_{1n}^-) + L_{1m}\ell_{0n}^- - \ell_{1m}L_{0n}^- \end{pmatrix}$$

□

**Lemma I.2.2.1-(BAc2r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( D_{0m}D_{3n}^{\leftrightarrow} - D_{3m}D_{0n}^\dagger - D_{2m}D_{1n}^{\leftrightarrow} + D_{1m}D_{2n}^\dagger \right) = \begin{pmatrix} (-L_{2m}L_{1n} + L_{1m}L_{2n}) - L_{2m}\ell_{1n} + \ell_{2m}L_{1n} - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) - L_{0m}\ell_{3n} + \ell_{0m}L_{3n} \\ (L_{0m}L_{3n} - L_{3m}L_{0n}) + L_{0m}\ell_{3n} - \ell_{0m}L_{3n} + L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) + L_{2m}\ell_{1n} - \ell_{2m}L_{1n} \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} -L_{2n}\ell_{1n} + \ell_{2n}L_{1n} - L_{1n}\ell_{2n} + \ell_{1n}L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & -L_{0n}\ell_{3n} + \ell_{0n}L_{3n} - L_{3n}\ell_{0n} + \ell_{3n}L_{0n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \\ +L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + L_{3n}\ell_{0n} - \ell_{3m}L_{0n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & +L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + L_{1n}\ell_{2n} - \ell_{1m}L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \\ (+\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & (+\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2m}L_{1n} + L_{1m}L_{2n}) + \ell_{2m}L_{1n} + \ell_{1m}L_{2n} + (-L_{2m}\ell_{1n} - L_{1m}\ell_{2n} + \ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) + \ell_{0m}L_{3n} \\ (L_{0m}L_{3n} - L_{3m}L_{0n}) - \ell_{0m}L_{3n} - \ell_{3m}L_{0n} + (+L_{0m}\ell_{3n} + L_{3m}\ell_{0n} - \ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) - \ell_{2m}L_{1n} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{0m}D_{3n}^{\leftrightarrow} - D_{3m}^{\leftrightarrow}D_{0n}^{\dagger} - D_{2m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{1m}D_{2n}^{\dagger}) &= \left( \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} - \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{1n}^- & 0 \\ 0 & D_{1n}^+ \end{pmatrix} \right) \\ &= \begin{pmatrix} -D_{2m}^-D_{1n}^+ + D_{1m}^+D_{2n}^- & D_{0m}^+D_{3n}^- - D_{3m}^-D_{0n}^+ \\ D_{0m}^-D_{3n}^+ - D_{3m}^+D_{0n}^- & -D_{2m}^+D_{1n}^- + D_{1m}^-D_{2n}^+ \end{pmatrix} \\ &= \begin{pmatrix} -(L_{2m} - \ell_{2m})(L_{1n} + \ell_{1n}) + (L_{1m} + \ell_{1m})(L_{2n} - \ell_{2n}) & (L_{0m} + \ell_{0m})(L_{3n} - \ell_{3n}) - (L_{3m} - \ell_{3m})(L_{0n} + \ell_{0n}) \\ (L_{0m} - \ell_{0m})(L_{3n} + \ell_{3n}) - (L_{3m} + \ell_{3m})(L_{0n} - \ell_{0n}) & -(L_{2m} + \ell_{2m})(L_{1n} - \ell_{1n}) + (L_{1m} - \ell_{1m})(L_{2n} + \ell_{2n}) \end{pmatrix} \\ &= \begin{pmatrix} (-L_{2m}L_{1n} + L_{1m}L_{2n}) - L_{2m}\ell_{1n} + \ell_{2m}L_{1n} - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) - L_{0m}\ell_{3n} \\ (L_{0m}L_{3n} - L_{3m}L_{0n}) + L_{0m}\ell_{3n} - \ell_{0m}L_{3n} + L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) + L_{2m}\ell_{1n} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} -L_{2n}\ell_{1n} + \ell_{2m}L_{1n} - L_{1n}\ell_{2n} + \ell_{1m}L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & -L_{0n}\ell_{3n} + \ell_{0m}L_{3n} - L_{3n}\ell_{0n} + \ell_{3m}L_{0n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \\ +L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + L_{3n}\ell_{0n} - \ell_{3m}L_{0n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & +L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + L_{1n}\ell_{2n} - \ell_{1m}L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \\ (+\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & (+\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2m}L_{1n} + L_{1m}L_{2n}) + \ell_{2m}L_{1n} + \ell_{1m}L_{2n} + (-L_{2m}\ell_{1n} - L_{1m}\ell_{2n} + \ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) + \ell_{0m}L_{3n} \\ (L_{0m}L_{3n} - L_{3m}L_{0n}) - \ell_{0m}L_{3n} - \ell_{3m}L_{0n} + (+L_{0m}\ell_{3n} + L_{3m}\ell_{0n} - \ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) - \ell_{2m}L_{1n} \end{pmatrix}$$

□

**Lemma I.2.2.2-(BAc2r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\dagger} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{3m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{0m}D_{0n}^{\dagger} + D_{1m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{2m}D_{2n}^{\dagger}) = \begin{pmatrix} (L_{3m}L_{3n} + L_{0m}L_{0n} + L_{1m}L_{1n} + L_{2m}L_{2n}) + L_{3m}\ell_{3n} - \ell_{3m}L_{3n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - L_{2m}\ell_{2n} + \ell_{2m}L_{2n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + L_{3n}\ell_{3n} - \ell_{3m}L_{3n} - L_{0n}\ell_{0n} + \ell_{0m}L_{0n} + L_{1n}\ell_{1n} - \ell_{1m}L_{1n} - L_{2n}\ell_{2n} + \ell_{2m}L_{2n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} - \ell_{3m}\ell_{3n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) - (\ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n} + \ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n}) \\ 0 \end{pmatrix} \quad (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{3m}L_{3n} + L_{0m}L_{0n} + L_{1m}L_{1n} + L_{2m}L_{2n}) + (-\ell_{3m}L_{3n} + \ell_{0m}L_{0n} - \ell_{1m}L_{1n} + \ell_{2m}L_{2n}) + ([L_{3m}\ell_{3n} - L_{0m}\ell_{0n} + L_{1m}\ell_{1n} - L_{2m}\ell_{2n}]) \\ 0 \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{3m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{0m}D_{0n}^{\dagger} + D_{1m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{2m}D_{2n}^{\dagger}) &= \left( \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{1n}^- & 0 \\ 0 & D_{1n}^+ \end{pmatrix} \right) \\ &= \begin{pmatrix} D_{3m}^-D_{3n}^+ + D_{0m}^+D_{0n}^- + D_{1m}^-D_{1n}^+ + D_{2m}^+D_{2n}^- & 0 \\ 0 & D_{3m}^+D_{3n}^- + D_{0m}^-D_{0n}^+ + D_{1m}^+D_{1n}^- + D_{2m}^-D_{2n}^+ \end{pmatrix} \\ &= \begin{pmatrix} (L_{3m} - \ell_{3m})(L_{3n} + \ell_{3n}) + (L_{0m} + \ell_{0m})(L_{0n} - \ell_{0n}) + (L_{1m} - \ell_{1m})(L_{1n} + \ell_{1n}) + (L_{2m} + \ell_{2m})(L_{2n} - \ell_{2n}) & 0 \\ 0 & (L_{3m} + \ell_{3m})(L_{3n} - \ell_{3n}) + (L_{0m} - \ell_{0m})(L_{0n} + \ell_{0n}) + (L_{1m} + \ell_{1m})(L_{1n} - \ell_{1n}) + (L_{2m} - \ell_{2m})(L_{2n} + \ell_{2n}) \end{pmatrix} \\ &= \begin{pmatrix} L_{3m}L_{3n} + L_{3m}\ell_{3n} - \ell_{3m}L_{3n} - \ell_{3m}\ell_{3n} + L_{0m}L_{0n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - \ell_{0m}\ell_{0n} + L_{1m}L_{1n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - \ell_{1m}\ell_{1n} \\ 0 \end{pmatrix} \end{aligned}$$



$$\begin{aligned}
&= \begin{pmatrix} (L_{3m}L_{3n} + L_{0m}L_{0n} + L_{1m}L_{1n} + L_{2m}L_{2n}) + L_{3m}\ell_{3n} - \ell_{3m}L_{3n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - L_{2m}\ell_{2n} + \ell_{2m}L_{2n} \\ 0 \end{pmatrix} \\
L_{jm} = L_{jn} : & \\
&= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + L_{3n}\ell_{3n} - \ell_{3m}L_{3n} - L_{0n}\ell_{0n} + \ell_{0m}L_{0n} + L_{1n}\ell_{1n} - \ell_{1m}L_{1n} - L_{2n}\ell_{2n} + \ell_{2m}L_{2n} \\ 0 \end{pmatrix} \\
L_{jm} = L_{jn} \text{ are linear/differential } \& \ \ell_{ij} \text{ are constants:} \\
&= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} \\ 0 \end{pmatrix} \\
L_{jm} = L_{jn} \text{ are linear/differential } \& \ \ell_{ij} \text{ are constants } \& \ \ell_{jm} = \ell_{jn} : \\
&= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) - (\ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n} + \ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n}) \\ 0 \end{pmatrix} \quad (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) \\
L_{jn} \text{ are functions or differential operators } \& \ \ell_{ij} \text{ are non-constant functions:} \\
&= \begin{pmatrix} (L_{3m}L_{3n} + L_{0m}L_{0n} + L_{1m}L_{1n} + L_{2m}L_{2n}) + (-\ell_{3m}L_{3n} + \ell_{0m}L_{0n} - \ell_{1m}L_{1n} + \ell_{2m}L_{2n}) + ([L_{3m}\ell_{3n} - L_{0m}\ell_{0n} + L_{1m}\ell_{1n} - L_{2m}\ell_{2n}]) \\ 0 \end{pmatrix} \\
\square
\end{aligned}$$

**Lemma 1.2.2.3-(BAc2r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$  and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \ , \ D_{ij}^- \equiv (L_{ij} - \ell_{ij}) \\
D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \ , \ D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(-D_{2m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{1m}^{\leftrightarrow}D_{0n}^\dagger - D_{0m}D_{1n}^{\leftrightarrow} + D_{3m}D_{2n}^\dagger) = \begin{pmatrix} (-L_{2m}L_{3n} + L_{3m}L_{2n}) - L_{2m}\ell_{3n} + \ell_{2m}L_{3n} - L_{3m}\ell_{2n} + \ell_{3m}L_{2n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & (L_{1m}L_{0n} - L_{0m}L_{1n}) + L_{1m}\ell_{0n} - \ell_{1m}L_{0n} \\ (L_{1m}L_{0n} - L_{0m}L_{1n}) - L_{1m}\ell_{0n} + \ell_{1m}L_{0n} - L_{0m}\ell_{1n} + \ell_{0m}L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & (-L_{2m}L_{3n} + L_{3m}L_{2n}) + L_{2m}\ell_{3n} - \ell_{2m}L_{3n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{2n} + (+\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (+\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \\ (+\ell_{1m} - \ell_{1n})L_{0n} + (+\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & (+\ell_{3n} - \ell_{3m})L_{2n} + (+\ell_{2n} - \ell_{2m})L_{3n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$ :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2m}L_{3n} + L_{3m}L_{2n}) + \ell_{2m}L_{3n} + \ell_{3m}L_{2n} + (-L_{2m}\ell_{3n} - L_{3m}\ell_{2n} + \ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & (L_{1m}L_{0n} - L_{0m}L_{1n}) - \ell_{1m}L_{0n} + \ell_{1m}\ell_{0n} \\ (L_{1m}L_{0n} - L_{0m}L_{1n}) + \ell_{1m}L_{0n} + \ell_{0m}L_{1n} + (-L_{1m}\ell_{0n} - L_{0m}\ell_{1n} - \ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & (-L_{2m}L_{3n} + L_{3m}L_{2n}) - \ell_{2m}L_{3n} + \ell_{2m}\ell_{3n} \end{pmatrix}$$

**Proof:**

$$\begin{aligned}
(-D_{2m}^{\leftrightarrow}D_{3n}^{\leftrightarrow} + D_{1m}^{\leftrightarrow}D_{0n}^\dagger - D_{0m}D_{1n}^{\leftrightarrow} + D_{3m}D_{2n}^\dagger) &= \left( -\begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \right) \\
&= \begin{pmatrix} -D_{2m}^-D_{3n}^+ + D_{3m}^+D_{2n}^- & D_{1m}^-D_{0n}^+ - D_{0m}^+D_{1n}^- \\ D_{1m}^+D_{0n}^- - D_{0m}^-D_{1n}^+ & -D_{2m}^+D_{3n}^- + D_{3m}^-D_{2n}^+ \end{pmatrix} \\
&= \begin{pmatrix} -(L_{2m} - \ell_{2m})(L_{3n} + \ell_{3n}) + (L_{3m} + \ell_{3m})(L_{2n} - \ell_{2n}) & (L_{1m} - \ell_{1m})(L_{0n} + \ell_{0n}) - (L_{0m} + \ell_{0m})(L_{1n} - \ell_{1n}) \\ (L_{1m} + \ell_{1m})(L_{0n} - \ell_{0n}) - (L_{0m} - \ell_{0m})(L_{1n} + \ell_{1n}) & -(L_{2m} + \ell_{2m})(L_{3n} - \ell_{3n}) + (L_{3m} - \ell_{3m})(L_{2n} + \ell_{2n}) \end{pmatrix} \\
&= \begin{pmatrix} -L_{2m}(L_{3n} + \ell_{3n}) + \ell_{2m}(L_{3n} + \ell_{3n}) + L_{3m}(L_{2n} - \ell_{2n}) + \ell_{3m}(L_{2n} - \ell_{2n}) & L_{1m}(L_{0n} + \ell_{0n}) - \ell_{1m}(L_{0n} + \ell_{0n}) - L_{0m} \\ L_{1m}(L_{0n} - \ell_{0n}) + \ell_{1m}(L_{0n} - \ell_{0n}) - L_{0m}(L_{1n} + \ell_{1n}) + \ell_{0m}(L_{1n} + \ell_{1n}) & -L_{2m}(L_{3n} - \ell_{3n}) - \ell_{2m}(L_{3n} - \ell_{3n}) + L_{3m} \end{pmatrix} \\
&= \begin{pmatrix} -L_{2m}L_{3n} - L_{2m}\ell_{3n} + \ell_{2m}L_{3n} + \ell_{2m}\ell_{3n} + L_{3m}L_{2n} - L_{3m}\ell_{2n} + \ell_{3m}L_{2n} - \ell_{3m}\ell_{2n} & L_{1m}L_{0n} + L_{1m}\ell_{0n} - \ell_{1m}L_{0n} - \ell_{1m}\ell_{0n} \\ L_{1m}L_{0n} - L_{1m}\ell_{0n} + \ell_{1m}L_{0n} - \ell_{1m}\ell_{0n} - L_{0m}L_{1n} - L_{0m}\ell_{1n} + \ell_{0m}L_{1n} + \ell_{0m}\ell_{1n} & -L_{2m}L_{3n} + L_{2m}\ell_{3n} - \ell_{2m}L_{3n} + \ell_{2m}\ell_{3n} \end{pmatrix} \\
&= \begin{pmatrix} (-L_{2m}L_{3n} + L_{3m}L_{2n}) - L_{2m}\ell_{3n} + \ell_{2m}L_{3n} - L_{3m}\ell_{2n} + \ell_{3m}L_{2n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & (L_{1m}L_{0n} - L_{0m}L_{1n}) + L_{1m}\ell_{0n} - \ell_{1m}L_{0n} \\ (L_{1m}L_{0n} - L_{0m}L_{1n}) - L_{1m}\ell_{0n} + \ell_{1m}L_{0n} - L_{0m}\ell_{1n} + \ell_{0m}L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & (-L_{2m}L_{3n} + L_{3m}L_{2n}) + L_{2m}\ell_{3n} - \ell_{2m}L_{3n} \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$ :

$$\begin{aligned}
&= \begin{pmatrix} (-L_{2n}L_{3n} + L_{3n}L_{2n}) - L_{2n}\ell_{3n} + \ell_{2m}L_{3n} - L_{3n}\ell_{2n} + \ell_{3m}L_{2n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & (L_{1n}L_{0n} - L_{0n}L_{1n}) + L_{1n}\ell_{0n} - \ell_{1n}L_{0n} \\ (L_{1n}L_{0n} - L_{0n}L_{1n}) - L_{1n}\ell_{0n} + \ell_{1m}L_{0n} - L_{0n}\ell_{1n} + \ell_{0m}L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & (-L_{2n}L_{3n} + L_{3n}L_{2n}) + L_{2n}\ell_{3n} - \ell_{2n}L_{3n} \end{pmatrix} \\
&= \begin{pmatrix} -L_{2n}\ell_{3n} + \ell_{2m}L_{3n} - L_{3n}\ell_{2n} + \ell_{3m}L_{2n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & +L_{1n}\ell_{0n} - \ell_{1m}L_{0n} + L_{0n}\ell_{1n} - \ell_{0m}L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \\ -L_{1n}\ell_{0n} + \ell_{1m}L_{0n} - L_{0n}\ell_{1n} + \ell_{0m}L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & +L_{2n}\ell_{3n} - \ell_{2m}L_{3n} + L_{3n}\ell_{2n} - \ell_{3m}L_{2n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{2n} + (+\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (+\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \\ (+\ell_{1m} - \ell_{1n})L_{0n} + (+\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & (+\ell_{3n} - \ell_{3m})L_{2n} + (+\ell_{2n} - \ell_{2m})L_{3n} + (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$ :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2m}L_{3n} + L_{3m}L_{2n}) + \ell_{2m}L_{3n} + \ell_{3m}L_{2n} + (-L_{2m}\ell_{3n} - L_{3m}\ell_{2n} + \ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & (L_{1m}L_{0n} - L_{0m}L_{1n}) - \ell_{1m}L_{0n} \\ (L_{1m}L_{0n} - L_{0m}L_{1n}) + \ell_{1m}L_{0n} + \ell_{0m}L_{1n} + (-L_{1m}\ell_{0n} - L_{0m}\ell_{1n} - \ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & (-L_{2m}L_{3n} + L_{3m}L_{2n}) - \ell_{2m}L_{3n} \end{pmatrix}$$

□

**Lemma I.2.2.4-(BAc2r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{1m}^\dagger D_{3n}^{\leftrightarrow} + D_{2m}^\dagger D_{0n}^\dagger - D_{3m}^\dagger D_{1n}^{\leftrightarrow} - D_{0m}^\dagger D_{2n}^\dagger) = \begin{pmatrix} (L_{2m}L_{0n} - L_{0m}L_{2n}) - L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + L_{0m}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (L_{1m}L_{3n} - L_{3m}L_{1n}) - L_{1m}\ell_{3n} - \ell_{1m}L_{3n} \\ (L_{1m}L_{3n} - L_{3m}L_{1n}) + L_{1m}\ell_{3n} + \ell_{1m}L_{3n} - L_{3m}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (L_{2m}L_{0n} - L_{0m}L_{2n}) + L_{2m}\ell_{0n} + \ell_{0m}L_{2n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} -L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & -L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + L_{3n}\ell_{1n} + \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \\ +L_{1n}\ell_{3n} + \ell_{1m}L_{3n} - L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +L_{2n}\ell_{0n} + \ell_{2m}L_{0n} - L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \\ (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{2m}L_{0n} - L_{0m}L_{2n}) - \ell_{2m}L_{0n} + \ell_{0m}L_{2n} + (-L_{2m}\ell_{0n} + L_{0m}\ell_{2n} + \ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (L_{1m}L_{3n} - L_{3m}L_{1n}) - \ell_{1m}L_{3n} \\ (L_{1m}L_{3n} - L_{3m}L_{1n}) + \ell_{1m}L_{3n} - \ell_{3m}L_{1n} + (+L_{1m}\ell_{3n} - L_{3m}\ell_{1n} + \ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (L_{2m}L_{0n} - L_{0m}L_{2n}) + \ell_{2m}L_{0n} \end{pmatrix}$$

**Proof:**

$$(D_{1m}^\dagger D_{3n}^{\leftrightarrow} + D_{2m}^\dagger D_{0n}^\dagger - D_{3m}^\dagger D_{1n}^{\leftrightarrow} - D_{0m}^\dagger D_{2n}^\dagger) = \left( \begin{pmatrix} D_{1m}^- & 0 \\ 0 & D_{1m}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{2m}^- & 0 \\ 0 & D_{2m}^+ \end{pmatrix} \begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} - \begin{pmatrix} D_{3m}^- & 0 \\ 0 & D_{3m}^+ \end{pmatrix} \begin{pmatrix} D_{1n}^- & 0 \\ 0 & D_{1n}^+ \end{pmatrix} \right) \begin{pmatrix} D_{2m}^- D_{0n}^- - D_{0m}^- D_{2n}^- & D_{1m}^- D_{3n}^- - D_{3m}^- D_{1n}^- \\ D_{1m}^+ D_{3n}^+ - D_{3m}^+ D_{1n}^+ & D_{2m}^+ D_{0n}^+ - D_{0m}^+ D_{2n}^+ \end{pmatrix}$$

$$= \begin{pmatrix} (L_{2m} - \ell_{2m})(L_{0n} - \ell_{0n}) - (L_{0m} - \ell_{0m})(L_{2n} - \ell_{2n}) & (L_{1m} - \ell_{1m})(L_{3n} - \ell_{3n}) - (L_{3m} - \ell_{3m})(L_{1n} - \ell_{1n}) \\ (L_{1m} + \ell_{1m})(L_{3n} + \ell_{3n}) - (L_{3m} + \ell_{3m})(L_{1n} + \ell_{1n}) & (L_{2m} + \ell_{2m})(L_{0n} + \ell_{0n}) - (L_{0m} + \ell_{0m})(L_{2n} + \ell_{2n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{2m}(L_{0n} - \ell_{0n}) - \ell_{2m}(L_{0n} - \ell_{0n}) - L_{0m}(L_{2n} - \ell_{2n}) + \ell_{0m}(L_{2n} - \ell_{2n}) & L_{1m}(L_{3n} - \ell_{3n}) - \ell_{1m}(L_{3n} - \ell_{3n}) - L_{3m}(L_{1n} - \ell_{1n}) \\ L_{1m}(L_{3n} + \ell_{3n}) + \ell_{1m}(L_{3n} + \ell_{3n}) - L_{3m}(L_{1n} + \ell_{1n}) - \ell_{3m}(L_{1n} + \ell_{1n}) & L_{2m}(L_{0n} + \ell_{0n}) + \ell_{2m}(L_{0n} + \ell_{0n}) - L_{0m}(L_{2n} + \ell_{2n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{2m}L_{0n} - L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + \ell_{0m}L_{2n} + L_{0m}\ell_{2n} + \ell_{0m}L_{2n} - \ell_{0m}\ell_{2n} & L_{1m}L_{3n} - L_{1m}\ell_{3n} - \ell_{1m}L_{3n} + \ell_{1m}\ell_{3n} \\ L_{1m}L_{3n} + L_{1m}\ell_{3n} + \ell_{1m}L_{3n} + \ell_{1m}\ell_{3n} - L_{3m}L_{1n} - L_{3m}\ell_{1n} - \ell_{3m}L_{1n} - \ell_{3m}\ell_{1n} & L_{2m}L_{0n} + L_{2m}\ell_{0n} + \ell_{2m}L_{0n} + \ell_{2m}\ell_{0n} \end{pmatrix}$$

$$= \begin{pmatrix} (L_{2m}L_{0n} - L_{0m}L_{2n}) - L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + L_{0m}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (L_{1m}L_{3n} - L_{3m}L_{1n}) - L_{1m}\ell_{3n} - \ell_{1m}L_{3n} \\ (L_{1m}L_{3n} - L_{3m}L_{1n}) + L_{1m}\ell_{3n} + \ell_{1m}L_{3n} - L_{3m}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (L_{2m}L_{0n} - L_{0m}L_{2n}) + L_{2m}\ell_{0n} + \ell_{0m}L_{2n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{0n} - L_{0n}L_{2n}) - L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (L_{1n}L_{3n} - L_{3n}L_{1n}) - L_{1n}\ell_{3n} - \ell_{1m}L_{3n} \\ (L_{1n}L_{3n} - L_{3n}L_{1n}) + L_{1n}\ell_{3n} + \ell_{1m}L_{3n} - L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (L_{2n}L_{0n} - L_{0n}L_{2n}) + L_{2n}\ell_{0n} + \ell_{0m}L_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} -L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + L_{0n}\ell_{2n} + \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & -L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + L_{3n}\ell_{1n} + \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \\ +L_{1n}\ell_{3n} + \ell_{1m}L_{3n} - L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +L_{2n}\ell_{0n} + \ell_{2m}L_{0n} - L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \\ (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{2m}L_{0n} - L_{0m}L_{2n}) - \ell_{2m}L_{0n} + \ell_{0m}L_{2n} + (-L_{2m}\ell_{0n} + L_{0m}\ell_{2n} + \ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & (L_{1m}L_{3n} - L_{3m}L_{1n}) - \ell_{1m}L_{3n} \\ (L_{1m}L_{3n} - L_{3m}L_{1n}) + \ell_{1m}L_{3n} - \ell_{3m}L_{1n} + (+L_{1m}\ell_{3n} - L_{3m}\ell_{1n} + \ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & (L_{2m}L_{0n} - L_{0m}L_{2n}) + \ell_{2m}L_{0n} \end{pmatrix}$$

□

**Lemma I.2.3.1-(BAc3r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\uparrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left(-D_{0m}D_{2n}^{\leftrightarrow} - D_{3m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{2m}^{\leftrightarrow}D_{0n}^\uparrow + D_{1m}D_{3n}^\uparrow\right) = \begin{pmatrix} (-L_{3m}L_{1n} + L_{1m}L_{3n}) - L_{3m}\ell_{1n} + \ell_{3m}L_{1n} - L_{1m}\ell_{3n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (-L_{0m}L_{2n} + L_{2m}L_{0n}) + L_{0m}\ell_{2n} - \ell_{0m}L_{2n} + L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ (-L_{0m}L_{2n} + L_{2m}L_{0n}) - L_{0m}\ell_{2n} + \ell_{0m}L_{2n} - L_{2m}\ell_{0n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (-L_{3m}L_{1n} + L_{1m}L_{3n}) + L_{3m}\ell_{1n} - \ell_{3m}L_{1n} + L_{1n}\ell_{3m} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} -L_{3n}\ell_{1n} + \ell_{3m}L_{1n} - L_{1n}\ell_{3n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & +L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ -L_{0n}\ell_{2n} + \ell_{0m}L_{2n} - L_{2n}\ell_{0n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & +L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3m}L_{1n} + L_{1m}L_{3n}) + \ell_{3m}L_{1n} + \ell_{1m}L_{3n} + (-L_{3m}\ell_{1n} - L_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (-L_{0m}L_{2n} + L_{2m}L_{0n}) - \ell_{0m}L_{2n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ (-L_{0m}L_{2n} + L_{2m}L_{0n}) + \ell_{0m}L_{2n} + \ell_{2m}L_{0n} + (-L_{0m}\ell_{2n} - L_{2m}\ell_{0n} + \ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (-L_{3m}L_{1n} + L_{1m}L_{3n}) - \ell_{3m}L_{1n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

**Proof:**

$$\begin{aligned} \left(-D_{0m}D_{2n}^{\leftrightarrow} - D_{3m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{2m}^{\leftrightarrow}D_{0n}^\uparrow + D_{1m}D_{3n}^\uparrow\right) &= \left(-\begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix}\begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} - \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{0n}^\uparrow \\ D_{0n}^\uparrow & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^\uparrow \\ D_{1m}^\uparrow & 0 \end{pmatrix}\begin{pmatrix} 0 & D_{3n}^\uparrow \\ D_{3n}^\uparrow & 0 \end{pmatrix}\right) \\ &= \begin{pmatrix} -D_{3m}^-D_{1n}^+ + D_{1m}^+D_{3n}^- & -D_{0m}^+D_{2n}^- + D_{2m}^-D_{0n}^\uparrow \\ -D_{0m}^-D_{2n}^+ + D_{2m}^+D_{0n}^\uparrow & -D_{3m}^+D_{1n}^- + D_{1m}^-D_{3n}^\uparrow \end{pmatrix} \\ &= \begin{pmatrix} -(L_{3m} - \ell_{3m})(L_{1n} + \ell_{1n}) + (L_{1m} + \ell_{1m})(L_{3n} - \ell_{3n}) & -(L_{0m} + \ell_{0m})(L_{2n} - \ell_{2n}) + (L_{2m} - \ell_{2m})(L_{0n} + \ell_{0n}) \\ -(L_{0m} - \ell_{0m})(L_{2n} + \ell_{2n}) + (L_{2m} + \ell_{2m})(L_{0n} - \ell_{0n}) & -(L_{3m} + \ell_{3m})(L_{1n} - \ell_{1n}) + (L_{1m} - \ell_{1m})(L_{3n} + \ell_{3n}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{3m}(L_{1n} + \ell_{1n}) + \ell_{3m}(L_{1n} + \ell_{1n}) + L_{1m}(L_{3n} - \ell_{3n}) + \ell_{1m}(L_{3n} - \ell_{3n}) & -L_{0m}(L_{2n} - \ell_{2n}) - \ell_{0m}(L_{2n} - \ell_{2n}) + L_{2m}(L_{0n} + \ell_{0n}) - \ell_{2m}(L_{0n} + \ell_{0n}) \\ -L_{0m}(L_{2n} + \ell_{2n}) + \ell_{0m}(L_{2n} + \ell_{2n}) + L_{2m}(L_{0n} - \ell_{0n}) + \ell_{2m}(L_{0n} - \ell_{0n}) & -L_{3m}(L_{1n} - \ell_{1n}) - \ell_{3m}(L_{1n} - \ell_{1n}) + L_{1m}(L_{3n} + \ell_{3n}) - \ell_{1m}(L_{3n} + \ell_{3n}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{3m}L_{1n} - L_{3m}\ell_{1n} + \ell_{3m}L_{1n} + \ell_{3m}\ell_{1n} + L_{1m}L_{3n} - L_{1m}\ell_{3n} + \ell_{1m}L_{3n} - \ell_{1m}\ell_{3n} & -L_{0m}L_{2n} + L_{0m}\ell_{2n} - \ell_{0m}L_{2n} + \ell_{0m}\ell_{2n} \\ -L_{0m}L_{2n} - L_{0m}\ell_{2n} + \ell_{0m}L_{2n} + \ell_{0m}\ell_{2n} + L_{2m}L_{0n} - L_{2m}\ell_{0n} + \ell_{2m}L_{0n} - \ell_{2m}\ell_{0n} & -L_{3m}L_{1n} + L_{3m}\ell_{1n} - \ell_{3m}L_{1n} + \ell_{3m}\ell_{1n} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{3m}L_{1n} + L_{1m}L_{3n}) - L_{3m}\ell_{1n} + \ell_{3m}L_{1n} - L_{1m}\ell_{3n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (-L_{0m}L_{2n} + L_{2m}L_{0n}) + L_{0m}\ell_{2n} - \ell_{0m}L_{2n} + L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ (-L_{0m}L_{2n} + L_{2m}L_{0n}) - L_{0m}\ell_{2n} + \ell_{0m}L_{2n} - L_{2m}\ell_{0n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (-L_{3m}L_{1n} + L_{1m}L_{3n}) + L_{3m}\ell_{1n} - \ell_{3m}L_{1n} + L_{1n}\ell_{3m} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix} \end{aligned}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (-L_{3n}L_{1n} + L_{1n}L_{3n}) - L_{3n}\ell_{1n} + \ell_{3m}L_{1n} - L_{1n}\ell_{3n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (-L_{0n}L_{2n} + L_{2n}L_{0n}) + L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ (-L_{0n}L_{2n} + L_{2n}L_{0n}) - L_{0n}\ell_{2n} + \ell_{0m}L_{2n} - L_{2n}\ell_{0n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (-L_{3n}L_{1n} + L_{1n}L_{3n}) + L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + L_{1n}\ell_{3m} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

$$= \begin{pmatrix} -L_{3n}\ell_{1n} + \ell_{3m}L_{1n} - L_{1n}\ell_{3n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & +L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ -L_{0n}\ell_{2n} + \ell_{0m}L_{2n} - L_{2n}\ell_{0n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & +L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3m}L_{1n} + L_{1m}L_{3n}) + \ell_{3m}L_{1n} + \ell_{1m}L_{3n} + (-L_{3m}\ell_{1n} - L_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (-L_{0m}L_{2n} + L_{2m}L_{0n}) - \ell_{0m}L_{2n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \\ (-L_{0m}L_{2n} + L_{2m}L_{0n}) + \ell_{0m}L_{2n} + \ell_{2m}L_{0n} + (-L_{0m}\ell_{2n} - L_{2m}\ell_{0n} + \ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (-L_{3m}L_{1n} + L_{1m}L_{3n}) - \ell_{3m}L_{1n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{pmatrix}$$

□

**Lemma 1.2.3.2-(BAc3r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\uparrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left(-D_{3m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{0m}D_{1n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{0n}^\uparrow + D_{2m}D_{3n}^\uparrow\right) = \begin{pmatrix} (-L_{3m}L_{2n} + L_{2m}L_{3n}) - L_{3m}\ell_{2n} + \ell_{3m}L_{2n} - L_{2m}\ell_{3n} + \ell_{2m}L_{3n} + (\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) - L_{0m}\ell_{1n} + \ell_{0m}L_{1n} - L_{1n}\ell_{0m} + \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \\ (L_{0m}L_{1n} - L_{1m}L_{0n}) + L_{0m}\ell_{1n} - \ell_{0m}L_{1n} + L_{1n}\ell_{0m} - \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) + L_{3m}\ell_{2n} - \ell_{3m}L_{2n} + L_{2n}\ell_{3m} - \ell_{2m}L_{3n} + (\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} -L_{3n}\ell_{2n} + \ell_{3m}L_{2n} - L_{2n}\ell_{3n} + \ell_{2m}L_{3n} + (\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & -L_{0n}\ell_{1n} + \ell_{0m}L_{1n} - L_{1n}\ell_{0n} + \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \\ +L_{0n}\ell_{1n} - \ell_{0m}L_{1n} + L_{1n}\ell_{0m} - \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & +L_{3n}\ell_{2n} - \ell_{3m}L_{2n} + L_{2n}\ell_{3m} - \ell_{2m}L_{3n} + (\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \\ (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & (+\ell_{2n} - \ell_{3m})L_{2n} + (\ell_{3n} - \ell_{2m})L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3m}L_{2n} + L_{2m}L_{3n}) + \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + (-L_{3m}\ell_{2n} - L_{2m}\ell_{3n} + \ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) + \ell_{0m}L_{1n} \\ (L_{0m}L_{1n} - L_{1m}L_{0n}) - \ell_{0m}L_{1n} - \ell_{1m}L_{0n} + (+L_{0m}\ell_{1n} + L_{1m}\ell_{0n} - \ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) - \ell_{3m}L_{2n} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (-D_{3m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{0m}D_{1n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{0n}^{\downarrow} + D_{2m}D_{3n}^{\downarrow}) &= \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix} - \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix} \\ &= \begin{pmatrix} -D_{3m}^-D_{2n}^+ + D_{2m}^+D_{3n}^- & D_{0m}^+D_{1n}^- - D_{1m}^-D_{0n}^+ \\ D_{0m}^-D_{1n}^+ - D_{1m}^+D_{0n}^- & -D_{3m}^+D_{2n}^- + D_{2m}^-D_{3n}^+ \end{pmatrix} \\ &= \begin{pmatrix} -(L_{3m} - \ell_{3m})(L_{2n} + \ell_{2n}) + (L_{2m} + \ell_{2m})(L_{3n} - \ell_{3n}) & (L_{0m} + \ell_{0m})(L_{1n} - \ell_{1n}) - (L_{1m} - \ell_{1m})(L_{0n} + \ell_{0n}) \\ (L_{0m} - \ell_{0m})(L_{1n} + \ell_{1n}) - (L_{1m} + \ell_{1m})(L_{0n} - \ell_{0n}) & -(L_{3m} + \ell_{3m})(L_{2n} - \ell_{2n}) + (L_{2m} - \ell_{2m})(L_{3n} + \ell_{3n}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{3m}(L_{2n} + \ell_{2n}) + \ell_{3m}(L_{2n} + \ell_{2n}) + L_{2m}(L_{3n} - \ell_{3n}) + \ell_{2m}(L_{3n} - \ell_{3n}) & L_{0m}(L_{1n} - \ell_{1n}) + \ell_{0m}(L_{1n} - \ell_{1n}) - L_{1m} \\ L_{0m}(L_{1n} + \ell_{1n}) - \ell_{0m}(L_{1n} + \ell_{1n}) - L_{1m}(L_{0n} - \ell_{0n}) - \ell_{1m}(L_{0n} - \ell_{0n}) & -L_{3m}(L_{2n} - \ell_{2n}) - \ell_{3m}(L_{2n} - \ell_{2n}) + L_{2m} \end{pmatrix} \\ &= \begin{pmatrix} -L_{3m}L_{2n} - L_{3m}\ell_{2n} + \ell_{3m}L_{2n} + \ell_{3m}\ell_{2n} + L_{2m}L_{3n} - L_{2m}\ell_{3n} + \ell_{2m}L_{3n} - \ell_{2m}\ell_{3n} & L_{0m}L_{1n} - L_{0m}\ell_{1n} + \ell_{0m}L_{1n} - \ell_{0m}\ell_{1n} \\ L_{0m}L_{1n} + L_{0m}\ell_{1n} - \ell_{0m}L_{1n} - \ell_{0m}\ell_{1n} - L_{1m}L_{0n} + L_{1m}\ell_{0n} - \ell_{1m}L_{0n} + \ell_{1m}\ell_{0n} & -L_{3m}L_{2n} + L_{3m}\ell_{2n} - \ell_{3m}L_{2n} + \ell_{3m}\ell_{2n} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{3m}L_{2n} + L_{2m}L_{3n}) - L_{3m}\ell_{2n} + \ell_{3m}L_{2n} - L_{2m}\ell_{3n} + \ell_{2m}L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) - L_{0m}\ell_{1n} \\ (L_{0m}L_{1n} - L_{1m}L_{0n}) + L_{0m}\ell_{1n} - \ell_{0m}L_{1n} + L_{1m}\ell_{0n} - \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) + L_{3m}\ell_{2n} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$\begin{aligned} &= \begin{pmatrix} (-L_{3n}L_{2n} + L_{2n}L_{3n}) - L_{3n}\ell_{2n} + \ell_{3m}L_{2n} - L_{2n}\ell_{3n} + \ell_{2m}L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & (L_{0n}L_{1n} - L_{1n}L_{0n}) - L_{0n}\ell_{1n} \\ (L_{0n}L_{1n} - L_{1n}L_{0n}) + L_{0n}\ell_{1n} - \ell_{0m}L_{1n} + L_{1n}\ell_{0n} - \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & (-L_{3n}L_{2n} + L_{2n}L_{3n}) + L_{3n}\ell_{2n} \end{pmatrix} \\ &= \begin{pmatrix} -L_{3n}\ell_{2n} + \ell_{3m}L_{2n} - L_{2n}\ell_{3n} + \ell_{2m}L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & -L_{0n}\ell_{1n} + \ell_{0m}L_{1n} - L_{1n}\ell_{0n} + \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \\ +L_{0n}\ell_{1n} - \ell_{0m}L_{1n} + L_{1n}\ell_{0n} - \ell_{1m}L_{0n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & +L_{3n}\ell_{2n} - \ell_{3m}L_{2n} + L_{2n}\ell_{3n} - \ell_{2m}L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \\ (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & (+\ell_{2n} - \ell_{3m})L_{2n} + (\ell_{3n} - \ell_{2m})L_{3n} + (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3m}L_{2n} + L_{2m}L_{3n}) + \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + (-L_{3m}\ell_{2n} - L_{2m}\ell_{3n} + \ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) + \ell_{0m}L_{1n} \\ (L_{0m}L_{1n} - L_{1m}L_{0n}) - \ell_{0m}L_{1n} - \ell_{1m}L_{0n} + (+L_{0m}\ell_{1n} + L_{1m}\ell_{0n} - \ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) - \ell_{3m}L_{2n} \end{pmatrix}$$

□

**Lemma I.2.3.3-(Bac3r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\downarrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\downarrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( D_{2m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{1m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{0m}D_{0n}^{\downarrow} + D_{3m}D_{3n}^{\downarrow} \right) = \begin{pmatrix} (L_{2m}L_{2n} + L_{1m}L_{1n} + L_{0m}L_{0n} + L_{3m}L_{3n}) + L_{2m}\ell_{2n} - \ell_{2m}L_{2n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - L_{3m}\ell_{3n} + \ell_{3m}L_{3n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + L_{2n}\ell_{2n} - \ell_{2m}L_{2n} + L_{1n}\ell_{1n} - \ell_{1m}L_{1n} - L_{0n}\ell_{0n} + \ell_{0m}L_{0n} - L_{3n}\ell_{3n} + \ell_{3m}L_{3n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} - \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) - (\ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n} + \ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n}) \\ 0 \end{pmatrix} \quad (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{2m}L_{2n} + L_{1m}L_{1n} + L_{0m}L_{0n} + L_{3m}L_{3n}) - \ell_{2m}L_{2n} - \ell_{1m}L_{1n} + \ell_{0m}L_{0n} + \ell_{3m}L_{3n} + [(+L_{2m}\ell_{2n} + L_{1m}\ell_{1n} - L_{0m}\ell_{0n})] & 0 \\ 0 & 0 \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{2m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{1m}^{\leftrightarrow}D_{1n}^{\leftrightarrow} + D_{0m}D_{0n}^{\updownarrow} + D_{3m}D_{3n}^{\updownarrow}) &= \left( \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \begin{pmatrix} 0 & D_{0n}^- \\ D_{0n}^+ & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} D_{2m}^-D_{2n}^+ + D_{1m}^-D_{1n}^+ + D_{0m}^+D_{0n}^- + D_{3m}^+D_{3n}^- & 0 \\ 0 & D_{2m}^+D_{2n}^- + D_{1m}^+D_{1n}^- + D_{0m}^-D_{0n}^+ + D_{3m}^-D_{3n}^+ \end{pmatrix} \\ &= \begin{pmatrix} (L_{2m} - \ell_{2m})(L_{2n} + \ell_{2n}) + (L_{1m} - \ell_{1m})(L_{1n} + \ell_{1n}) + (L_{0m} + \ell_{0m})(L_{0n} - \ell_{0n}) + (L_{3m} + \ell_{3m})(L_{3n} - \ell_{3n}) & 0 \\ 0 & (L_{2m} + \ell_{2m})(L_{2n} - \ell_{2n}) + (L_{1m} + \ell_{1m})(L_{1n} - \ell_{1n}) + (L_{0m} - \ell_{0m})(L_{0n} + \ell_{0n}) + (L_{3m} - \ell_{3m})(L_{3n} + \ell_{3n}) \end{pmatrix} \\ &= \begin{pmatrix} L_{2m}(L_{2n} + \ell_{2n}) - \ell_{2m}(L_{2n} + \ell_{2n}) + L_{1m}(L_{1n} + \ell_{1n}) - \ell_{1m}(L_{1n} + \ell_{1n}) + L_{0m}(L_{0n} - \ell_{0n}) + \ell_{0m}(L_{0n} - \ell_{0n}) + L_{3m}(L_{3n} - \ell_{3n}) - \ell_{3m}(L_{3n} - \ell_{3n}) & 0 \\ 0 & L_{2m}(L_{2n} - \ell_{2n}) + \ell_{2m}(L_{2n} - \ell_{2n}) + L_{1m}(L_{1n} - \ell_{1n}) + \ell_{1m}(L_{1n} - \ell_{1n}) + L_{0m}(L_{0n} + \ell_{0n}) - \ell_{0m}(L_{0n} + \ell_{0n}) - L_{3m}(L_{3n} + \ell_{3n}) + \ell_{3m}(L_{3n} + \ell_{3n}) \end{pmatrix} \\ &= \begin{pmatrix} L_{2m}L_{2n} + L_{2m}\ell_{2n} - \ell_{2m}L_{2n} - \ell_{2m}\ell_{2n} + L_{1m}L_{1n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - \ell_{1m}\ell_{1n} + L_{0m}L_{0n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - \ell_{0m}\ell_{0n} & 0 \\ 0 & L_{2m}L_{2n} + L_{2m}\ell_{2n} - \ell_{2m}L_{2n} - \ell_{2m}\ell_{2n} + L_{1m}L_{1n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - \ell_{1m}\ell_{1n} + L_{0m}L_{0n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - \ell_{0m}\ell_{0n} \end{pmatrix} \\ &= \begin{pmatrix} (L_{2m}L_{2n} + L_{1m}L_{1n} + L_{0m}L_{0n} + L_{3m}L_{3n}) + L_{2m}\ell_{2n} - \ell_{2m}L_{2n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - L_{3m}\ell_{3n} + \ell_{3m}L_{3n} & 0 \\ 0 & (L_{2m}L_{2n} + L_{1m}L_{1n} + L_{0m}L_{0n} + L_{3m}L_{3n}) + L_{2m}\ell_{2n} - \ell_{2m}L_{2n} + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} - L_{0m}\ell_{0n} + \ell_{0m}L_{0n} - L_{3m}\ell_{3n} + \ell_{3m}L_{3n} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + L_{2n}\ell_{2n} - \ell_{2n}L_{2n} + L_{1n}\ell_{1n} - \ell_{1n}L_{1n} - L_{0n}\ell_{0n} + \ell_{0n}L_{0n} - L_{3n}\ell_{3n} + \ell_{3n}L_{3n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) - (\ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n} + \ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n}) & 0 \\ 0 & (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) - (\ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n} + \ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n}) \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{2m}L_{2n} + L_{1m}L_{1n} + L_{0m}L_{0n} + L_{3m}L_{3n}) - \ell_{2m}L_{2n} - \ell_{1m}L_{1n} + \ell_{0m}L_{0n} + \ell_{3m}L_{3n} + [(+L_{2m}\ell_{2n} + L_{1m}\ell_{1n} - L_{0m}\ell_{0n})] & 0 \\ 0 & 0 \end{pmatrix}$$

□

**Lemma 1.2.3.4-(BAc3r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\updownarrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\updownarrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(-D_{1m}^{\updownarrow}D_{2n}^{\leftrightarrow} + D_{2m}^{\updownarrow}D_{1n}^{\leftrightarrow} + D_{3m}^{\updownarrow}D_{0n}^{\updownarrow} - D_{0m}^{\updownarrow}D_{3n}^{\updownarrow}) = \begin{pmatrix} (L_{3m}L_{0n} - L_{0m}L_{3n}) - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) + L_{1m}\ell_{2n} - \ell_{1m}L_{2n} \\ (-L_{1m}L_{2n} + L_{2m}L_{1n}) - L_{1m}\ell_{2n} - \ell_{1m}L_{2n} + L_{2m}\ell_{1n} + \ell_{2m}L_{1n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & (L_{3m}L_{0n} - L_{0m}L_{3n}) + L_{3m}\ell_{0n} - \ell_{3m}L_{0n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{0n} - L_{0n}L_{3n}) - L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} + \ell_{0n}L_{3n} + (+\ell_{3n}\ell_{0n} - \ell_{0n}\ell_{3n}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + L_{1n}\ell_{2n} - \ell_{1n}L_{2n} \\ (-L_{1n}L_{2n} + L_{2n}L_{1n}) - L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + L_{2n}\ell_{1n} + \ell_{2n}L_{1n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & (L_{3n}L_{0n} - L_{0n}L_{3n}) + L_{3n}\ell_{0n} - \ell_{3n}L_{0n} \end{pmatrix}$$

$$= \begin{pmatrix} -L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} + \ell_{0n}L_{3n} + (+\ell_{3n}\ell_{0n} - \ell_{0n}\ell_{3n}) & +L_{1n}\ell_{2n} + \ell_{1m}L_{2n} - L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \\ -L_{1n}\ell_{2n} - \ell_{1m}L_{2n} + L_{2n}\ell_{1n} + \ell_{2m}L_{1n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & +L_{3n}\ell_{0n} + \ell_{3m}L_{0n} - L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} +(\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & +(\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \\ +(\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & +(\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{3m}L_{0n} - L_{0m}L_{3n}) - \ell_{3m}L_{0n} + \ell_{0m}L_{3n} + (-L_{3m}\ell_{0n} + L_{0m}\ell_{3n} + \ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) + \ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n} \\ (-L_{1m}L_{2n} + L_{2m}L_{1n}) - \ell_{1m}L_{2n} + \ell_{2m}L_{1n} + (-L_{1m}\ell_{2n} + L_{2m}\ell_{1n} - \ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & (L_{3m}L_{0n} - L_{0m}L_{3n}) + \ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (-D_{1m}^{\updownarrow}D_{2n}^{\leftrightarrow} + D_{2m}^{\updownarrow}D_{1n}^{\leftrightarrow} + D_{3m}^{\updownarrow}D_{0n}^{\updownarrow} - D_{0m}^{\updownarrow}D_{3n}^{\updownarrow}) &= \left( -\begin{pmatrix} D_{1m}^- & 0 \\ 0 & D_{1m}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{2m}^- & 0 \\ 0 & D_{2m}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix} + \begin{pmatrix} D_{3m}^- & 0 \\ 0 & D_{3m}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{0n}^- \\ D_{0n}^+ & 0 \end{pmatrix} - \begin{pmatrix} D_{0m}^- & 0 \\ 0 & D_{0m}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{3n}^- \\ D_{3n}^+ & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} D_{3m}^-D_{0n}^- - D_{0m}^-D_{3n}^- & -D_{1m}^-D_{2n}^- + D_{2m}^-D_{1n}^- \\ -D_{1m}^+D_{2n}^+ + D_{2m}^+D_{1n}^+ & D_{3m}^+D_{0n}^+ - D_{0m}^+D_{3n}^+ \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} L_{3m}L_{0n} - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + \ell_{3m}\ell_{0n} - L_{0m}L_{3n} + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} - \ell_{0m}\ell_{3n} & -L_{1m}L_{2n} + L_{1m}\ell_{2n} + \ell_{1m}L_{2n} - \ell_{1m}\ell_{2n} \\ -L_{1m}L_{2n} - L_{1m}\ell_{2n} - \ell_{1m}L_{2n} - \ell_{1m}\ell_{2n} + L_{2m}L_{1n} + L_{2m}\ell_{1n} + \ell_{2m}L_{1n} + \ell_{2m}\ell_{1n} & L_{3m}L_{0n} + L_{3m}\ell_{0n} + \ell_{3m}L_{0n} + \ell_{3m}\ell_{0n} \end{pmatrix} \\
&= \begin{pmatrix} (L_{3m}L_{0n} - L_{0m}L_{3n}) - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + (\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) + L_{1m}\ell_{2n} - \ell_{1m}\ell_{2n} \\ (-L_{1m}L_{2n} + L_{2m}L_{1n}) - L_{1m}\ell_{2n} - \ell_{1m}L_{2n} + L_{2m}\ell_{1n} + \ell_{2m}L_{1n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & (L_{3m}L_{0n} - L_{0m}L_{3n}) + L_{3m}\ell_{0n} - \ell_{3m}\ell_{0n} \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  :

$$\begin{aligned}
&= \begin{pmatrix} (L_{3n}L_{0n} - L_{0n}L_{3n}) - L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} + \ell_{0n}L_{3n} + (\ell_{3n}\ell_{0n} - \ell_{0n}\ell_{3n}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + L_{1n}\ell_{2n} - \ell_{1n}\ell_{2n} \\ (-L_{1n}L_{2n} + L_{2n}L_{1n}) - L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + L_{2n}\ell_{1n} + \ell_{2n}L_{1n} + (-\ell_{1n}\ell_{2n} + \ell_{2n}\ell_{1n}) & (L_{3n}L_{0n} - L_{0n}L_{3n}) + L_{3n}\ell_{0n} - \ell_{3n}\ell_{0n} \end{pmatrix} \\
&= \begin{pmatrix} -L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} + \ell_{0n}L_{3n} + (\ell_{3n}\ell_{0n} - \ell_{0n}\ell_{3n}) & +L_{1n}\ell_{2n} + \ell_{1n}L_{2n} - L_{2n}\ell_{1n} - \ell_{2n}L_{1n} + (-\ell_{1n}\ell_{2n} + \ell_{2n}\ell_{1n}) \\ -L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + L_{2n}\ell_{1n} + \ell_{2n}L_{1n} + (-\ell_{1n}\ell_{2n} + \ell_{2n}\ell_{1n}) & +L_{3n}\ell_{0n} + \ell_{3n}L_{0n} - L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + (\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} +(\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & +(\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \\ +(\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & +(\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{3m}L_{0n} - L_{0m}L_{3n}) - \ell_{3m}L_{0n} + \ell_{0m}L_{3n} + (-L_{3m}\ell_{0n} + L_{0m}\ell_{3n} + \ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) + \ell_{1m}\ell_{2n} - \ell_{1m}\ell_{2n} \\ (-L_{1m}L_{2n} + L_{2m}L_{1n}) - \ell_{1m}L_{2n} + \ell_{2m}L_{1n} + (-L_{1m}\ell_{2n} + L_{2m}\ell_{1n} - \ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & (L_{3m}L_{0n} - L_{0m}L_{3n}) + \ell_{3m}\ell_{0n} - \ell_{3m}\ell_{0n} \end{pmatrix}$$

□

**Lemma I.2.4.1-(BAc4r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{0m}D_{1n} - D_{3m}^{\leftrightarrow}D_{2n} + D_{2m}^{\leftrightarrow}D_{3n} - D_{1m}D_{0n}) = \begin{pmatrix} (L_{0m}L_{1n} - L_{1m}L_{0n}) + L_{0m}\ell_{1n} + \ell_{0m}L_{1n} - \ell_{1m}L_{0n} - L_{1m}\ell_{0n} + (\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) + L_{3m}\ell_{2n} - \ell_{3m}\ell_{2n} \\ (-L_{3m}L_{2n} + L_{2m}L_{3n}) - L_{3m}\ell_{2n} - \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + L_{2m}\ell_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) - L_{0m}\ell_{1n} + \ell_{0m}\ell_{1n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} +L_{0n}\ell_{1n} + \ell_{0m}L_{1n} - \ell_{1m}L_{0n} - L_{1n}\ell_{0n} + (\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & +L_{3n}\ell_{2n} + \ell_{3m}L_{2n} - L_{2n}\ell_{3n} - \ell_{2m}L_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \\ -L_{3n}\ell_{2n} - \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + L_{2n}\ell_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & -L_{0n}\ell_{1n} - \ell_{0m}L_{1n} + L_{1n}\ell_{0n} + \ell_{1m}L_{0n} + (\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} +(\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & +(\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \\ +(\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & +(\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0m}L_{1n} - L_{1m}L_{0n}) + \ell_{0m}L_{1n} - \ell_{1m}L_{0n} + (+L_{0m}\ell_{1n} - L_{1m}\ell_{0n} + \ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) + \ell_{3m}\ell_{2n} - \ell_{3m}\ell_{2n} \\ (-L_{3m}L_{2n} + L_{2m}L_{3n}) - \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + (-L_{3m}\ell_{2n} + L_{2m}\ell_{3n} - \ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) - \ell_{0m}\ell_{1n} + \ell_{0m}\ell_{1n} \end{pmatrix}$$

**Proof:**

$$(D_{0m}D_{1n} - D_{3m}^{\leftrightarrow}D_{2n} + D_{2m}^{\leftrightarrow}D_{3n} - D_{1m}D_{0n}) = \left( \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \begin{pmatrix} D_{1n}^+ & 0 \\ 0 & D_{1n}^- \end{pmatrix} - \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{2n}^+ & 0 \\ 0 & D_{2n}^- \end{pmatrix} + \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{3n}^+ & 0 \\ 0 & D_{3n}^- \end{pmatrix} \right)$$

$$= \begin{pmatrix} D_{0m}^+D_{1n}^+ - D_{1m}^+D_{0n}^+ & -D_{3m}^-D_{2n}^- + D_{2m}^-D_{3n}^- \\ -D_{3m}^+D_{2n}^+ + D_{2m}^+D_{3n}^+ & D_{0m}^-D_{1n}^- - D_{1m}^-D_{0n}^- \end{pmatrix}$$

$$= \begin{pmatrix} (L_{0m} + \ell_{0m})(L_{1n} + \ell_{1n}) - (L_{1m} + \ell_{1m})(L_{0n} + \ell_{0n}) & -(L_{3m} - \ell_{3m})(L_{2n} - \ell_{2n}) + (L_{2m} - \ell_{2m})(L_{3n} - \ell_{3n}) \\ -(L_{3m} + \ell_{3m})(L_{2n} + \ell_{2n}) + (L_{2m} + \ell_{2m})(L_{3n} + \ell_{3n}) & (L_{0m} - \ell_{0m})(L_{1n} - \ell_{1n}) - (L_{1m} - \ell_{1m})(L_{0n} - \ell_{0n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{0m}(L_{1n} + \ell_{1n}) + \ell_{0m}(L_{1n} + \ell_{1n}) - L_{1m}(L_{0n} + \ell_{0n}) - \ell_{1m}(L_{0n} + \ell_{0n}) & -L_{3m}(L_{2n} - \ell_{2n}) + \ell_{3m}(L_{2n} - \ell_{2n}) + L_{2m}(L_{3n} - \ell_{3n}) - \ell_{2m}(L_{3n} - \ell_{3n}) \\ -L_{3m}(L_{2n} + \ell_{2n}) - \ell_{3m}(L_{2n} + \ell_{2n}) + L_{2m}(L_{3n} + \ell_{3n}) + \ell_{2m}(L_{3n} + \ell_{3n}) & L_{0m}(L_{1n} - \ell_{1n}) - \ell_{0m}(L_{1n} - \ell_{1n}) - L_{1m}(L_{0n} - \ell_{0n}) + \ell_{1m}(L_{0n} - \ell_{0n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{0m}L_{1n} + L_{0m}\ell_{1n} + \ell_{0m}L_{1n} + \ell_{0m}\ell_{1n} - L_{1m}L_{0n} - L_{1m}\ell_{0n} - \ell_{1m}L_{0n} - \ell_{1m}\ell_{0n} & -L_{3m}L_{2n} + L_{3m}\ell_{2n} + \ell_{3m}L_{2n} - \ell_{3m}\ell_{2n} \\ -L_{3m}L_{2n} - L_{3m}\ell_{2n} - \ell_{3m}L_{2n} - \ell_{3m}\ell_{2n} + L_{2m}L_{3n} + L_{2m}\ell_{3n} + \ell_{2m}L_{3n} + \ell_{2m}\ell_{3n} & L_{0m}L_{1n} - L_{0m}\ell_{1n} - \ell_{0m}L_{1n} + \ell_{0m}\ell_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} (L_{0m}L_{1n} - L_{1m}L_{0n}) + L_{0m}\ell_{1n} + \ell_{0m}L_{1n} - \ell_{1m}L_{0n} - L_{1m}\ell_{0n} + (\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) + L_{3m}\ell_{2n} - \ell_{3m}\ell_{2n} \\ (-L_{3m}L_{2n} + L_{2m}L_{3n}) - L_{3m}\ell_{2n} - \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + L_{2m}\ell_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) - L_{0m}\ell_{1n} + \ell_{0m}\ell_{1n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{1n} - L_{1n}L_{0n}) + L_{0n}\ell_{1n} + \ell_{0n}L_{1n} + \ell_{0n}\ell_{1n} - L_{1n}\ell_{0n} + (-\ell_{1m}L_{0n} - \ell_{1m}\ell_{0n}) & (-L_{3n}L_{2n} + L_{2n}L_{3n}) + L_{3n}\ell_{2n} - \ell_{3n}\ell_{2n} \\ (-L_{3n}L_{2n} + L_{2n}L_{3n}) - L_{3n}\ell_{2n} - \ell_{3n}L_{2n} - \ell_{3n}\ell_{2n} + L_{2n}\ell_{3n} + (\ell_{2m}L_{3n} + \ell_{2m}\ell_{3n}) & (L_{0n}L_{1n} - L_{1n}L_{0n}) - L_{0n}\ell_{1n} + \ell_{0n}\ell_{1n} \end{pmatrix}$$

$$= \begin{pmatrix} +L_{0n}\ell_{1n} + \ell_{0m}L_{1n} - \ell_{1m}L_{0n} - L_{1n}\ell_{0n} + (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & +L_{3n}\ell_{2n} + \ell_{3m}L_{2n} - L_{2n}\ell_{3n} - \ell_{2m}L_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \\ -L_{3n}\ell_{2n} - \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + L_{2n}\ell_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & -L_{0n}\ell_{1n} - \ell_{0m}L_{1n} + L_{1n}\ell_{0n} + \ell_{1m}L_{0n} + (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \\ (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0m}L_{1n} - L_{1m}L_{0n}) + \ell_{0m}L_{1n} - \ell_{1m}L_{0n} + (+L_{0m}\ell_{1n} - L_{1m}\ell_{0n} + \ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & (-L_{3m}L_{2n} + L_{2m}L_{3n}) + \ell_{3m}L_{2n} - \ell_{2m}L_{3n} \\ (-L_{3m}L_{2n} + L_{2m}L_{3n}) - \ell_{3m}L_{2n} + \ell_{2m}L_{3n} + (-L_{3m}\ell_{2n} + L_{2m}\ell_{3n} - \ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & (L_{0m}L_{1n} - L_{1m}L_{0n}) - \ell_{0m}L_{1n} + \ell_{1m}L_{0n} \end{pmatrix}$$

□

**Lemma I.2.4.2-(BAc4r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\updownarrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\updownarrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{3m}^{\leftrightarrow}D_{1n} + D_{0m}D_{2n} - D_{1m}^{\leftrightarrow}D_{3n} - D_{2m}D_{0n}) = \begin{pmatrix} (L_{0m}L_{2n} - L_{2m}L_{0n}) + L_{0m}\ell_{2n} + \ell_{0m}L_{2n} - L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (L_{3m}L_{1n} - L_{1m}L_{3n}) - L_{3m}\ell_{1n} - \ell_{3m}L_{1n} \\ (L_{3m}L_{1n} - L_{1m}L_{3n}) + L_{3m}\ell_{1n} + \ell_{3m}L_{1n} - L_{1m}\ell_{3n} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (L_{0m}L_{2n} - L_{2m}L_{0n}) - L_{0m}\ell_{2n} - \ell_{0m}L_{2n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} +L_{0n}\ell_{2n} + \ell_{0m}L_{2n} - L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & -L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + L_{1n}\ell_{3n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \\ +L_{3n}\ell_{1n} + \ell_{3m}L_{1n} - L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & -L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + L_{2n}\ell_{0n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \\ (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (+\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0m}L_{2n} - L_{2m}L_{0n}) + \ell_{0m}L_{2n} - \ell_{2m}L_{0n} + (+L_{0m}\ell_{2n} - L_{2m}\ell_{0n} + \ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (L_{3m}L_{1n} - L_{1m}L_{3n}) - \ell_{3m}L_{1n} + \ell_{1m}L_{3n} \\ (L_{3m}L_{1n} - L_{1m}L_{3n}) + \ell_{3m}L_{1n} - \ell_{1m}L_{3n} + (+L_{3m}\ell_{1n} - L_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (L_{0m}L_{2n} - L_{2m}L_{0n}) - \ell_{0m}L_{2n} + \ell_{2m}L_{0n} \end{pmatrix}$$

**Proof:**

$$(D_{3m}^{\leftrightarrow}D_{1n} + D_{0m}D_{2n} - D_{1m}^{\leftrightarrow}D_{3n} - D_{2m}D_{0n}) = \left( \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{1n}^+ & 0 \\ 0 & D_{1n}^- \end{pmatrix} + \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \begin{pmatrix} D_{2n}^+ & 0 \\ 0 & D_{2n}^- \end{pmatrix} - \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{3n}^+ & 0 \\ 0 & D_{3n}^- \end{pmatrix} \right)$$

$$= \begin{pmatrix} D_{0m}^+D_{2n}^+ - D_{2m}^+D_{0n}^+ & D_{3m}^-D_{1n}^- - D_{1m}^-D_{3n}^- \\ D_{3m}^+D_{1n}^+ - D_{1m}^+D_{3n}^+ & D_{0m}^-D_{2n}^- - D_{2m}^-D_{0n}^- \end{pmatrix}$$

$$= \begin{pmatrix} (L_{0m} + \ell_{0m})(L_{2n} + \ell_{2n}) - (L_{2m} + \ell_{2m})(L_{0n} + \ell_{0n}) & (L_{3m} - \ell_{3m})(L_{1n} - \ell_{1n}) - (L_{1m} - \ell_{1m})(L_{3n} - \ell_{3n}) \\ (L_{3m} + \ell_{3m})(L_{1n} + \ell_{1n}) - (L_{1m} + \ell_{1m})(L_{3n} + \ell_{3n}) & (L_{0m} - \ell_{0m})(L_{2n} - \ell_{2n}) - (L_{2m} - \ell_{2m})(L_{0n} - \ell_{0n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{0m}(L_{2n} + \ell_{2n}) + \ell_{0m}(L_{2n} + \ell_{2n}) - L_{2m}(L_{0n} + \ell_{0n}) - \ell_{2m}(L_{0n} + \ell_{0n}) & L_{3m}(L_{1n} - \ell_{1n}) - \ell_{3m}(L_{1n} - \ell_{1n}) - L_{1m}(L_{3n} - \ell_{3n}) \\ L_{3m}(L_{1n} + \ell_{1n}) + \ell_{3m}(L_{1n} + \ell_{1n}) - L_{1m}(L_{3n} + \ell_{3n}) - \ell_{1m}(L_{3n} + \ell_{3n}) & L_{0m}(L_{2n} - \ell_{2n}) - \ell_{0m}(L_{2n} - \ell_{2n}) - L_{2m}(L_{0n} - \ell_{0n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{0m}L_{2n} + L_{0m}\ell_{2n} + \ell_{0m}L_{2n} + \ell_{0m}\ell_{2n} - L_{2m}L_{0n} - L_{2m}\ell_{0n} - \ell_{2m}L_{0n} - \ell_{2m}\ell_{0n} & L_{3m}L_{1n} - L_{3m}\ell_{1n} - \ell_{3m}L_{1n} + \ell_{3m}\ell_{1n} \\ L_{3m}L_{1n} + L_{3m}\ell_{1n} + \ell_{3m}L_{1n} + \ell_{3m}\ell_{1n} - L_{1m}L_{3n} - L_{1m}\ell_{3n} - \ell_{1m}L_{3n} - \ell_{1m}\ell_{3n} & L_{0m}L_{2n} - L_{0m}\ell_{2n} - \ell_{0m}L_{2n} + \ell_{0m}\ell_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} (L_{0m}L_{2n} - L_{2m}L_{0n}) + L_{0m}\ell_{2n} + \ell_{0m}L_{2n} - L_{2m}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (L_{3m}L_{1n} - L_{1m}L_{3n}) - L_{3m}\ell_{1n} - \ell_{3m}L_{1n} \\ (L_{3m}L_{1n} - L_{1m}L_{3n}) + L_{3m}\ell_{1n} + \ell_{3m}L_{1n} - L_{1m}\ell_{3n} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (L_{0m}L_{2n} - L_{2m}L_{0n}) - L_{0m}\ell_{2n} - \ell_{0m}L_{2n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{2n} - L_{2n}L_{0n}) + L_{0n}\ell_{2n} + \ell_{0m}L_{2n} - L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (L_{3n}L_{1n} - L_{1n}L_{3n}) - L_{3n}\ell_{1n} - \ell_{3m}L_{1n} \\ (L_{3n}L_{1n} - L_{1n}L_{3n}) + L_{3n}\ell_{1n} + \ell_{3m}L_{1n} - L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (L_{0n}L_{2n} - L_{2n}L_{0n}) - L_{0n}\ell_{2n} - \ell_{0m}L_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} +L_{0n}\ell_{2n} + \ell_{0m}L_{2n} - L_{2n}\ell_{0n} - \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & -L_{3n}\ell_{1n} - \ell_{3m}L_{1n} + L_{1n}\ell_{3n} + \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \\ +L_{3n}\ell_{1n} + \ell_{3m}L_{1n} - L_{1n}\ell_{3n} - \ell_{1m}L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & -L_{0n}\ell_{2n} - \ell_{0m}L_{2n} + L_{2n}\ell_{0n} + \ell_{2m}L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \\ (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (+\ell_{0n} - \ell_{0m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0m}L_{2n} - L_{2m}L_{0n}) + \ell_{0m}L_{2n} - \ell_{2m}L_{0n} + (+L_{0m}\ell_{2n} - L_{2m}\ell_{0n} + \ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & (L_{3m}L_{1n} - L_{1m}L_{3n}) - \ell_{3m}L_{1n} \\ (L_{3m}L_{1n} - L_{1m}L_{3n}) + \ell_{3m}L_{1n} - \ell_{1m}L_{3n} + (+L_{3m}\ell_{1n} - L_{1m}\ell_{3n} + \ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & (L_{0m}L_{2n} - L_{2m}L_{0n}) - \ell_{0m}L_{2n} \end{pmatrix}$$

□

**Lemma I.2.4.3-(BAc4r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(-D_{2m}^{\leftrightarrow}D_{1n} + D_{1m}^{\leftrightarrow}D_{2n} + D_{0m}D_{3n} - D_{3m}D_{0n}) = \begin{pmatrix} (L_{0m}L_{3n} - L_{3m}L_{0n}) + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) + L_{2m}\ell_{1n} \\ (-L_{2m}L_{1n} + L_{1m}L_{2n}) - L_{2m}\ell_{1n} - \ell_{2m}L_{1n} + L_{1m}\ell_{2n} + \ell_{1m}L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) - L_{0m}\ell_{3n} - \ell_{3m}L_{0n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} +L_{0n}\ell_{3n} + \ell_{0m}L_{3n} - L_{3n}\ell_{0n} - \ell_{3m}L_{0n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & +L_{2n}\ell_{1n} + \ell_{2m}L_{1n} - L_{1n}\ell_{2n} - \ell_{1m}L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\ -L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + L_{1n}\ell_{2n} + \ell_{1m}L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & -L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + L_{3n}\ell_{0n} + \ell_{3m}L_{0n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & (+\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\ (+\ell_{1m} - \ell_{1n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0m}L_{3n} - L_{3m}L_{0n}) + \ell_{0m}L_{3n} - \ell_{3m}L_{0n} + (+L_{0m}\ell_{3n} - L_{3m}\ell_{0n} + \ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) + \ell_{2m}L_{1n} \\ (-L_{2m}L_{1n} + L_{1m}L_{2n}) - \ell_{2m}L_{1n} + \ell_{1m}L_{2n} + (-L_{2m}\ell_{1n} + L_{1m}\ell_{2n} + \ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) - \ell_{0m}L_{3n} - \ell_{3m}L_{0n} \end{pmatrix}$$

**Proof:**

$$(-D_{2m}^{\leftrightarrow}D_{1n} + D_{1m}^{\leftrightarrow}D_{2n} + D_{0m}D_{3n} - D_{3m}D_{0n}) = \left( - \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{1n}^+ & 0 \\ 0 & D_{1n}^- \end{pmatrix} + \begin{pmatrix} 0 & D_{1m}^- \\ D_{1m}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{2n}^+ & 0 \\ 0 & D_{2n}^- \end{pmatrix} + \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} \right)$$

$$= \begin{pmatrix} D_{0m}^+D_{3n}^+ - D_{3m}^+D_{0n}^+ & -D_{2m}^-D_{1n}^- + D_{1m}^-D_{2n}^- \\ -D_{2m}^+D_{1n}^+ + D_{1m}^+D_{2n}^+ & D_{0m}^-D_{3n}^- - D_{3m}^-D_{0n}^- \end{pmatrix}$$

$$= \begin{pmatrix} (L_{0m} + \ell_{0m})(L_{3n} + \ell_{3n}) - (L_{3m} + \ell_{3m})(L_{0n} + \ell_{0n}) & -(L_{2m} - \ell_{2m})(L_{1n} - \ell_{1n}) + (L_{1m} - \ell_{1m})(L_{2n} - \ell_{2n}) \\ -(L_{2m} + \ell_{2m})(L_{1n} + \ell_{1n}) + (L_{1m} + \ell_{1m})(L_{2n} + \ell_{2n}) & (L_{0m} - \ell_{0m})(L_{3n} - \ell_{3n}) - (L_{3m} - \ell_{3m})(L_{0n} - \ell_{0n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{0m}(L_{3n} + \ell_{3n}) + \ell_{0m}(L_{3n} + \ell_{3n}) - L_{3m}(L_{0n} + \ell_{0n}) - \ell_{3m}(L_{0n} + \ell_{0n}) & -L_{2m}(L_{1n} - \ell_{1n}) + \ell_{2m}(L_{1n} - \ell_{1n}) + L_{1m}(L_{2n} - \ell_{2n}) - \ell_{1m}(L_{2n} - \ell_{2n}) \\ -L_{2m}(L_{1n} + \ell_{1n}) - \ell_{2m}(L_{1n} + \ell_{1n}) + L_{1m}(L_{2n} + \ell_{2n}) + \ell_{1m}(L_{2n} + \ell_{2n}) & L_{0m}(L_{3n} - \ell_{3n}) - \ell_{0m}(L_{3n} - \ell_{3n}) - L_{3m}(L_{0n} - \ell_{0n}) + \ell_{3m}(L_{0n} - \ell_{0n}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{0m}L_{3n} + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + \ell_{0m}\ell_{3n} - L_{3m}L_{0n} - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} - \ell_{3m}\ell_{0n} & -L_{2m}L_{1n} + L_{2m}\ell_{1n} + \ell_{2m}L_{1n} - \ell_{2m}\ell_{1n} \\ -L_{2m}L_{1n} - L_{2m}\ell_{1n} - \ell_{2m}L_{1n} - \ell_{2m}\ell_{1n} + L_{1m}L_{2n} + L_{1m}\ell_{2n} + \ell_{1m}L_{2n} + \ell_{1m}\ell_{2n} & L_{0m}L_{3n} - L_{0m}\ell_{3n} - \ell_{0m}L_{3n} + \ell_{0m}\ell_{3n} \end{pmatrix}$$

$$= \begin{pmatrix} (L_{0m}L_{3n} - L_{3m}L_{0n}) + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) + L_{2m}\ell_{1n} \\ (-L_{2m}L_{1n} + L_{1m}L_{2n}) - L_{2m}\ell_{1n} - \ell_{2m}L_{1n} + L_{1m}\ell_{2n} + \ell_{1m}L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) - L_{0m}\ell_{3n} - \ell_{3m}L_{0n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{3n} - L_{3n}L_{0n}) + L_{0n}\ell_{3n} + \ell_{0m}L_{3n} - L_{3n}\ell_{0n} - \ell_{3m}L_{0n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & (-L_{2n}L_{1n} + L_{1n}L_{2n}) + L_{2n}\ell_{1n} \\ (-L_{2n}L_{1n} + L_{1n}L_{2n}) - L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + L_{1n}\ell_{2n} + \ell_{1m}L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{0n}L_{3n} - L_{3n}L_{0n}) - L_{0n}\ell_{3n} - \ell_{3m}L_{0n} \end{pmatrix}$$

$$= \begin{pmatrix} +L_{0n}\ell_{3n} + \ell_{0m}L_{3n} - L_{3n}\ell_{0n} - \ell_{3m}L_{0n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & +L_{2n}\ell_{1n} + \ell_{2m}L_{1n} - L_{1n}\ell_{2n} - \ell_{1m}L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\ -L_{2n}\ell_{1n} - \ell_{2m}L_{1n} + L_{1n}\ell_{2n} + \ell_{1m}L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & -L_{0n}\ell_{3n} - \ell_{0m}L_{3n} + L_{3n}\ell_{0n} + \ell_{3m}L_{0n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & (+\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\ (+\ell_{1m} - \ell_{1n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0m}L_{3n} - L_{3m}L_{0n}) + \ell_{0m}L_{3n} - \ell_{3m}L_{0n} + (+L_{0m}\ell_{3n} - L_{3m}\ell_{0n} + \ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & (-L_{2m}L_{1n} + L_{1m}L_{2n}) + \ell_{2m}L_{1n} \\ (-L_{2m}L_{1n} + L_{1m}L_{2n}) - \ell_{2m}L_{1n} + \ell_{1m}L_{2n} + (-L_{2m}\ell_{1n} + L_{1m}\ell_{2n} + \ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (L_{0m}L_{3n} - L_{3m}L_{0n}) - \ell_{0m}L_{3n} - \ell_{3m}L_{0n} \end{pmatrix}$$

□

**Lemma I.2.4.4-(BAc4r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$



$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{1m}^{\uparrow}D_{1n} + D_{2m}^{\uparrow}D_{2n} + D_{3m}^{\uparrow}D_{3n} + D_{0m}^{\uparrow}D_{0n}) = \begin{pmatrix} (L_{1m}L_{1n} + L_{2m}L_{2n} + L_{3m}L_{3n} + L_{0m}L_{0n}) + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} + L_{2m}\ell_{2n} - \ell_{2m}L_{2n} + L_{3m}\ell_{3n} - \ell_{3m}L_{3n} + L_{0m}\ell_{0n} - \ell_{0m}L_{0n} \\ 0 \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + L_{1n}\ell_{1n} - \ell_{1n}L_{1n} + L_{2n}\ell_{2n} - \ell_{2n}L_{2n} + L_{3n}\ell_{3n} - \ell_{3n}L_{3n} + L_{0n}\ell_{0n} - \ell_{0n}L_{0n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} - \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) - (\ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n} + \ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n}) \\ 0 \end{pmatrix} \quad (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1m}L_{1n} + L_{2m}L_{2n} + L_{3m}L_{3n} + L_{0m}L_{0n}) - (\ell_{1m}L_{1n} + \ell_{2m}L_{2n} + \ell_{3m}L_{3n} + \ell_{0m}L_{0n}) + [(L_{1m}\ell_{1n} + L_{2m}\ell_{2n} + L_{3m}\ell_{3n} + L_{0m}\ell_{0n})] \\ 0 \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{1m}^{\uparrow}D_{1n} + D_{2m}^{\uparrow}D_{2n} + D_{3m}^{\uparrow}D_{3n} + D_{0m}^{\uparrow}D_{0n}) &= \left( \begin{pmatrix} D_{1m}^- & 0 \\ 0 & D_{1m}^+ \end{pmatrix} \begin{pmatrix} D_{1n}^+ & 0 \\ 0 & D_{1n}^- \end{pmatrix} + \begin{pmatrix} D_{2m}^- & 0 \\ 0 & D_{2m}^+ \end{pmatrix} \begin{pmatrix} D_{2n}^+ & 0 \\ 0 & D_{2n}^- \end{pmatrix} + \begin{pmatrix} D_{3m}^- & 0 \\ 0 & D_{3m}^+ \end{pmatrix} \begin{pmatrix} D_{3n}^+ & 0 \\ 0 & D_{3n}^- \end{pmatrix} + \begin{pmatrix} D_{0m}^- & 0 \\ 0 & D_{0m}^+ \end{pmatrix} \begin{pmatrix} D_{0n}^+ & 0 \\ 0 & D_{0n}^- \end{pmatrix} \right) \\ &= \begin{pmatrix} D_{1m}^-D_{1n}^+ + D_{2m}^-D_{2n}^+ + D_{3m}^-D_{3n}^+ + D_{0m}^-D_{0n}^+ & 0 \\ 0 & D_{1m}^+D_{1n}^- + D_{2m}^+D_{2n}^- + D_{3m}^+D_{3n}^- + D_{0m}^+D_{0n}^- \end{pmatrix} \\ &= \begin{pmatrix} (L_{1m} - \ell_{1m})(L_{1n} + \ell_{1n}) + (L_{2m} - \ell_{2m})(L_{2n} + \ell_{2n}) + (L_{3m} - \ell_{3m})(L_{3n} + \ell_{3n}) + (L_{0m} - \ell_{0m})(L_{0n} + \ell_{0n}) \\ 0 \end{pmatrix} \quad (L_{1m} + \ell_{1m}) \\ &= \begin{pmatrix} L_{1m}(L_{1n} + \ell_{1n}) - \ell_{1m}(L_{1n} + \ell_{1n}) + L_{2m}(L_{2n} + \ell_{2n}) - \ell_{2m}(L_{2n} + \ell_{2n}) + L_{3m}(L_{3n} + \ell_{3n}) - \ell_{3m}(L_{3n} + \ell_{3n}) + L_{0m}(L_{0n} + \ell_{0n}) - \ell_{0m}(L_{0n} + \ell_{0n}) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{1m}L_{1n} + L_{2m}L_{2n} + L_{3m}L_{3n} + L_{0m}L_{0n}) + L_{1m}\ell_{1n} - \ell_{1m}L_{1n} + L_{2m}\ell_{2n} - \ell_{2m}L_{2n} + L_{3m}\ell_{3n} - \ell_{3m}L_{3n} + L_{0m}\ell_{0n} - \ell_{0m}L_{0n} \\ 0 \end{pmatrix} \end{aligned}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + L_{1n}\ell_{1n} - \ell_{1n}L_{1n} + L_{2n}\ell_{2n} - \ell_{2n}L_{2n} + L_{3n}\ell_{3n} - \ell_{3n}L_{3n} + L_{0n}\ell_{0n} - \ell_{0n}L_{0n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \ell_{1n}L_{1n} - \ell_{1m}L_{1n} + \ell_{2n}L_{2n} - \ell_{2m}L_{2n} + \ell_{3n}L_{3n} - \ell_{3m}L_{3n} + \ell_{0n}L_{0n} - \ell_{0m}L_{0n} \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} - \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) - (\ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n} + \ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n}) \\ 0 \end{pmatrix} \quad (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1m}L_{1n} + L_{2m}L_{2n} + L_{3m}L_{3n} + L_{0m}L_{0n}) - (\ell_{1m}L_{1n} + \ell_{2m}L_{2n} + \ell_{3m}L_{3n} + \ell_{0m}L_{0n}) + [(L_{1m}\ell_{1n} + L_{2m}\ell_{2n} + L_{3m}\ell_{3n} + L_{0m}\ell_{0n})] \\ 0 \end{pmatrix}$$

□

**Lemma I.3.1.1-(ABc1r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{0n}^{\uparrow}D_{0m} + D_{3n}^{\leftrightarrow}D_{3m}^{\leftrightarrow} + D_{2n}^{\leftrightarrow}D_{2m}^{\leftrightarrow} + D_{1n}^{\uparrow}D_{1m}^{\uparrow}) = \begin{pmatrix} (L_{0n}L_{0m} + L_{3n}L_{3m} + L_{2n}L_{2m} + L_{1n}L_{1m}) + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} - L_{1n}\ell_{1m} + \ell_{1n}L_{1m} \\ 0 \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + L_{0n}\ell_{0m} - \ell_{0n}L_{0n} + L_{3n}\ell_{3m} - \ell_{3n}L_{3n} + L_{2n}\ell_{2m} - \ell_{2n}L_{2n} - L_{1n}\ell_{1m} + \ell_{1n}L_{1n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} - \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) - (\ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n} + \ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n}) \\ 0 \end{pmatrix} \quad (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{0m} + L_{3n}L_{3m} + L_{2n}L_{2m} + L_{1n}L_{1m}) + ([+\ell_{1n}L_{1m} - \ell_{0n}L_{0m} - \ell_{3n}L_{3m} - \ell_{2n}L_{2m}]) + (-[+L_{1n}\ell_{1m} - L_{0n}\ell_{0m} - \\ 0 \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{0n}^\dagger D_{0m} + D_{3n}^{\leftrightarrow} D_{3m}^{\leftrightarrow} + D_{2n}^{\leftrightarrow} D_{2m}^{\leftrightarrow} + D_{1n} D_{1m}^\dagger) &= \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \begin{pmatrix} (L_{0m} + \ell_{0m}) & 0 \\ 0 & (L_{0m} - \ell_{0m}) \end{pmatrix} + \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n} - \ell_{0n})(L_{0m} + \ell_{0m}) + (L_{3n} - \ell_{3n})(L_{3m} + \ell_{3m}) + (L_{2n} - \ell_{2n})(L_{2m} + \ell_{2m}) + (L_{1n} + \ell_{1n})(L_{1m} - \ell_{1m}) \\ 0 \end{pmatrix} \quad (L_{0n} + \ell_{0n}) \\ &= \begin{pmatrix} L_{0n}L_{0m} + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} - \ell_{0n}\ell_{0m} + L_{3n}L_{3m} + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} - \ell_{3n}\ell_{3m} + L_{2n}L_{2m} + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} - \ell_{2n}\ell_{2m} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n}L_{0m} + L_{3n}L_{3m} + L_{2n}L_{2m} + L_{1n}L_{1m}) + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} - L_{1n}\ell_{1m} + \ell_{1n}L_{1m} \\ 0 \end{pmatrix} \end{aligned}$$

$L_{jn} = L_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + L_{0n}\ell_{0m} - \ell_{0n}L_{0n} + L_{3n}\ell_{3m} - \ell_{3n}L_{3n} + L_{2n}\ell_{2m} - \ell_{2n}L_{2n} - L_{1n}\ell_{1m} + \ell_{1n}L_{1n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} - \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) - (\ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n} + \ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n}) \\ 0 \end{pmatrix} \quad (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{0m} + L_{3n}L_{3m} + L_{2n}L_{2m} + L_{1n}L_{1m}) + ([+\ell_{1n}L_{1m} - \ell_{0n}L_{0m} - \ell_{3n}L_{3m} - \ell_{2n}L_{2m}]) + (-[+L_{1n}\ell_{1m} - L_{0n}\ell_{0m} - \\ 0 \end{pmatrix}$$

□

**Lemma I.3.1.2-(ABc1r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(-D_{3n}^{\leftrightarrow} D_{0m} + D_{0n}^\dagger D_{3m}^{\leftrightarrow} - D_{1n}^{\leftrightarrow} D_{2m}^{\leftrightarrow} + D_{2n} D_{1m}^\dagger) = \begin{pmatrix} (-L_{1n}L_{2m} + L_{2n}L_{1m}) - L_{1n}\ell_{2m} + \ell_{1n}L_{2m} - L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & (-L_{3n}L_{0m} + L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} \\ (-L_{3n}L_{0m} + L_{0n}L_{3m}) - L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}L_{2m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} -L_{1n}\ell_{2m} + \ell_{1n}L_{2n} - L_{2n}\ell_{1m} + \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +L_{3n}\ell_{0m} + \ell_{3n}L_{0n} - L_{0n}\ell_{3m} - \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \\ -L_{3n}\ell_{0m} - \ell_{3n}L_{0n} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +L_{1n}\ell_{2m} - \ell_{1n}L_{2n} + L_{2n}\ell_{1m} - \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & (\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \\ (\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1n}L_{2m} + L_{2n}L_{1m}) + ([+\ell_{2n}L_{1m} + \ell_{1n}L_{2m}]) + (-[+L_{1n}\ell_{2m} + L_{2n}\ell_{1m}] + [+ \ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}]) & (-L_{3n}L_{0m} + L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} \\ (-L_{3n}L_{0m} + L_{0n}L_{3m}) + (-[+\ell_{3n}L_{0m} - \ell_{0n}L_{3m}]) + ([+L_{0n}\ell_{3m} - L_{3n}\ell_{0m}] + [-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}]) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}L_{2m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (-D_{3n}^{\leftrightarrow} D_{0m} + D_{0n}^\dagger D_{3m}^{\leftrightarrow} - D_{1n}^{\leftrightarrow} D_{2m}^{\leftrightarrow} + D_{2n} D_{1m}^\dagger) &= - \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \begin{pmatrix} (L_{0m} + \ell_{0m}) & 0 \\ 0 & (L_{0m} - \ell_{0m}) \end{pmatrix} + \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \\ &= \begin{pmatrix} -(L_{1n} - \ell_{1n})(L_{2m} + \ell_{2m}) + (L_{2n} + \ell_{2n})(L_{1m} - \ell_{1m}) & -(L_{3n} - \ell_{3n})(L_{0m} - \ell_{0m}) + (L_{0n} - \ell_{0n})(L_{3m} - \ell_{3m}) \\ -(L_{3n} + \ell_{3n})(L_{0m} + \ell_{0m}) + (L_{0n} + \ell_{0n})(L_{3m} + \ell_{3m}) & -(L_{1n} + \ell_{1n})(L_{2m} - \ell_{2m}) + (L_{2n} - \ell_{2n})(L_{1m} + \ell_{1m}) \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} -L_{1n}L_{2m} - L_{1n}\ell_{2m} + \ell_{1n}L_{2m} + \ell_{1n}\ell_{2m} + L_{2n}L_{1m} - L_{2n}\ell_{1m} + \ell_{2n}L_{1m} - \ell_{2n}\ell_{1m} & -L_{3n}L_{0m} + L_{3n}\ell_{0m} + \ell_{3n}L_{0m} - \ell_{3n}\ell_{0m} \\ -L_{3n}L_{0m} - L_{3n}\ell_{0m} - \ell_{3n}L_{0m} - \ell_{3n}\ell_{0m} + L_{0n}L_{3m} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} + \ell_{0n}\ell_{3m} & -L_{1n}L_{2m} + L_{1n}\ell_{2m} - \ell_{1n}L_{2m} + \ell_{1n}\ell_{2m} \end{pmatrix} \\
&= \begin{pmatrix} (-L_{1n}L_{2m} + L_{2n}L_{1m}) - L_{1n}\ell_{2m} + \ell_{1n}L_{2m} - L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & (-L_{3n}L_{0m} + L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}\ell_{0m} \\ (-L_{3n}L_{0m} + L_{0n}L_{3m}) - L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} + (+\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}\ell_{2m} \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  :

$$\begin{aligned}
&= \begin{pmatrix} (-L_{1n}L_{2n} + L_{2n}L_{1n}) - L_{1n}\ell_{2m} + \ell_{1n}L_{2n} - L_{2n}\ell_{1m} + \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & (-L_{3n}L_{0n} + L_{0n}L_{3n}) + L_{3n}\ell_{0m} - \ell_{3n}\ell_{0m} \\ (-L_{3n}L_{0n} + L_{0n}L_{3n}) - L_{3n}\ell_{0m} - \ell_{3n}L_{0n} + L_{0n}\ell_{3m} + \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + L_{1n}\ell_{2m} - \ell_{1n}\ell_{2m} \end{pmatrix} \\
&= \begin{pmatrix} -L_{1n}\ell_{2m} + \ell_{1n}L_{2n} - L_{2n}\ell_{1m} + \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +L_{3n}\ell_{0m} + \ell_{3n}L_{0n} - L_{0n}\ell_{3m} - \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \\ -L_{3n}\ell_{0m} - \ell_{3n}L_{0n} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +L_{1n}\ell_{2m} - \ell_{1n}L_{2n} + L_{2n}\ell_{1m} - \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & (\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \\ (\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1n}L_{2m} + L_{2n}L_{1m}) + (+[\ell_{2n}L_{1m} + \ell_{1n}L_{2m}]) + (-[+L_{1n}\ell_{2m} + L_{2n}\ell_{1m}] + [+ \ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}]) & (-L_{3n}L_{0m} + L_{0n}L_{3m}) + (-[\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}]) \\ (-L_{3n}L_{0m} + L_{0n}L_{3m}) + (-[+\ell_{3n}L_{0m} - \ell_{0n}L_{3m}]) + ([+L_{0n}\ell_{3m} - L_{3n}\ell_{0m}] + [-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}]) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + (+[\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}]) \end{pmatrix}$$

□

**Lemma I.3.1.2-(ABc1r2)a:** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(-D_{3m}^{\leftrightarrow}D_{0n} + D_{0m}^\dagger D_{3n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{2m}D_{1n}^\dagger) = \begin{pmatrix} (-L_{1m}L_{2n} + L_{2m}L_{1n}) - L_{1n}\ell_{2m} + \ell_{1n}L_{2m} - L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & (-L_{3n}L_{0m} + L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}\ell_{0m} \\ (-L_{3n}L_{0m} + L_{0n}L_{3m}) - L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}\ell_{2m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} -L_{1n}\ell_{2m} + \ell_{1n}L_{2n} - L_{2n}\ell_{1m} + \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +L_{3n}\ell_{0m} + \ell_{3n}L_{0n} - L_{0n}\ell_{3m} - \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \\ -L_{3n}\ell_{0m} - \ell_{3n}L_{0n} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +L_{1n}\ell_{2m} - \ell_{1n}L_{2n} + L_{2n}\ell_{1m} - \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & (\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \\ (\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1n}L_{2m} + L_{2n}L_{1m}) + (+[\ell_{2n}L_{1m} + \ell_{1n}L_{2m}]) + (-[+L_{1n}\ell_{2m} + L_{2n}\ell_{1m}] + [+ \ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}]) & (-L_{3n}L_{0m} + L_{0n}L_{3m}) + (-[\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}]) \\ (-L_{3n}L_{0m} + L_{0n}L_{3m}) + (-[+\ell_{3n}L_{0m} - \ell_{0n}L_{3m}]) + ([+L_{0n}\ell_{3m} - L_{3n}\ell_{0m}] + [-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}]) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + (+[\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}]) \end{pmatrix}$$

**Proof:**

$$\begin{aligned}
(-D_{3m}^{\leftrightarrow}D_{0n} + D_{0m}^\dagger D_{3n}^{\leftrightarrow} - D_{1m}^{\leftrightarrow}D_{2n}^{\leftrightarrow} + D_{2m}D_{1n}^\dagger) &= \begin{pmatrix} 0 & (L_{3m} - \ell_{3m}) \\ (L_{3m} + \ell_{3m}) & 0 \end{pmatrix} \begin{pmatrix} (L_{0n} + \ell_{0n}) & 0 \\ 0 & (L_{0n} - \ell_{0n}) \end{pmatrix} + \begin{pmatrix} (L_{0m} - \ell_{0m}) & 0 \\ 0 & (L_{0m} + \ell_{0m}) \end{pmatrix} \\
&= \begin{pmatrix} -(L_{1m} - \ell_{1m})(L_{2n} + \ell_{2n}) + (L_{2m} + \ell_{2m})(L_{1n} - \ell_{1n}) & -(L_{3m} - \ell_{3m})(L_{0n} - \ell_{0n}) + (L_{0m} - \ell_{0m})(L_{3n} - \ell_{3n}) \\ -(L_{3m} + \ell_{3m})(L_{0n} + \ell_{0n}) + (L_{0m} + \ell_{0m})(L_{3n} + \ell_{3n}) & -(L_{1m} + \ell_{1m})(L_{2n} - \ell_{2n}) + (L_{2m} - \ell_{2m})(L_{1n} + \ell_{1n}) \end{pmatrix} \\
&= \begin{pmatrix} -L_{1m}L_{2n} - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} + \ell_{1m}\ell_{2n} + L_{2m}L_{1n} - L_{2m}\ell_{1n} + \ell_{2m}L_{1n} - \ell_{2m}\ell_{1n} & -L_{3m}L_{0n} + L_{3m}\ell_{0n} + \ell_{3m}L_{0n} - \ell_{3m}\ell_{0n} \\ -L_{3m}L_{0n} - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} - \ell_{3m}\ell_{0n} + L_{0m}L_{3n} + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + \ell_{0m}\ell_{3n} & -L_{1m}L_{2n} + L_{1m}\ell_{2n} - \ell_{1m}L_{2n} + \ell_{1m}\ell_{2n} \end{pmatrix} \\
&= \begin{pmatrix} (-L_{1m}L_{2n} + L_{2m}L_{1n}) - L_{1m}\ell_{2n} + \ell_{1m}L_{2n} - L_{2m}\ell_{1n} + \ell_{2m}L_{1n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (-L_{3m}L_{0n} + L_{0m}L_{3n}) + L_{3m}\ell_{0n} - \ell_{3m}\ell_{0n} \\ (-L_{3m}L_{0n} + L_{0m}L_{3n}) - L_{3m}\ell_{0n} - \ell_{3m}L_{0n} + L_{0m}\ell_{3n} + \ell_{0m}L_{3n} + (+\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (-L_{1m}L_{2n} + L_{2m}L_{1n}) + L_{1m}\ell_{2n} - \ell_{1m}\ell_{2n} \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  :

$$\begin{aligned}
&= \begin{pmatrix} (-L_{1n}L_{2n} + L_{2n}L_{1n}) - L_{1n}\ell_{2n} + \ell_{1n}L_{2n} - L_{2n}\ell_{1n} + \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2n} - \ell_{2n}\ell_{1n}) & (-L_{3n}L_{0n} + L_{0n}L_{3n}) + L_{3n}\ell_{0n} - \ell_{3n}\ell_{0n} \\ (-L_{3n}L_{0n} + L_{0n}L_{3n}) - L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} + \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0n} + \ell_{0n}\ell_{3n}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + L_{1n}\ell_{2n} - \ell_{1n}\ell_{2n} \end{pmatrix} \\
&= \begin{pmatrix} -L_{1n}\ell_{2n} + \ell_{1n}L_{2n} - L_{2n}\ell_{1n} + \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2n} - \ell_{2n}\ell_{1n}) & +L_{3n}\ell_{0n} + \ell_{3n}L_{0n} - L_{0n}\ell_{3n} - \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0n} + \ell_{0n}\ell_{3n}) \\ -L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} + \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0n} + \ell_{0n}\ell_{3n}) & +L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + L_{2n}\ell_{1n} - \ell_{2n}L_{1n} + (+\ell_{1n}\ell_{2n} - \ell_{2n}\ell_{1n}) \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2m} - \ell_{2n})L_{1n} + (+\ell_{1m} - \ell_{1n})L_{2n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & (+\ell_{3m} - \ell_{3n})L_{0n} + (+\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ (+\ell_{3n} - \ell_{3m})L_{0n} + (+\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & (+\ell_{2n} - \ell_{2m})L_{1n} + (+\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1n}L_{2m} + L_{2n}L_{1m}) + ([\ell_{2n}L_{1m} + \ell_{1n}L_{2m}]) + (-[+L_{1n}\ell_{2m} + L_{2n}\ell_{1m}] + [+ \ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}]) & (-L_{3n}L_{0m} + L_{0n}L_{3m}) \\ (-L_{3n}L_{0m} + L_{0n}L_{3m}) + (-[+\ell_{3n}L_{0m} - \ell_{0n}L_{3m}]) + ([+L_{0n}\ell_{3m} - L_{3n}\ell_{0m}] + [-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}]) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) \end{pmatrix}$$

□

**Lemma I.3.1.3-(ABc1r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( D_{2n}^{\leftrightarrow} D_{0m} - D_{1n}^{\leftrightarrow} D_{3m} - D_{0n}^\dagger D_{2m}^{\leftrightarrow} + D_{3n} D_{1m}^\dagger \right) = \begin{pmatrix} (-L_{1n}L_{3m} + L_{3n}L_{1m}) - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} - L_{3n}\ell_{1m} + \ell_{3n}L_{1m} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) - L_{2n}\ell_{0m} + \ell_{2n}L_{0m} \\ (L_{2n}L_{0m} - L_{0n}L_{2m}) + L_{2n}\ell_{0m} + \ell_{2n}L_{0m} - L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & (-L_{1n}L_{3m} + L_{3n}L_{1m}) + L_{1n}\ell_{3m} - \ell_{1n}L_{3m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} -L_{1n}\ell_{3m} + \ell_{1n}L_{3n} - L_{3n}\ell_{1m} + \ell_{3n}L_{1n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & -L_{2n}\ell_{0m} - \ell_{2n}L_{0n} + L_{0n}\ell_{2m} + \ell_{0n}L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \\ +L_{2n}\ell_{0m} + \ell_{2n}L_{0n} - L_{0n}\ell_{2m} - \ell_{0n}L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & +L_{1n}\ell_{3m} - \ell_{1n}L_{3n} + L_{3n}\ell_{1m} - \ell_{3n}L_{1n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \\ (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1n}L_{3m} + L_{3n}L_{1m}) + \ell_{1n}L_{3m} + \ell_{3n}L_{1m} + (-L_{1n}\ell_{3m} - L_{3n}\ell_{1m} + \ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) - L_{2n}\ell_{0m} + \ell_{2n}L_{0m} \\ (L_{2n}L_{0m} - L_{0n}L_{2m}) + \ell_{2n}L_{0m} - \ell_{0n}L_{2m} + (+L_{2n}\ell_{0m} - L_{0n}\ell_{2m} + \ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & (-L_{1n}L_{3m} + L_{3n}L_{1m}) - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} \left( D_{2n}^{\leftrightarrow} D_{0m} - D_{1n}^{\leftrightarrow} D_{3m} - D_{0n}^\dagger D_{2m}^{\leftrightarrow} + D_{3n} D_{1m}^\dagger \right) &= \left( \begin{pmatrix} 0 & D_{2n}^- \\ D_{2n}^+ & 0 \end{pmatrix} \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} - \begin{pmatrix} 0 & D_{1n}^- \\ D_{1n}^+ & 0 \end{pmatrix} \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} - \begin{pmatrix} D_{0n}^- & 0 \\ 0 & D_{0n}^+ \end{pmatrix} \begin{pmatrix} D_{2m}^+ & 0 \\ 0 & D_{2m}^- \end{pmatrix} \right) \\ &= \begin{pmatrix} -D_{1n}^- D_{3m}^+ + D_{3n}^+ D_{1m}^- & D_{2n}^- D_{0m}^- - D_{0n}^- D_{2m}^- \\ D_{2n}^+ D_{0m}^+ - D_{0n}^+ D_{2m}^+ & -D_{1n}^+ D_{3m}^- + D_{3n}^- D_{1m}^+ \end{pmatrix} \\ &= \begin{pmatrix} -(L_{1n} - \ell_{1n})(L_{3m} + \ell_{3m}) + (L_{3n} + \ell_{3n})(L_{1m} - \ell_{1m}) & (L_{2n} - \ell_{2n})(L_{0m} - \ell_{0m}) - (L_{0n} - \ell_{0n})(L_{2m} - \ell_{2m}) \\ (L_{2n} + \ell_{2n})(L_{0m} + \ell_{0m}) - (L_{0n} + \ell_{0n})(L_{2m} + \ell_{2m}) & -(L_{1n} + \ell_{1n})(L_{3m} - \ell_{3m}) + (L_{3n} - \ell_{3n})(L_{1m} + \ell_{1m}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{1n}(L_{3m} + \ell_{3m}) + \ell_{1n}(L_{3m} + \ell_{3m}) + L_{3n}(L_{1m} - \ell_{1m}) + \ell_{3n}(L_{1m} - \ell_{1m}) & L_{2n}(L_{0m} - \ell_{0m}) - \ell_{2n}(L_{0m} - \ell_{0m}) - \\ L_{2n}(L_{0m} + \ell_{0m}) + \ell_{2n}(L_{0m} + \ell_{0m}) - L_{0n}(L_{2m} + \ell_{2m}) - \ell_{0n}(L_{2m} + \ell_{2m}) & -L_{1n}(L_{3m} - \ell_{3m}) - \ell_{1n}(L_{3m} - \ell_{3m}) + \\ -L_{1n}L_{3m} - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} + \ell_{1n}\ell_{3m} + L_{3n}L_{1m} - L_{3n}\ell_{1m} + \ell_{3n}L_{1m} - \ell_{3n}\ell_{1m} & L_{2n}L_{0m} - L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + \ell_{2n}\ell_{0m} \\ L_{2n}L_{0m} + L_{2n}\ell_{0m} + \ell_{2n}L_{0m} + \ell_{2n}\ell_{0m} - L_{0n}L_{2m} - L_{0n}\ell_{2m} - \ell_{0n}L_{2m} - \ell_{0n}\ell_{2m} & -L_{1n}L_{3m} + L_{1n}\ell_{3m} - \ell_{1n}L_{3m} + \ell_{1n}\ell_{3m} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{1n}L_{3m} + L_{3n}L_{1m}) - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} - L_{3n}\ell_{1m} + \ell_{3n}L_{1m} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) - L_{2n}\ell_{0m} + \ell_{2n}L_{0m} \\ (L_{2n}L_{0m} - L_{0n}L_{2m}) + L_{2n}\ell_{0m} + \ell_{2n}L_{0m} - L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & (-L_{1n}L_{3m} + L_{3n}L_{1m}) + L_{1n}\ell_{3m} - \ell_{1n}L_{3m} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (-L_{1n}L_{3n} + L_{3n}L_{1n}) - L_{1n}\ell_{3m} + \ell_{1n}L_{3n} - L_{3n}\ell_{1m} + \ell_{3n}L_{1n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0n} - L_{0n}L_{2n}) - L_{2n}\ell_{0n} + \ell_{2n}L_{0n} \\ (L_{2n}L_{0n} - L_{0n}L_{2n}) + L_{2n}\ell_{0n} + \ell_{2n}L_{0n} - L_{0n}\ell_{2n} - \ell_{0n}L_{2n} + (+\ell_{2n}\ell_{0n} - \ell_{0n}\ell_{2n}) & (-L_{1n}L_{3n} + L_{3n}L_{1n}) + L_{1n}\ell_{3n} - \ell_{1n}L_{3n} \\ -L_{1n}\ell_{3m} + \ell_{1n}L_{3n} - L_{3n}\ell_{1m} + \ell_{3n}L_{1n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & -L_{2n}\ell_{0m} - \ell_{2n}L_{0n} + L_{0n}\ell_{2m} + \ell_{0n}L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \\ +L_{2n}\ell_{0m} + \ell_{2n}L_{0n} - L_{0n}\ell_{2m} - \ell_{0n}L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & +L_{1n}\ell_{3m} - \ell_{1n}L_{3n} + L_{3n}\ell_{1m} - \ell_{3n}L_{1n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \\ (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (+\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{1n}L_{3m} + L_{3n}L_{1m}) + \ell_{1n}L_{3m} + \ell_{3n}L_{1m} + (-L_{1n}\ell_{3m} - L_{3n}\ell_{1m} + \ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) - L_{2n}\ell_{0m} + \ell_{2n}L_{0m} \\ (L_{2n}L_{0m} - L_{0n}L_{2m}) + \ell_{2n}L_{0m} - \ell_{0n}L_{2m} + (+L_{2n}\ell_{0m} - L_{0n}\ell_{2m} + \ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & (-L_{1n}L_{3m} + L_{3n}L_{1m}) - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} \end{pmatrix}$$

□

**Lemma I.3.1.4-(ABc1r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\updownarrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\updownarrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( D_{1n}^{\updownarrow} D_{0m} + D_{2n}^{\updownarrow} D_{3m}^{\leftrightarrow} - D_{3n}^{\updownarrow} D_{2m}^{\leftrightarrow} - D_{0n} D_{1m}^{\updownarrow} \right) = \begin{pmatrix} (L_{1n}L_{0m} - L_{0n}L_{1m}) + L_{1n}\ell_{0m} - \ell_{1n}L_{0m} + L_{0n}\ell_{1m} - \ell_{0n}L_{1m} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (L_{2n}L_{3m} - L_{3n}L_{2m}) - L_{2n}\ell_{3m} - \\ (L_{2n}L_{3m} - L_{3n}L_{2m}) + L_{2n}\ell_{3m} + \ell_{2n}L_{3m} - L_{3n}\ell_{2m} - \ell_{3n}L_{2m} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (L_{1n}L_{0m} - L_{0n}L_{1m}) - L_{1n}\ell_{0m} + \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} +L_{1n}\ell_{0m} - \ell_{1n}L_{0n} + L_{0n}\ell_{1m} - \ell_{0n}L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & -L_{2n}\ell_{3m} - \ell_{2n}L_{3n} + L_{3n}\ell_{2m} + \ell_{3n}L_{2n} + (+\ell_{2n}\ell_{3m} - \\ +L_{2n}\ell_{3m} + \ell_{2n}L_{3n} - L_{3n}\ell_{2m} - \ell_{3n}L_{2n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & -L_{1n}\ell_{0m} + \ell_{1n}L_{0n} - L_{0n}\ell_{1m} + \ell_{0n}L_{1n} + (-\ell_{1n}\ell_{0m} + \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \\ (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1n}L_{0m} - L_{0n}L_{1m}) - \ell_{1n}L_{0m} - \ell_{0n}L_{1m} + (+L_{1n}\ell_{0m} + L_{0n}\ell_{1m} - \ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (L_{2n}L_{3m} - L_{3n}L_{2m}) - \ell_{2n}L_{3m} - \\ (L_{2n}L_{3m} - L_{3n}L_{2m}) + \ell_{2n}L_{3m} - \ell_{3n}L_{2m} + (+L_{2n}\ell_{3m} - L_{3n}\ell_{2m} + \ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (L_{1n}L_{0m} - L_{0n}L_{1m}) + \ell_{1n}L_{0m} \end{pmatrix}$$

**Proof:**

$$\left( D_{1n}^{\updownarrow} D_{0m} + D_{2n}^{\updownarrow} D_{3m}^{\leftrightarrow} - D_{3n}^{\updownarrow} D_{2m}^{\leftrightarrow} - D_{0n} D_{1m}^{\updownarrow} \right) = \left( \begin{pmatrix} D_{1n}^- & 0 \\ 0 & D_{1n}^+ \end{pmatrix} \begin{pmatrix} D_{0m}^+ & 0 \\ 0 & D_{0m}^- \end{pmatrix} + \begin{pmatrix} D_{2n}^- & 0 \\ 0 & D_{2n}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{3m}^- \\ D_{3m}^+ & 0 \end{pmatrix} - \begin{pmatrix} D_{3n}^- & 0 \\ 0 & D_{3n}^+ \end{pmatrix} \begin{pmatrix} 0 & D_{2m}^- \\ D_{2m}^+ & 0 \end{pmatrix} \right)$$

$$= \begin{pmatrix} D_{1n}^- D_{0m}^+ - D_{0n}^+ D_{1m}^- & D_{2n}^- D_{3m}^- - D_{3n}^- D_{2m}^- \\ D_{2n}^+ D_{3m}^+ - D_{3n}^+ D_{2m}^+ & D_{1n}^+ D_{0m}^- - D_{0n}^- D_{1m}^+ \end{pmatrix}$$

$$= \begin{pmatrix} (L_{1n} - \ell_{1n})(L_{0m} + \ell_{0m}) - (L_{0n} + \ell_{0n})(L_{1m} - \ell_{1m}) & (L_{2n} - \ell_{2n})(L_{3m} - \ell_{3m}) - (L_{3n} - \ell_{3n})(L_{2m} - \ell_{2m}) \\ (L_{2n} + \ell_{2n})(L_{3m} + \ell_{3m}) - (L_{3n} + \ell_{3n})(L_{2m} + \ell_{2m}) & (L_{1n} + \ell_{1n})(L_{0m} - \ell_{0m}) - (L_{0n} - \ell_{0n})(L_{1m} + \ell_{1m}) \end{pmatrix}$$

$$= \begin{pmatrix} L_{1n}(L_{0m} + \ell_{0m}) - \ell_{1n}(L_{0m} + \ell_{0m}) - L_{0n}(L_{1m} - \ell_{1m}) - \ell_{0n}(L_{1m} - \ell_{1m}) & L_{2n}(L_{3m} - \ell_{3m}) - \ell_{2n}(L_{3m} - \ell_{3m}) - L_{3n} \\ L_{2n}(L_{3m} + \ell_{3m}) + \ell_{2n}(L_{3m} + \ell_{3m}) - L_{3n}(L_{2m} + \ell_{2m}) - \ell_{3n}(L_{2m} + \ell_{2m}) & L_{1n}(L_{0m} - \ell_{0m}) + \ell_{1n}(L_{0m} - \ell_{0m}) - L_{0n} \end{pmatrix}$$

$$= \begin{pmatrix} L_{1n}L_{0m} + L_{1n}\ell_{0m} - \ell_{1n}L_{0m} - \ell_{1n}\ell_{0m} - L_{0n}L_{1m} + L_{0n}\ell_{1m} - \ell_{0n}L_{1m} + \ell_{0n}\ell_{1m} & L_{2n}L_{3m} - L_{2n}\ell_{3m} - \ell_{2n}L_{3m} + \ell_{2n}\ell_{3m} \\ L_{2n}L_{3m} + L_{2n}\ell_{3m} + \ell_{2n}L_{3m} + \ell_{2n}\ell_{3m} - L_{3n}L_{2m} - L_{3n}\ell_{2m} - \ell_{3n}L_{2m} - \ell_{3n}\ell_{2m} & L_{1n}L_{0m} - L_{1n}\ell_{0m} + \ell_{1n}L_{0m} - \ell_{1n}\ell_{0m} \end{pmatrix}$$

$$= \begin{pmatrix} (L_{1n}L_{0m} - L_{0n}L_{1m}) + L_{1n}\ell_{0m} - \ell_{1n}L_{0m} + L_{0n}\ell_{1m} - \ell_{0n}L_{1m} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (L_{2n}L_{3m} - L_{3n}L_{2m}) - L_{2n}\ell_{3m} - \\ (L_{2n}L_{3m} - L_{3n}L_{2m}) + \ell_{2n}\ell_{3m} + \ell_{2n}L_{3m} - L_{3n}\ell_{2m} - \ell_{3n}L_{2m} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (L_{1n}L_{0m} - L_{0n}L_{1m}) - L_{1n}\ell_{0m} \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{1n}L_{0n} - L_{0n}L_{1n}) + L_{1n}\ell_{0m} - \ell_{1n}L_{0n} + L_{0n}\ell_{1m} - \ell_{0n}L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (L_{2n}L_{3n} - L_{3n}L_{2n}) - L_{2n}\ell_{3m} - \\ (L_{2n}L_{3n} - L_{3n}L_{2n}) + \ell_{2n}\ell_{3m} + \ell_{2n}L_{3n} - L_{3n}\ell_{2m} - \ell_{3n}L_{2n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (L_{1n}L_{0n} - L_{0n}L_{1n}) - L_{1n}\ell_{0m} \end{pmatrix}$$

$$= \begin{pmatrix} +L_{1n}\ell_{0m} - \ell_{1n}L_{0n} + L_{0n}\ell_{1m} - \ell_{0n}L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & -L_{2n}\ell_{3m} - \ell_{2n}L_{3n} + L_{3n}\ell_{2m} + \ell_{3n}L_{2n} + (+\ell_{2n}\ell_{3m} - \\ +L_{2n}\ell_{3m} + \ell_{2n}L_{3n} - L_{3n}\ell_{2m} - \ell_{3n}L_{2n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & -L_{1n}\ell_{0m} + \ell_{1n}L_{0n} - L_{0n}\ell_{1m} + \ell_{0n}L_{1n} + (-\ell_{1n}\ell_{0m} + \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \\ (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1n}L_{0m} - L_{0n}L_{1m}) - \ell_{1n}L_{0m} - \ell_{0n}L_{1m} + (+L_{1n}\ell_{0m} + L_{0n}\ell_{1m} - \ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (L_{2n}L_{3m} - L_{3n}L_{2m}) - \ell_{2n}L_{3m} - \\ (L_{2n}L_{3m} - L_{3n}L_{2m}) + \ell_{2n}L_{3m} - \ell_{3n}L_{2m} + (+L_{2n}\ell_{3m} - L_{3n}\ell_{2m} + \ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (L_{1n}L_{0m} - L_{0n}L_{1m}) + \ell_{1n}L_{0m} \end{pmatrix}$$

□

**Lemma I.3.2.1-(ABc2r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\updownarrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\updownarrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( -D_{0n}^{\updownarrow} D_{3m}^{\leftrightarrow} + D_{3n}^{\updownarrow} D_{0m} - D_{2n}^{\updownarrow} D_{1m}^{\leftrightarrow} + D_{1n} D_{2m}^{\updownarrow} \right) = \begin{pmatrix} (-L_{2n}L_{1m} + L_{1n}L_{2m}) - L_{2n}\ell_{1m} + \ell_{2n}L_{1m} - L_{1n}\ell_{2m} + \ell_{1n}L_{2m} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (-L_{0n}L_{3m} + L_{3n}L_{0m}) + L_{0n}\ell_{3m} - \\ (-L_{0n}L_{3m} + L_{3n}L_{0m}) - L_{0n}\ell_{3m} - \ell_{0n}L_{3m} + L_{3n}\ell_{0m} + \ell_{3n}L_{0m} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (-L_{2n}L_{1m} + L_{1n}L_{2m}) + L_{2n}\ell_{1m} - \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} -L_{2n}\ell_{1m} + \ell_{2n}L_{1n} - L_{1n}\ell_{2m} + \ell_{1n}L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & +L_{0n}\ell_{3m} + \ell_{0n}L_{3n} - L_{3n}\ell_{0m} - \ell_{3n}L_{0n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \\ -L_{0n}\ell_{3m} - \ell_{0n}L_{3n} + L_{3n}\ell_{0m} + \ell_{3n}L_{0n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & +L_{2n}\ell_{1m} - \ell_{2n}L_{1n} + L_{1n}\ell_{2m} - \ell_{1n}L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2n} - \ell_{2m})L_{1n} + (+\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \\ (+\ell_{3n} - \ell_{3m})L_{0n} + (-\ell_{0n} + \ell_{0m})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (-\ell_{2n} + \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2n}L_{1m} + L_{1n}L_{2m}) + \ell_{2n}L_{1m} + \ell_{1n}L_{2m} + (-L_{2n}\ell_{1m} - L_{1n}\ell_{2m} + \ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (-L_{0n}L_{3m} + L_{3n}L_{0m}) + \ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m} \\ (-L_{0n}L_{3m} + L_{3n}L_{0m}) - \ell_{0n}L_{3m} + \ell_{3n}L_{0m} + (-L_{0n}\ell_{3m} + L_{3n}\ell_{0m} - \ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (-L_{2n}L_{1m} + L_{1n}L_{2m}) - \ell_{2n}L_{1m} + \ell_{1n}L_{2m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (-D_{0n}^{\hat{\circ}}D_{3m}^{\leftrightarrow} + D_{3n}^{\leftrightarrow}D_{0m} - D_{2n}^{\leftrightarrow}D_{1m}^{\leftrightarrow} + D_{1n}D_{2m}^{\hat{\circ}}) &= - \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \begin{pmatrix} 0 & (L_{3m} - \ell_{3m}) \\ (L_{3m} + \ell_{3m}) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} -(L_{2n} - \ell_{2n})(L_{1m} + \ell_{1m}) + (L_{1n} + \ell_{1n})(L_{2m} - \ell_{2m}) & -(L_{0n} - \ell_{0n})(L_{3m} - \ell_{3m}) + (L_{3n} - \ell_{3n})(L_{0m} - \ell_{0m}) \\ -(L_{0n} + \ell_{0n})(L_{3m} + \ell_{3m}) + (L_{3n} + \ell_{3n})(L_{0m} + \ell_{0m}) & -(L_{2n} + \ell_{2n})(L_{1m} - \ell_{1m}) + (L_{1n} - \ell_{1n})(L_{2m} + \ell_{2m}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{2n}L_{1m} - L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + \ell_{2n}\ell_{1m} + L_{1n}L_{2m} - L_{1n}\ell_{2m} + \ell_{1n}L_{2m} - \ell_{1n}\ell_{2m} & -L_{0n}L_{3m} + L_{0n}\ell_{3m} + \ell_{0n}L_{3m} - \ell_{0n}\ell_{3m} \\ -L_{0n}L_{3m} - L_{0n}\ell_{3m} - \ell_{0n}L_{3m} - \ell_{0n}\ell_{3m} + L_{3n}L_{0m} + L_{3n}\ell_{0m} + \ell_{3n}L_{0m} + \ell_{3n}\ell_{0m} & -L_{2n}L_{1m} + L_{2n}\ell_{1m} - \ell_{2n}L_{1m} + \ell_{2n}\ell_{1m} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{2n}L_{1m} + L_{1n}L_{2m}) - L_{2n}\ell_{1m} + \ell_{2n}L_{1m} - L_{1n}\ell_{2m} + \ell_{1n}L_{2m} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (-L_{0n}L_{3m} + L_{3n}L_{0m}) + \ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m} \\ (-L_{0n}L_{3m} + L_{3n}L_{0m}) - L_{0n}\ell_{3m} - \ell_{0n}L_{3m} + L_{3n}\ell_{0m} + \ell_{3n}L_{0m} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (-L_{2n}L_{1m} + L_{1n}L_{2m}) - \ell_{2n}L_{1m} + \ell_{1n}L_{2m} \end{pmatrix} \end{aligned}$$

$$L_{jm} = L_{jn} :$$

$$\begin{aligned} &= \begin{pmatrix} (-L_{2n}L_{1n} + L_{1n}L_{2n}) - L_{2n}\ell_{1m} + \ell_{2n}L_{1n} - L_{1n}\ell_{2m} + \ell_{1n}L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (-L_{0n}L_{3n} + L_{3n}L_{0n}) + \ell_{0n}\ell_{3n} - \ell_{3n}\ell_{0n} \\ (-L_{0n}L_{3n} + L_{3n}L_{0n}) - L_{0n}\ell_{3n} - \ell_{0n}L_{3n} + L_{3n}\ell_{0n} + \ell_{3n}L_{0n} + (-\ell_{0n}\ell_{3n} + \ell_{3n}\ell_{0n}) & (-L_{2n}L_{1n} + L_{1n}L_{2n}) + \ell_{2n}\ell_{1n} - \ell_{1n}\ell_{2n} \end{pmatrix} \\ &= \begin{pmatrix} -L_{2n}\ell_{1m} + \ell_{2n}L_{1n} - L_{1n}\ell_{2m} + \ell_{1n}L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & +L_{0n}\ell_{3m} + \ell_{0n}L_{3n} - L_{3n}\ell_{0m} - \ell_{3n}L_{0n} + (-\ell_{0n}\ell_{3m} + \ell_{3m}\ell_{0n}) \\ -L_{0n}\ell_{3m} - \ell_{0n}L_{3n} + L_{3n}\ell_{0m} + \ell_{3n}L_{0n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & +L_{2n}\ell_{1m} - \ell_{2n}L_{1n} + L_{1n}\ell_{2m} - \ell_{1n}L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2n} - \ell_{2m})L_{1n} + (+\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3m}\ell_{0n}) \\ (+\ell_{3n} - \ell_{3m})L_{0n} + (-\ell_{0n} + \ell_{0m})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3m}\ell_{0n}) & (-\ell_{2n} + \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1m}\ell_{2n}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2n}L_{1m} + L_{1n}L_{2m}) + \ell_{2n}L_{1m} + \ell_{1n}L_{2m} + (-L_{2n}\ell_{1m} - L_{1n}\ell_{2m} + \ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (-L_{0n}L_{3m} + L_{3n}L_{0m}) + \ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m} \\ (-L_{0n}L_{3m} + L_{3n}L_{0m}) - \ell_{0n}L_{3m} + \ell_{3n}L_{0m} + (-L_{0n}\ell_{3m} + L_{3n}\ell_{0m} - \ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (-L_{2n}L_{1m} + L_{1n}L_{2m}) - \ell_{2n}L_{1m} + \ell_{1n}L_{2m} \end{pmatrix}$$

□

**Lemma I.3.2.2-(ABc2r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\hat{\circ}} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\hat{\circ}\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{3n}^{\leftrightarrow}D_{3m}^{\leftrightarrow} + D_{0n}^{\hat{\circ}}D_{0m} + D_{1n}^{\leftrightarrow}D_{1m}^{\leftrightarrow} + D_{2n}D_{2m}^{\hat{\circ}}) = \begin{pmatrix} (L_{3n}L_{3m} + L_{0n}L_{0m} + L_{1n}L_{1m} + L_{2n}L_{2m}) + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} - L_{2n}\ell_{2m} + \ell_{2n}L_{2m} & 0 \\ 0 & 0 \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + L_{3n}\ell_{3n} - \ell_{3n}L_{3n} + L_{0n}\ell_{0n} - \ell_{0n}L_{0n} + L_{1n}\ell_{1n} - \ell_{1n}L_{1n} - L_{2n}\ell_{2n} + \ell_{2n}L_{2n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} - \ell_{0n}\ell_{3m} + \ell_{3m}\ell_{0n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) - (\ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n} + \ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n}) & 0 \\ 0 & (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) - (\ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n} + \ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n}) \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{3n}L_{3m} + L_{0n}L_{0m} + L_{1n}L_{1m} + L_{2n}L_{2m}) - \ell_{3n}L_{3m} - \ell_{0n}L_{0m} - \ell_{1n}L_{1m} + \ell_{2n}L_{2m} + [+(L_{3n}\ell_{3m} + L_{0n}\ell_{0m} + L_{1n}\ell_{1m} - \ell_{2n}\ell_{2m})] & 0 \\ 0 & 0 \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{3n}^{\leftrightarrow} D_{3m}^{\leftrightarrow} + D_{0n}^{\uparrow} D_{0m} + D_{1n}^{\leftrightarrow} D_{1m}^{\leftrightarrow} + D_{2n} D_{2m}^{\uparrow}) &= \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \begin{pmatrix} 0 & (L_{3m} - \ell_{3m}) \\ (L_{3m} + \ell_{3m}) & 0 \end{pmatrix} + \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \\ &= \begin{pmatrix} (L_{3n} - \ell_{3n})(L_{3m} + \ell_{3m}) + (L_{0n} - \ell_{0n})(L_{0m} + \ell_{0m}) + (L_{1n} - \ell_{1n})(L_{1m} + \ell_{1m}) + (L_{2n} + \ell_{2n})(L_{2m} - \ell_{2m}) & \\ & 0 \end{pmatrix} \\ &= \begin{pmatrix} L_{3n}(L_{3m} + \ell_{3m}) - \ell_{3n}(L_{3m} + \ell_{3m}) + L_{0n}(L_{0m} + \ell_{0m}) - \ell_{0n}(L_{0m} + \ell_{0m}) + L_{1n}(L_{1m} + \ell_{1m}) - \ell_{1n}(L_{1m} + \ell_{1m}) + L_{2n}(L_{2m} - \ell_{2m}) - \ell_{2n}(L_{2m} - \ell_{2m}) & \\ & 0 \end{pmatrix} \\ &= \begin{pmatrix} L_{3n}L_{3m} + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} - \ell_{3n}\ell_{3m} + L_{0n}L_{0m} + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} - \ell_{0n}\ell_{0m} + L_{1n}L_{1m} + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} - \ell_{1n}\ell_{1m} + L_{2n}L_{2m} - L_{2n}\ell_{2m} - \ell_{2n}L_{2m} - \ell_{2n}\ell_{2m} & \\ & 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{3n}L_{3m} + L_{0n}L_{0m} + L_{1n}L_{1m} + L_{2n}L_{2m}) + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} - L_{2n}\ell_{2m} + \ell_{2n}L_{2m} & \\ & 0 \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + L_{3n}\ell_{3m} - \ell_{3n}L_{3n} + L_{0n}\ell_{0m} - \ell_{0n}L_{0n} + L_{1n}\ell_{1m} - \ell_{1n}L_{1n} - L_{2n}\ell_{2m} + \ell_{2n}L_{2n} & \\ & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} & \\ & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) - (\ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n} + \ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n}) & \\ & 0 \end{pmatrix} \quad (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n})$$

$L_{jm}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{3n}L_{3m} + L_{0n}L_{0m} + L_{1n}L_{1m} + L_{2n}L_{2m}) - \ell_{3n}L_{3m} - \ell_{0n}L_{0m} - \ell_{1n}L_{1m} + \ell_{2n}L_{2m} + [(+L_{3n}\ell_{3m} + L_{0n}\ell_{0m} + L_{1n}\ell_{1m} + L_{2n}\ell_{2m})] & \\ & 0 \end{pmatrix}$$

□

**Lemma I.3.2.3-(ABc2r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$\begin{aligned} D_{ij}^+ &\equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij}) \\ D_{ij} &\equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix} \end{aligned}$$

then:

$$\begin{aligned} (-D_{2n}^{\leftrightarrow} D_{3m}^{\leftrightarrow} - D_{1n}^{\leftrightarrow} D_{0m} + D_{0n}^{\uparrow} D_{1m}^{\leftrightarrow} + D_{3n} D_{2m}^{\uparrow}) &= \begin{pmatrix} (-L_{2n}L_{3m} + L_{3n}L_{2m}) - L_{2n}\ell_{3m} + \ell_{2n}L_{3m} - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (-L_{1n}L_{0m} + L_{0n}L_{1m}) + L_{1n}\ell_{0m} - \ell_{1n}\ell_{0m} \\ (-L_{1n}L_{0m} + L_{0n}L_{1m}) - L_{1n}\ell_{0m} - \ell_{1n}L_{0m} + L_{0n}\ell_{1m} + \ell_{0n}L_{1m} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (-L_{2n}L_{3m} + L_{3n}L_{2m}) + L_{2n}\ell_{3m} - \ell_{2n}L_{3m} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (-L_{2n}L_{3n} + L_{3n}L_{2n}) - L_{2n}\ell_{3m} + \ell_{2n}L_{3n} - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (-L_{1n}L_{0n} + L_{0n}L_{1n}) + L_{1n}\ell_{0m} - \ell_{1n}\ell_{0m} \\ (-L_{1n}L_{0m} + L_{0n}L_{1n}) - L_{1n}\ell_{0m} - \ell_{1n}L_{0n} + L_{0n}\ell_{1m} + \ell_{0n}L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (-L_{2n}L_{3n} + L_{3n}L_{2n}) + L_{2n}\ell_{3m} - \ell_{2n}L_{3n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \\ (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2n}L_{3m} + L_{3n}L_{2m}) + \ell_{2n}L_{3m} + \ell_{3n}L_{2m} + (-L_{2n}\ell_{3m} - L_{3n}\ell_{2m} + \ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (-L_{1n}L_{0m} + L_{0n}L_{1m}) + L_{1n}\ell_{0m} - \ell_{1n}\ell_{0m} \\ (-L_{1n}L_{0m} + L_{0n}L_{1m}) - \ell_{1n}L_{0m} + \ell_{0n}L_{1m} + (-L_{1n}\ell_{0m} + L_{0n}\ell_{1m} - \ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (-L_{2n}L_{3m} + L_{3n}L_{2m}) + L_{2n}\ell_{3m} - \ell_{2n}L_{3m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (-D_{2n}^{\leftrightarrow} D_{3m}^{\leftrightarrow} - D_{1n}^{\leftrightarrow} D_{0m} + D_{0n}^{\uparrow} D_{1m}^{\leftrightarrow} + D_{3n} D_{2m}^{\uparrow}) &= - \begin{pmatrix} 0 & (L_{2n} - \ell_{2n}) \\ (L_{2n} + \ell_{2n}) & 0 \end{pmatrix} \begin{pmatrix} 0 & (L_{3m} - \ell_{3m}) \\ (L_{3m} + \ell_{3m}) & 0 \end{pmatrix} - \begin{pmatrix} 0 & (L_{1n} - \ell_{1n}) \\ (L_{1n} + \ell_{1n}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} -(L_{2n} - \ell_{2n})(L_{3m} + \ell_{3m}) + (L_{3n} + \ell_{3n})(L_{2m} - \ell_{2m}) & -(L_{1n} - \ell_{1n})(L_{0m} - \ell_{0m}) + (L_{0n} - \ell_{0n})(L_{1m} - \ell_{1m}) \\ -(L_{1n} + \ell_{1n})(L_{0m} + \ell_{0m}) + (L_{0n} + \ell_{0n})(L_{1m} + \ell_{1m}) & -(L_{2n} + \ell_{2n})(L_{3m} - \ell_{3m}) + (L_{3n} - \ell_{3n})(L_{2m} + \ell_{2m}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{2n}L_{3m} - L_{2n}\ell_{3m} + \ell_{2n}L_{3m} + \ell_{2n}\ell_{3m} + L_{3n}L_{2m} - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} - \ell_{3n}\ell_{2m} & -L_{1n}L_{0m} + L_{1n}\ell_{0m} + \ell_{1n}L_{0m} - \ell_{1n}\ell_{0m} \\ -L_{1n}L_{0m} - L_{1n}\ell_{0m} - \ell_{1n}L_{0m} - \ell_{1n}\ell_{0m} + L_{0n}L_{1m} + L_{0n}\ell_{1m} + \ell_{0n}L_{1m} + \ell_{0n}\ell_{1m} & -L_{2n}L_{3m} + L_{2n}\ell_{3m} - \ell_{2n}L_{3m} + \ell_{2n}\ell_{3m} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{2n}L_{3m} + L_{3n}L_{2m}) - L_{2n}\ell_{3m} + \ell_{2n}L_{3m} - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (-L_{1n}L_{0m} + L_{0n}L_{1m}) + L_{1n}\ell_{0m} - \ell_{1n}\ell_{0m} \\ (-L_{1n}L_{0m} + L_{0n}L_{1m}) - L_{1n}\ell_{0m} - \ell_{1n}L_{0m} + L_{0n}\ell_{1m} + \ell_{0n}L_{1m} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (-L_{2n}L_{3m} + L_{3n}L_{2m}) + L_{2n}\ell_{3m} - \ell_{2n}L_{3m} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (-L_{2n}L_{3n} + L_{3n}L_{2n}) - L_{2n}\ell_{3m} + \ell_{2n}L_{3n} - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (-L_{1n}L_{0n} + L_{0n}L_{1n}) + L_{1n}\ell_{0m} - \ell_{1n}\ell_{0m} \\ (-L_{1n}L_{0m} + L_{0n}L_{1n}) - L_{1n}\ell_{0m} - \ell_{1n}L_{0n} + L_{0n}\ell_{1m} + \ell_{0n}L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (-L_{2n}L_{3n} + L_{3n}L_{2n}) + L_{2n}\ell_{3m} - \ell_{2n}L_{3n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \\ (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{2n}L_{3m} + L_{3n}L_{2m}) + \ell_{2n}L_{3m} + \ell_{3n}L_{2m} + (-L_{2n}\ell_{3m} - L_{3n}\ell_{2m} + \ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & (-L_{1n}L_{0m} + L_{0n}L_{1m}) + \ell_{1n}L_{0m} + \ell_{0n}L_{1m} \\ (-L_{1n}L_{0m} + L_{0n}L_{1m}) - \ell_{1n}L_{0m} + \ell_{0n}L_{1m} + (-L_{1n}\ell_{0m} + L_{0n}\ell_{1m} - \ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & (-L_{2n}L_{3m} + L_{3n}L_{2m}) + \ell_{2n}L_{3m} + \ell_{3n}L_{2m} \end{pmatrix}$$

□

**Lemma I.3.2.4-(ABc2r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\dagger} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left(-D_{1n}^{\dagger}D_{3m}^{\leftrightarrow} + D_{2n}^{\dagger}D_{0m} - D_{3n}^{\dagger}D_{1m}^{\leftrightarrow} - D_{0n}D_{2m}^{\dagger}\right) = \begin{pmatrix} (L_{2n}L_{0m} - L_{0n}L_{2m}) + L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + (-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) & (L_{1n}L_{3m} - L_{3n}L_{1m}) - L_{1n}\ell_{3m} + \ell_{1n}\ell_{3m} \\ (L_{1n}L_{3m} - L_{3n}L_{1m}) + L_{1n}\ell_{3m} + \ell_{1n}L_{3m} - L_{3n}\ell_{1m} - \ell_{3n}L_{1m} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) - L_{2n}\ell_{0m} + \ell_{2n}\ell_{0m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{0n} - L_{0n}L_{2n}) + L_{2n}\ell_{0n} - \ell_{2n}L_{0n} + L_{0n}\ell_{2n} - \ell_{0n}L_{2n} + (-\ell_{2n}\ell_{0n} + \ell_{0n}\ell_{2n}) & (L_{1n}L_{3n} - L_{3n}L_{1n}) - L_{1n}\ell_{3n} + \ell_{1n}\ell_{3n} \\ (L_{1n}L_{3n} - L_{3n}L_{1n}) + L_{1n}\ell_{3n} + \ell_{1n}L_{3n} - L_{3n}\ell_{1n} - \ell_{3n}L_{1n} + (+\ell_{1n}\ell_{3n} - \ell_{3n}\ell_{1n}) & (L_{2n}L_{0n} - L_{0n}L_{2n}) - L_{2n}\ell_{0n} + \ell_{2n}\ell_{0n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \\ (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{2n}L_{0m} - L_{0n}L_{2m}) - \ell_{2n}L_{0m} - \ell_{0n}L_{2m} + (+L_{2n}\ell_{0m} + L_{0n}\ell_{2m} - \ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) & (L_{1n}L_{3m} - L_{3n}L_{1m}) - \ell_{1n}\ell_{3m} + \ell_{1n}\ell_{3m} \\ (L_{1n}L_{3m} - L_{3n}L_{1m}) + \ell_{1n}L_{3m} - \ell_{3n}L_{1m} + (+L_{1n}\ell_{3m} - L_{3n}\ell_{1m} + \ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) + \ell_{2n}L_{0m} - \ell_{2n}\ell_{0m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} \left(-D_{1n}^{\dagger}D_{3m}^{\leftrightarrow} + D_{2n}^{\dagger}D_{0m} - D_{3n}^{\dagger}D_{1m}^{\leftrightarrow} - D_{0n}D_{2m}^{\dagger}\right) &= \begin{pmatrix} (L_{1n} - \ell_{1n}) & 0 \\ 0 & (L_{1n} + \ell_{1n}) \end{pmatrix} \begin{pmatrix} 0 & (L_{3m} - \ell_{3m}) \\ (L_{3m} + \ell_{3m}) & 0 \end{pmatrix} + \begin{pmatrix} (L_{2n} - \ell_{2n}) & 0 \\ 0 & (L_{2n} + \ell_{2n}) \end{pmatrix} \\ &= \begin{pmatrix} (L_{2n} - \ell_{2n})(L_{0m} + \ell_{0m}) - (L_{0n} + \ell_{0n})(L_{2m} - \ell_{2m}) & (L_{1n} - \ell_{1n})(L_{3m} - \ell_{3m}) - (L_{3n} - \ell_{3n})(L_{1m} - \ell_{1m}) \\ (L_{1n} + \ell_{1n})(L_{3m} + \ell_{3m}) - (L_{3n} + \ell_{3n})(L_{1m} + \ell_{1m}) & (L_{2n} + \ell_{2n})(L_{0m} - \ell_{0m}) - (L_{0n} - \ell_{0n})(L_{2m} + \ell_{2m}) \end{pmatrix} \\ &= \begin{pmatrix} L_{2n}L_{0m} + L_{2n}\ell_{0m} - \ell_{2n}L_{0m} - \ell_{2n}\ell_{0m} - L_{0n}L_{2m} + L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + \ell_{0n}\ell_{2m} & L_{1n}L_{3m} - L_{1n}\ell_{3m} - \ell_{1n}L_{3m} + \ell_{1n}\ell_{3m} \\ L_{1n}L_{3m} + L_{1n}\ell_{3m} + \ell_{1n}L_{3m} + \ell_{1n}\ell_{3m} - L_{3n}L_{1m} - L_{3n}\ell_{1m} - \ell_{3n}L_{1m} - \ell_{3n}\ell_{1m} & L_{2n}L_{0m} - L_{2n}\ell_{0m} + \ell_{2n}L_{0m} - \ell_{2n}\ell_{0m} \end{pmatrix} \\ &= \begin{pmatrix} (L_{2n}L_{0m} - L_{0n}L_{2m}) + L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + (-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) & (L_{1n}L_{3m} - L_{3n}L_{1m}) - L_{1n}\ell_{3m} + \ell_{1n}\ell_{3m} \\ (L_{1n}L_{3m} - L_{3n}L_{1m}) + L_{1n}\ell_{3m} + \ell_{1n}L_{3m} - L_{3n}\ell_{1m} - \ell_{3n}L_{1m} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) - L_{2n}\ell_{0m} + \ell_{2n}\ell_{0m} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{0n} - L_{0n}L_{2n}) + L_{2n}\ell_{0n} - \ell_{2n}L_{0n} + L_{0n}\ell_{2n} - \ell_{0n}L_{2n} + (-\ell_{2n}\ell_{0n} + \ell_{0n}\ell_{2n}) & (L_{1n}L_{3n} - L_{3n}L_{1n}) - L_{1n}\ell_{3n} + \ell_{1n}\ell_{3n} \\ (L_{1n}L_{3n} - L_{3n}L_{1n}) + L_{1n}\ell_{3n} + \ell_{1n}L_{3n} - L_{3n}\ell_{1n} - \ell_{3n}L_{1n} + (+\ell_{1n}\ell_{3n} - \ell_{3n}\ell_{1n}) & (L_{2n}L_{0n} - L_{0n}L_{2n}) - L_{2n}\ell_{0n} + \ell_{2n}\ell_{0n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \\ (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (+\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{2n}L_{0m} - L_{0n}L_{2m}) - \ell_{2n}L_{0m} - \ell_{0n}L_{2m} + (+L_{2n}\ell_{0m} + L_{0n}\ell_{2m} - \ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) & (L_{1n}L_{3m} - L_{3n}L_{1m}) - \ell_{1n}\ell_{3m} + \ell_{1n}\ell_{3m} \\ (L_{1n}L_{3m} - L_{3n}L_{1m}) + \ell_{1n}L_{3m} - \ell_{3n}L_{1m} + (+L_{1n}\ell_{3m} - L_{3n}\ell_{1m} + \ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & (L_{2n}L_{0m} - L_{0n}L_{2m}) + \ell_{2n}L_{0m} - \ell_{2n}\ell_{0m} \end{pmatrix}$$

□

**Lemma I.3.3.1-(ABc3r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$



$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\uparrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( D_{0n}^\uparrow D_{2m}^{\leftrightarrow} - D_{3n}^{\leftrightarrow} D_{1m}^{\leftrightarrow} - D_{2n}^{\leftrightarrow} D_{0m} + D_{1n} D_{3m}^\uparrow \right) = \begin{pmatrix} (-L_{3n}L_{1m} + L_{1n}L_{3m}) - L_{3n}\ell_{1m} + \ell_{3n}L_{1m} - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) - L_{0n}\ell_{2m} \\ (L_{0n}L_{2m} - L_{2n}L_{0m}) + L_{0n}\ell_{2m} + \ell_{0n}L_{2m} - L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) + L_{3n}\ell_{1m} \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (-L_{3n}L_{1n} + L_{1n}L_{3n}) - L_{3n}\ell_{1m} + \ell_{3n}L_{1n} - L_{1n}\ell_{3m} + \ell_{1n}L_{3n} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (L_{0n}L_{2n} - L_{2n}L_{0n}) - L_{0n}\ell_{2m} \\ (L_{0n}L_{2n} - L_{2n}L_{0n}) + L_{0n}\ell_{2m} + \ell_{0n}L_{2n} - L_{2n}\ell_{0m} - \ell_{2n}L_{0n} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1n} + L_{1n}L_{3n}) + L_{3n}\ell_{1m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \\ (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3n}L_{1m} + L_{1n}L_{3m}) + \ell_{3n}L_{1m} + \ell_{1n}L_{3m} + (-L_{3n}\ell_{1m} - L_{1n}\ell_{3m} + \ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) - \ell_{0n}L_{2m} \\ (L_{0n}L_{2m} - L_{2n}L_{0m}) + \ell_{0n}L_{2m} - \ell_{2n}L_{0m} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) - \ell_{3n}L_{1m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} \left( D_{0n}^\uparrow D_{2m}^{\leftrightarrow} - D_{3n}^{\leftrightarrow} D_{1m}^{\leftrightarrow} - D_{2n}^{\leftrightarrow} D_{0m} + D_{1n} D_{3m}^\uparrow \right) &= \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \begin{pmatrix} 0 & (L_{2m} - \ell_{2m}) \\ (L_{2m} + \ell_{2m}) & 0 \end{pmatrix} - \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} -(L_{3n} - \ell_{3n})(L_{1m} + \ell_{1m}) + (L_{1n} + \ell_{1n})(L_{3m} - \ell_{3m}) & (L_{0n} - \ell_{0n})(L_{2m} - \ell_{2m}) - (L_{2n} - \ell_{2n})(L_{0m} - \ell_{0m}) \\ (L_{0n} + \ell_{0n})(L_{2m} + \ell_{2m}) - (L_{2n} + \ell_{2n})(L_{0m} + \ell_{0m}) & -(L_{3n} + \ell_{3n})(L_{1m} - \ell_{1m}) + (L_{1n} - \ell_{1n})(L_{3m} + \ell_{3m}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{3n}L_{1m} - L_{3n}\ell_{1m} + \ell_{3n}L_{1m} + \ell_{3n}\ell_{1m} + L_{1n}L_{3m} - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} - \ell_{1n}\ell_{3m} & L_{0n}L_{2m} - L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + \ell_{0n}\ell_{2m} \\ L_{0n}L_{2m} + L_{0n}\ell_{2m} + \ell_{0n}L_{2m} + \ell_{0n}\ell_{2m} - L_{2n}L_{0m} - L_{2n}\ell_{0m} - \ell_{2n}L_{0m} - \ell_{2n}\ell_{0m} & -L_{3n}L_{1m} + L_{3n}\ell_{1m} - \ell_{3n}L_{1m} + \ell_{3n}\ell_{1m} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{3n}L_{1m} + L_{1n}L_{3m}) - L_{3n}\ell_{1m} + \ell_{3n}L_{1m} - L_{1n}\ell_{3m} + \ell_{1n}L_{3m} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) - L_{0n}\ell_{2m} \\ (L_{0n}L_{2m} - L_{2n}L_{0m}) + L_{0n}\ell_{2m} + \ell_{0n}L_{2m} - L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) + L_{3n}\ell_{1m} \end{pmatrix} \end{aligned}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (-L_{3n}L_{1n} + L_{1n}L_{3n}) - L_{3n}\ell_{1m} + \ell_{3n}L_{1n} - L_{1n}\ell_{3m} + \ell_{1n}L_{3n} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (L_{0n}L_{2n} - L_{2n}L_{0n}) - L_{0n}\ell_{2m} \\ (L_{0n}L_{2n} - L_{2n}L_{0n}) + L_{0n}\ell_{2m} + \ell_{0n}L_{2n} - L_{2n}\ell_{0m} - \ell_{2n}L_{0n} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1n} + L_{1n}L_{3n}) + L_{3n}\ell_{1m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \\ (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (+\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3n}L_{1m} + L_{1n}L_{3m}) + \ell_{3n}L_{1m} + \ell_{1n}L_{3m} + (-L_{3n}\ell_{1m} - L_{1n}\ell_{3m} + \ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) - \ell_{0n}L_{2m} \\ (L_{0n}L_{2m} - L_{2n}L_{0m}) + \ell_{0n}L_{2m} - \ell_{2n}L_{0m} + (+\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) - \ell_{3n}L_{1m} \end{pmatrix}$$

□

**Lemma I.3.3.2-(ABc3r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}), \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix}, \quad D_{ij}^\uparrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix}, \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix}, \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( -D_{3n}^{\leftrightarrow} D_{2m}^{\leftrightarrow} - D_{0n}^\uparrow D_{1m}^{\leftrightarrow} + D_{1n}^{\leftrightarrow} D_{0m} + D_{2n} D_{3m}^\uparrow \right) = \begin{pmatrix} (-L_{3n}L_{2m} + L_{2n}L_{3m}) - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} - L_{2n}\ell_{3m} + \ell_{2n}L_{3m} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (-L_{0n}L_{1m} + L_{1n}L_{0m}) + L_{0n}\ell_{1m} \\ (-L_{0n}L_{1m} + L_{1n}L_{0m}) - L_{0n}\ell_{1m} - \ell_{0n}L_{1m} + L_{1n}\ell_{0m} + \ell_{1n}L_{0m} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (-L_{3n}L_{2m} + L_{2n}L_{3m}) + L_{3n}\ell_{2m} \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (-L_{3n}L_{2n} + L_{2n}L_{3n}) - L_{3n}\ell_{2m} + \ell_{3n}L_{2n} - L_{2n}\ell_{3m} + \ell_{2n}L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (-L_{0n}L_{1n} + L_{1n}L_{0n}) + L_{0n}\ell_{1m} \\ (-L_{0n}L_{1n} + L_{1n}L_{0n}) - L_{0n}\ell_{1m} - \ell_{0n}L_{1n} + L_{1n}\ell_{0m} + \ell_{1n}L_{0n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (-L_{3n}L_{2n} + L_{2n}L_{3n}) + L_{3n}\ell_{2m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \\ (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3n}L_{2m} + L_{2n}L_{3m}) + \ell_{3n}L_{2m} + \ell_{2n}L_{3m} + (-L_{3n}\ell_{2m} - L_{2n}\ell_{3m} + \ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (-L_{0n}L_{1m} + L_{1n}L_{0m}) + \ell_{0n}L_{1m} + \ell_{1n}L_{0m} \\ (-L_{0n}L_{1m} + L_{1n}L_{0m}) - \ell_{0n}L_{1m} + \ell_{1n}L_{0m} + (-L_{0n}\ell_{1m} + L_{1n}\ell_{0m} - \ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (-L_{3n}L_{2m} + L_{2n}L_{3m}) - \ell_{3n}L_{2m} + \ell_{2n}L_{3m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (-D_{3n}^{\leftrightarrow} D_{2m}^{\leftrightarrow} - D_{0n}^{\uparrow} D_{1m}^{\leftrightarrow} + D_{1n}^{\leftrightarrow} D_{0m} + D_{2n} D_{3m}^{\uparrow}) &= - \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \begin{pmatrix} 0 & (L_{2m} - \ell_{2m}) \\ (L_{2m} + \ell_{2m}) & 0 \end{pmatrix} - \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \\ &= \begin{pmatrix} -(L_{3n} - \ell_{3n})(L_{2m} + \ell_{2m}) + (L_{2n} + \ell_{2n})(L_{3m} - \ell_{3m}) & -(L_{0n} - \ell_{0n})(L_{1m} - \ell_{1m}) + (L_{1n} - \ell_{1n})(L_{0m} - \ell_{0m}) \\ -(L_{0n} + \ell_{0n})(L_{1m} + \ell_{1m}) + (L_{1n} + \ell_{1n})(L_{0m} + \ell_{0m}) & -(L_{3n} + \ell_{3n})(L_{2m} - \ell_{2m}) + (L_{2n} - \ell_{2n})(L_{3m} + \ell_{3m}) \end{pmatrix} \\ &= \begin{pmatrix} -L_{3n}L_{2m} - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} + \ell_{3n}\ell_{2m} + L_{2n}L_{3m} - L_{2n}\ell_{3m} + \ell_{2n}L_{3m} - \ell_{2n}\ell_{3m} & -L_{0n}L_{1m} + L_{0n}\ell_{1m} + \ell_{0n}L_{1m} - \ell_{0n}\ell_{1m} \\ -L_{0n}L_{1m} - L_{0n}\ell_{1m} - \ell_{0n}L_{1m} - \ell_{0n}\ell_{1m} + L_{1n}L_{0m} + L_{1n}\ell_{0m} + \ell_{1n}L_{0m} + \ell_{1n}\ell_{0m} & -L_{3n}L_{2m} + L_{3n}\ell_{2m} - \ell_{3n}L_{2m} + \ell_{3n}\ell_{2m} \end{pmatrix} \\ &= \begin{pmatrix} (-L_{3n}L_{2m} + L_{2n}L_{3m}) - L_{3n}\ell_{2m} + \ell_{3n}L_{2m} - L_{2n}\ell_{3m} + \ell_{2n}L_{3m} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (-L_{0n}L_{1m} + L_{1n}L_{0m}) + \ell_{0n}L_{1m} + \ell_{1n}L_{0m} \\ (-L_{0n}L_{1m} + L_{1n}L_{0m}) - L_{0n}\ell_{1m} - \ell_{0n}L_{1m} + L_{1n}\ell_{0m} + \ell_{1n}L_{0m} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (-L_{3n}L_{2m} + L_{2n}L_{3m}) - \ell_{3n}L_{2m} + \ell_{2n}L_{3m} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (-L_{3n}L_{2n} + L_{2n}L_{3n}) - L_{3n}\ell_{2n} + \ell_{3n}L_{2n} - L_{2n}\ell_{3n} + \ell_{2n}L_{3n} + (+\ell_{3n}\ell_{2n} - \ell_{2n}\ell_{3n}) & (-L_{0n}L_{1n} + L_{1n}L_{0n}) + L_{0n}\ell_{1n} + \ell_{1n}L_{0n} \\ (-L_{0n}L_{1n} + L_{1n}L_{0n}) - L_{0n}\ell_{1n} - \ell_{0n}L_{1n} + L_{1n}\ell_{0n} + \ell_{1n}L_{0n} + (-\ell_{0n}\ell_{1n} + \ell_{1n}\ell_{0n}) & (-L_{3n}L_{2n} + L_{2n}L_{3n}) + L_{3n}\ell_{2n} - \ell_{3n}L_{2n} + \ell_{2n}L_{3n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3n} - \ell_{3n})L_{2n} + (\ell_{2n} - \ell_{2n})L_{3n} + (+\ell_{3n}\ell_{2n} - \ell_{2n}\ell_{3n}) & (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \\ (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (-L_{3n}L_{2m} + L_{2n}L_{3m}) + \ell_{3n}L_{2m} + \ell_{2n}L_{3m} + (-L_{3n}\ell_{2m} - L_{2n}\ell_{3m} + \ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (-L_{0n}L_{1m} + L_{1n}L_{0m}) + \ell_{0n}L_{1m} + \ell_{1n}L_{0m} \\ (-L_{0n}L_{1m} + L_{1n}L_{0m}) - \ell_{0n}L_{1m} + \ell_{1n}L_{0m} + (-L_{0n}\ell_{1m} + L_{1n}\ell_{0m} - \ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (-L_{3n}L_{2m} + L_{2n}L_{3m}) - \ell_{3n}L_{2m} + \ell_{2n}L_{3m} \end{pmatrix}$$

□

**Lemma I.3.3.3-(ABc3r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{2n}^{\leftrightarrow} D_{2m}^{\leftrightarrow} + D_{1n}^{\leftrightarrow} D_{1m}^{\leftrightarrow} + D_{0n}^{\uparrow} D_{0m} + D_{3n} D_{3m}^{\uparrow}) = \begin{pmatrix} (L_{2n}L_{2m} + L_{1n}L_{1m} + L_{0n}L_{0m} + L_{3n}L_{3m}) + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} - L_{3n}\ell_{3m} + \ell_{3n}L_{3m} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + L_{2n}\ell_{2n} - \ell_{2n}L_{2n} + L_{1n}\ell_{1n} - \ell_{1n}L_{1n} + L_{0n}\ell_{0n} - \ell_{0n}L_{0n} - L_{3n}\ell_{3n} + \ell_{3n}L_{3n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + (\ell_{2n} - \ell_{2n})L_{2n} + (\ell_{1n} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0n})L_{0n} + (\ell_{3n} - \ell_{3n})L_{3n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) - (\ell_{2n}\ell_{2n} + \ell_{1n}\ell_{1n} + \ell_{0n}\ell_{0n} + \ell_{3n}\ell_{3n}) & 0 \\ 0 & (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{2n}L_{2m} + L_{1n}L_{1m} + L_{0n}L_{0m} + L_{3n}L_{3m}) - \ell_{2n}L_{2m} - \ell_{1n}L_{1m} - \ell_{0n}L_{0m} + \ell_{3n}L_{3m} + [+(+L_{2n}\ell_{2m} + L_{1n}\ell_{1m} + L_{0n}\ell_{0m} - \ell_{3n}\ell_{3m})] & 0 \\ 0 & 0 \end{pmatrix}$$

**Proof:**

$$\begin{aligned} D_{An} D_{Bm} \Rightarrow (D_{2n}^{\leftrightarrow} D_{2m}^{\leftrightarrow} + D_{1n}^{\leftrightarrow} D_{1m}^{\leftrightarrow} + D_{0n}^{\uparrow} D_{0m} + D_{3n} D_{3m}^{\uparrow}) &= \begin{pmatrix} 0 & (L_{2n} - \ell_{2n}) \\ (L_{2n} + \ell_{2n}) & 0 \end{pmatrix} \begin{pmatrix} 0 & (L_{2m} - \ell_{2m}) \\ (L_{2m} + \ell_{2m}) & 0 \end{pmatrix} + \begin{pmatrix} 0 & (L_{1n} - \ell_{1n}) \\ (L_{1n} + \ell_{1n}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{2n} - \ell_{2n})(L_{2m} + \ell_{2m}) + (L_{1n} - \ell_{1n})(L_{1m} + \ell_{1m}) + (L_{0n} - \ell_{0n})(L_{0m} + \ell_{0m}) + (L_{3n} + \ell_{3n})(L_{3m} - \ell_{3m}) & 0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{2n}L_{2m} + L_{1n}L_{1m} + L_{0n}L_{0m} + L_{3n}L_{3m}) + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} + L_{0n}\ell_{0m} - \ell_{0n}L_{0m} - L_{3n}\ell_{3m} + \ell_{3n}L_{3m} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + L_{2n}\ell_{2n} - \ell_{2n}L_{2n} + L_{1n}\ell_{1n} - \ell_{1n}L_{1n} + L_{0n}\ell_{0n} - \ell_{0n}L_{0n} - L_{3n}\ell_{3n} + \ell_{3n}L_{3n} & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$\begin{aligned}
&= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} & 0 \\ 0 & (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) - (\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m}) \end{pmatrix} \\
L_{jm} = L_{jn} \text{ are linear/differential } \& \ \ell_{ij} \text{ are constants } \& \ \ell_{jm} = \ell_{jn} : \\
&= \begin{pmatrix} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) - (\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m}) & 0 \\ 0 & (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} \end{pmatrix} \\
L_{jn} \text{ are functions or differential operators } \& \ \ell_{ij} \text{ are non-constant functions:} \\
&= \begin{pmatrix} (L_{2n}L_{2m} + L_{1n}L_{1m} + L_{0n}L_{0m} + L_{3n}L_{3m}) - \ell_{2n}L_{2m} - \ell_{1n}L_{1m} - \ell_{0n}L_{0m} + \ell_{3n}L_{3m} + [+(\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m})] & 0 \\ 0 & (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} \end{pmatrix}
\end{aligned}$$

□

**Lemma I.3.3.4-(ABc3r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$\begin{aligned}
D_{ij}^+ &\equiv (L_{ij} + \ell_{ij}) \ , \ D_{ij}^- \equiv (L_{ij} - \ell_{ij}) \\
D_{ij} &\equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \ , \ D_{ij}^\uparrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}
\end{aligned}$$

then:

$$(-D_{1n}^\uparrow D_{2m}^{\leftrightarrow} + D_{2n}^\uparrow D_{1m}^{\leftrightarrow} + D_{3n}^\uparrow D_{0m} - D_{0n} D_{3m}^\uparrow) = \begin{pmatrix} (L_{3n}L_{0m} - L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + L_{0n}\ell_{3m} - \ell_{0n}L_{3m} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}L_{2m} \\ (-L_{1n}L_{2m} + L_{2n}L_{1m}) - L_{1n}\ell_{2m} - \ell_{1n}L_{2m} + L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + (-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & (L_{3n}L_{0m} - L_{0n}L_{3m}) - L_{3n}\ell_{0m} + \ell_{3n}L_{0m} - \ell_{0n}L_{3m} + L_{0n}\ell_{3m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{0n} - L_{0n}L_{3n}) + L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} - \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0n} + \ell_{0n}\ell_{3n}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + L_{1n}\ell_{2n} - \ell_{1n}L_{2n} \\ (-L_{1n}L_{2n} + L_{2n}L_{1n}) - L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + L_{2n}\ell_{1n} + \ell_{2n}L_{1n} + (-\ell_{1n}\ell_{2n} + \ell_{2n}\ell_{1n}) & (L_{3n}L_{0n} - L_{0n}L_{3n}) - L_{3n}\ell_{0n} + \ell_{3n}L_{0n} - \ell_{0n}L_{3n} + L_{0n}\ell_{3n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} +(\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +(\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \\ +(\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & +(\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{3n}L_{0m} - L_{0n}L_{3m}) - \ell_{3n}L_{0m} - \ell_{0n}L_{3m} + (+L_{3n}\ell_{0m} + L_{0n}\ell_{3m} - \ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}L_{2m} \\ (-L_{1n}L_{2m} + L_{2n}L_{1m}) - \ell_{1n}L_{2m} + \ell_{2n}L_{1m} + (-L_{1n}\ell_{2m} + L_{2n}\ell_{1m} - \ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & (L_{3n}L_{0m} - L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} - \ell_{0n}L_{3m} + L_{0n}\ell_{3m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned}
(-D_{1n}^\uparrow D_{2m}^{\leftrightarrow} + D_{2n}^\uparrow D_{1m}^{\leftrightarrow} + D_{3n}^\uparrow D_{0m} - D_{0n} D_{3m}^\uparrow) &= - \begin{pmatrix} (L_{1n} - \ell_{1n}) & 0 \\ 0 & (L_{1n} + \ell_{1n}) \end{pmatrix} \begin{pmatrix} 0 & (L_{2m} - \ell_{2m}) \\ (L_{2m} + \ell_{2m}) & 0 \end{pmatrix} + \begin{pmatrix} (L_{2n} - \ell_{2n}) & 0 \\ 0 & (L_{2n} + \ell_{2n}) \end{pmatrix} \\
&= \begin{pmatrix} (L_{3n} - \ell_{3n})(L_{0m} + \ell_{0m}) - (L_{0n} + \ell_{0n})(L_{3m} - \ell_{3m}) & -(L_{1n} - \ell_{1n})(L_{2m} - \ell_{2m}) + (L_{2n} - \ell_{2n})(L_{1m} - \ell_{1m}) \\ -(L_{1n} + \ell_{1n})(L_{2m} + \ell_{2m}) + (L_{2n} + \ell_{2n})(L_{1m} + \ell_{1m}) & (L_{3n} + \ell_{3n})(L_{0m} - \ell_{0m}) - (L_{0n} - \ell_{0n})(L_{3m} + \ell_{3m}) \end{pmatrix} \\
&= \begin{pmatrix} L_{3n}L_{0m} + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} - \ell_{3n}\ell_{0m} - L_{0n}L_{3m} + L_{0n}\ell_{3m} - \ell_{0n}L_{3m} + \ell_{0n}\ell_{3m} & -L_{1n}L_{2m} + L_{1n}\ell_{2m} + \ell_{1n}L_{2m} - \ell_{1n}\ell_{2m} \\ -L_{1n}L_{2m} - L_{1n}\ell_{2m} - \ell_{1n}L_{2m} - \ell_{1n}\ell_{2m} + L_{2n}L_{1m} + L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + \ell_{2n}\ell_{1m} & L_{3n}L_{0m} - L_{3n}\ell_{0m} + \ell_{3n}L_{0m} - \ell_{0n}L_{3m} + L_{0n}\ell_{3m} \end{pmatrix} \\
&= \begin{pmatrix} (L_{3n}L_{0m} - L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + L_{0n}\ell_{3m} - \ell_{0n}L_{3m} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}L_{2m} \\ (-L_{1n}L_{2m} + L_{2n}L_{1m}) - L_{1n}\ell_{2m} - \ell_{1n}L_{2m} + L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + (-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & (L_{3n}L_{0m} - L_{0n}L_{3m}) - L_{3n}\ell_{0m} + \ell_{3n}L_{0m} - \ell_{0n}L_{3m} + L_{0n}\ell_{3m} \end{pmatrix}
\end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{3n}L_{0n} - L_{0n}L_{3n}) + L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + L_{0n}\ell_{3n} - \ell_{0n}L_{3n} + (-\ell_{3n}\ell_{0n} + \ell_{0n}\ell_{3n}) & (-L_{1n}L_{2n} + L_{2n}L_{1n}) + L_{1n}\ell_{2n} - \ell_{1n}L_{2n} \\ (-L_{1n}L_{2n} + L_{2n}L_{1n}) - L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + L_{2n}\ell_{1n} + \ell_{2n}L_{1n} + (-\ell_{1n}\ell_{2n} + \ell_{2n}\ell_{1n}) & (L_{3n}L_{0n} - L_{0n}L_{3n}) - L_{3n}\ell_{0n} + \ell_{3n}L_{0n} - \ell_{0n}L_{3n} + L_{0n}\ell_{3n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} +(\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +(\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \\ +(\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & +(\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{3n}L_{0m} - L_{0n}L_{3m}) - \ell_{3n}L_{0m} - \ell_{0n}L_{3m} + (+L_{3n}\ell_{0m} + L_{0n}\ell_{3m} - \ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & (-L_{1n}L_{2m} + L_{2n}L_{1m}) + L_{1n}\ell_{2m} - \ell_{1n}L_{2m} \\ (-L_{1n}L_{2m} + L_{2n}L_{1m}) - \ell_{1n}L_{2m} + \ell_{2n}L_{1m} + (-L_{1n}\ell_{2m} + L_{2n}\ell_{1m} - \ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & (L_{3n}L_{0m} - L_{0n}L_{3m}) + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} - \ell_{0n}L_{3m} + L_{0n}\ell_{3m} \end{pmatrix}$$

□

**Lemma I.3.4.1-(ABc4r1):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$\begin{aligned}
D_{ij}^+ &\equiv (L_{ij} + \ell_{ij}) \ , \ D_{ij}^- \equiv (L_{ij} - \ell_{ij}) \\
D_{ij} &\equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \ , \ D_{ij}^\uparrow \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \ , \ D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}
\end{aligned}$$

then:

$$\left( D_{0n}^\dagger D_{1m} + D_{3n}^{\leftrightarrow} D_{2m} - D_{2n}^{\leftrightarrow} D_{3m} - D_{1n} D_{0m}^\dagger \right) = \begin{pmatrix} (L_{0n}L_{1m} - L_{1n}L_{0m}) + L_{0n}\ell_{1m} - \ell_{0n}L_{1m} + L_{1n}\ell_{0m} - \ell_{1n}L_{0m} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (L_{3n}L_{2m} - L_{2n}L_{3m}) - L_{3n}\ell_{2m} - \\ (L_{3n}L_{2m} - L_{2n}L_{3m}) + L_{3n}\ell_{2m} + \ell_{3n}L_{2m} - L_{2n}\ell_{3m} - \ell_{2n}L_{3m} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (L_{0n}L_{1m} - L_{1n}L_{0m}) - L_{0n}\ell_{1m} + \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{0n}L_{1n} - L_{1n}L_{0n}) + L_{0n}\ell_{1m} - \ell_{0n}L_{1n} + L_{1n}\ell_{0m} - \ell_{1n}L_{0n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (L_{3n}L_{2n} - L_{2n}L_{3n}) - L_{3n}\ell_{2m} - \\ (L_{3n}L_{2n} - L_{2n}L_{3n}) + L_{3n}\ell_{2m} + \ell_{3n}L_{2n} - L_{2n}\ell_{3m} - \ell_{2n}L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (L_{0n}L_{1n} - L_{1n}L_{0n}) - L_{0n}\ell_{1m} + \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \\ (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{1m} - L_{1n}L_{0m}) - \ell_{0n}L_{1m} - \ell_{1n}L_{0m} + (+L_{0n}\ell_{1m} + L_{1n}\ell_{0m} - \ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (L_{3n}L_{2m} - L_{2n}L_{3m}) - \ell_{3n}L_{2m} - \\ (L_{3n}L_{2m} - L_{2n}L_{3m}) + \ell_{3n}L_{2m} - \ell_{2n}L_{3m} + (+L_{3n}\ell_{2m} - L_{2n}\ell_{3m} + \ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (L_{0n}L_{1m} - L_{1n}L_{0m}) + \ell_{0n}L_{1m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} \left( D_{0n}^\dagger D_{1m} + D_{3n}^{\leftrightarrow} D_{2m} - D_{2n}^{\leftrightarrow} D_{3m} - D_{1n} D_{0m}^\dagger \right) &= \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \begin{pmatrix} (L_{1m} + \ell_{1m}) & 0 \\ 0 & (L_{1m} - \ell_{1m}) \end{pmatrix} + \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n} - \ell_{0n})(L_{1m} + \ell_{1m}) - (L_{1n} + \ell_{1n})(L_{0m} - \ell_{0m}) & (L_{3n} - \ell_{3n})(L_{2m} - \ell_{2m}) - (L_{2n} - \ell_{2n})(L_{3m} - \ell_{3m}) \\ (L_{3n} + \ell_{3n})(L_{2m} + \ell_{2m}) - (L_{2n} + \ell_{2n})(L_{3m} + \ell_{3m}) & (L_{0n} + \ell_{0n})(L_{1m} - \ell_{1m}) - (L_{1n} - \ell_{1n})(L_{0m} + \ell_{0m}) \end{pmatrix} \\ &= \begin{pmatrix} L_{0n}L_{1m} + L_{0n}\ell_{1m} - \ell_{0n}L_{1m} - \ell_{0n}\ell_{1m} - L_{1n}L_{0m} + L_{1n}\ell_{0m} - \ell_{1n}L_{0m} + \ell_{1n}\ell_{0m} & L_{3n}L_{2m} - L_{3n}\ell_{2m} - \ell_{3n}L_{2m} + \ell_{3n}\ell_{2m} \\ L_{3n}L_{2m} + L_{3n}\ell_{2m} + \ell_{3n}L_{2m} + \ell_{3n}\ell_{2m} - L_{2n}L_{3m} - L_{2n}\ell_{3m} - \ell_{2n}L_{3m} - \ell_{2n}\ell_{3m} & L_{0n}L_{1m} - L_{0n}\ell_{1m} + \ell_{0n}L_{1m} - \ell_{0n}\ell_{1m} \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n}L_{1m} - L_{1n}L_{0m}) + L_{0n}\ell_{1m} - \ell_{0n}L_{1m} + L_{1n}\ell_{0m} - \ell_{1n}L_{0m} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (L_{3n}L_{2m} - L_{2n}L_{3m}) - L_{3n}\ell_{2m} - \\ (L_{3n}L_{2m} - L_{2n}L_{3m}) + L_{3n}\ell_{2m} + \ell_{3n}L_{2m} - L_{2n}\ell_{3m} - \ell_{2n}L_{3m} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (L_{0n}L_{1m} - L_{1n}L_{0m}) - L_{0n}\ell_{1m} \end{pmatrix} \end{aligned}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{0n}L_{1n} - L_{1n}L_{0n}) + L_{0n}\ell_{1m} - \ell_{0n}L_{1n} + L_{1n}\ell_{0m} - \ell_{1n}L_{0n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (L_{3n}L_{2n} - L_{2n}L_{3n}) - L_{3n}\ell_{2m} - \\ (L_{3n}L_{2n} - L_{2n}L_{3n}) + L_{3n}\ell_{2m} + \ell_{3n}L_{2n} - L_{2n}\ell_{3m} - \ell_{2n}L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (L_{0n}L_{1n} - L_{1n}L_{0n}) - L_{0n}\ell_{1m} + \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \\ (+\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n} + (+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (+\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n} + (-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{1m} - L_{1n}L_{0m}) - \ell_{0n}L_{1m} - \ell_{1n}L_{0m} + (+L_{0n}\ell_{1m} + L_{1n}\ell_{0m} - \ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & (L_{3n}L_{2m} - L_{2n}L_{3m}) - \ell_{3n}L_{2m} - \\ (L_{3n}L_{2m} - L_{2n}L_{3m}) + \ell_{3n}L_{2m} - \ell_{2n}L_{3m} + (+L_{3n}\ell_{2m} - L_{2n}\ell_{3m} + \ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & (L_{0n}L_{1m} - L_{1n}L_{0m}) + \ell_{0n}L_{1m} \end{pmatrix}$$

□

**Lemma 1.3.4.2-(ABc4r2):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$\left( -D_{3n}^{\leftrightarrow} D_{1m} + D_{0n}^\dagger D_{2m} + D_{1n}^{\leftrightarrow} D_{3m} - D_{2n} D_{0m}^\dagger \right) = \begin{pmatrix} (L_{0n}L_{2m} - L_{2n}L_{0m}) + L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + (-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) + L_{3n}\ell_{1m} - \\ (-L_{3n}L_{1m} + L_{1n}L_{3m}) - L_{3n}\ell_{1m} - \ell_{3n}L_{1m} + L_{1n}\ell_{3m} + \ell_{1n}\ell_{3m} + (-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) - L_{0n}\ell_{2m} + \end{pmatrix}$$

$$L_{jm} = L_{jn} :$$

$$= \begin{pmatrix} (L_{0n}L_{2n} - L_{2n}L_{0n}) + L_{0n}\ell_{2m} - \ell_{0n}L_{2n} + L_{2n}\ell_{0m} - \ell_{2n}L_{0n} + (-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1n} + L_{1n}L_{3n}) + L_{3n}\ell_{1n} - \\ (-L_{3n}L_{1n} + L_{1n}L_{3n}) - L_{3n}\ell_{1n} - \ell_{3n}L_{1n} + L_{1n}\ell_{3n} + \ell_{1n}\ell_{3n} + (-\ell_{3n}\ell_{1n} + \ell_{1n}\ell_{3n}) & (L_{0n}L_{2n} - L_{2n}L_{0n}) - L_{0n}\ell_{2m} + \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \\ (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jm}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{2m} - L_{2n}L_{0m}) - \ell_{0n}L_{2m} - \ell_{2n}L_{0m} + (+L_{0n}\ell_{2m} + L_{2n}\ell_{0m} - \ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) + \ell_{3n}L_{1m} - \ell_{1n}L_{3m} \\ (-L_{3n}L_{1m} + L_{1n}L_{3m}) - \ell_{3n}L_{1m} + \ell_{1n}L_{3m} + (-L_{3n}\ell_{1m} + L_{1n}\ell_{3m} - \ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) + \ell_{0n}L_{2m} - \ell_{2n}L_{0m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (-D_{3n}^{\leftrightarrow}D_{1m} + D_{0n}^{\uparrow}D_{2m} + D_{1n}^{\leftrightarrow}D_{3m} - D_{2n}D_{0m}^{\uparrow}) &= - \begin{pmatrix} 0 & (L_{3n} - \ell_{3n}) \\ (L_{3n} + \ell_{3n}) & 0 \end{pmatrix} \begin{pmatrix} (L_{1m} + \ell_{1m}) & 0 \\ 0 & (L_{1m} - \ell_{1m}) \end{pmatrix} + \begin{pmatrix} (L_{0n} - \ell_{0n}) & 0 \\ 0 & (L_{0n} + \ell_{0n}) \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n} - \ell_{0n})(L_{2m} + \ell_{2m}) - (L_{2n} + \ell_{2n})(L_{0m} - \ell_{0m}) & -(L_{3n} - \ell_{3n})(L_{1m} - \ell_{1m}) + (L_{1n} - \ell_{1n})(L_{3m} - \ell_{3m}) \\ -(L_{3n} + \ell_{3n})(L_{1m} + \ell_{1m}) + (L_{1n} + \ell_{1n})(L_{3m} + \ell_{3m}) & (L_{0n} + \ell_{0n})(L_{2m} - \ell_{2m}) - (L_{2n} - \ell_{2n})(L_{0m} + \ell_{0m}) \end{pmatrix} \\ &= \begin{pmatrix} L_{0n}L_{2m} + L_{0n}\ell_{2m} - \ell_{0n}L_{2m} - \ell_{0n}\ell_{2m} - L_{2n}L_{0m} + L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + \ell_{2n}\ell_{0m} & -L_{3n}L_{1m} + L_{3n}\ell_{1m} + \ell_{3n}L_{1m} - \ell_{3n}\ell_{1m} \\ -L_{3n}L_{1m} - L_{3n}\ell_{1m} - \ell_{3n}L_{1m} - \ell_{3n}\ell_{1m} + L_{1n}L_{3m} + L_{1n}\ell_{3m} + \ell_{1n}L_{3m} + \ell_{1n}\ell_{3m} & L_{0n}L_{2m} - L_{0n}\ell_{2m} + \ell_{0n}L_{2m} - \ell_{0n}\ell_{2m} \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n}L_{2m} - L_{2n}L_{0m}) + L_{0n}\ell_{2m} - \ell_{0n}L_{2m} + L_{2n}\ell_{0m} - \ell_{2n}L_{0m} + (-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) + \ell_{3n}L_{1m} - \ell_{1n}L_{3m} \\ (-L_{3n}L_{1m} + L_{1n}L_{3m}) - L_{3n}\ell_{1m} - \ell_{3n}L_{1m} + L_{1n}\ell_{3m} + \ell_{1n}L_{3m} + (-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) - L_{0n}\ell_{2m} + \ell_{0n}L_{2m} - \ell_{0n}\ell_{2m} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{2n} - L_{2n}L_{0n}) + L_{0n}\ell_{2n} - \ell_{0n}L_{2n} + L_{2n}\ell_{0n} - \ell_{2n}L_{0n} + (-\ell_{0n}\ell_{2n} + \ell_{2n}\ell_{0n}) & (-L_{3n}L_{1n} + L_{1n}L_{3n}) + \ell_{3n}L_{1n} - \ell_{1n}L_{3n} \\ (-L_{3n}L_{1n} + L_{1n}L_{3n}) - L_{3n}\ell_{1n} - \ell_{3n}L_{1n} + L_{1n}\ell_{3n} + \ell_{1n}L_{3n} + (-\ell_{3n}\ell_{1n} + \ell_{1n}\ell_{3n}) & (L_{0n}L_{2n} - L_{2n}L_{0n}) - L_{0n}\ell_{2n} + \ell_{0n}L_{2n} - \ell_{0n}\ell_{2n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n} + (-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & (+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n} + (-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \\ (+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n} + (-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & (+\ell_{2n} - \ell_{2m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n} + (-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{2m} - L_{2n}L_{0m}) - \ell_{0n}L_{2m} - \ell_{2n}L_{0m} + (+L_{0n}\ell_{2m} + L_{2n}\ell_{0m} - \ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & (-L_{3n}L_{1m} + L_{1n}L_{3m}) + \ell_{3n}L_{1m} - \ell_{1n}L_{3m} \\ (-L_{3n}L_{1m} + L_{1n}L_{3m}) - \ell_{3n}L_{1m} + \ell_{1n}L_{3m} + (-L_{3n}\ell_{1m} + L_{1n}\ell_{3m} - \ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & (L_{0n}L_{2m} - L_{2n}L_{0m}) + \ell_{0n}L_{2m} - \ell_{2n}L_{0m} \end{pmatrix}$$

□

**Lemma I.3.4.3-(ABc4r3):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^{\uparrow} \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\uparrow} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{2n}^{\leftrightarrow}D_{1m} - D_{1n}^{\leftrightarrow}D_{2m} + D_{0n}^{\uparrow}D_{3m} - D_{3n}D_{0m}^{\uparrow}) = \begin{pmatrix} (L_{0n}L_{3m} - L_{3n}L_{0m}) + L_{0n}\ell_{3m} - \ell_{0n}L_{3m} + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (L_{2n}L_{1m} - L_{1n}L_{2m}) - L_{2n}\ell_{1m} + \ell_{1n}L_{2m} \\ (L_{2n}L_{1m} - L_{1n}L_{2m}) + L_{2n}\ell_{1m} + \ell_{2n}L_{1m} - L_{1n}\ell_{2m} - \ell_{1n}L_{2m} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (L_{0n}L_{3m} - L_{3n}L_{0m}) - L_{0n}\ell_{3m} + \ell_{3n}L_{0m} - \ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m} \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{3n} - L_{3n}L_{0n}) + L_{0n}\ell_{3n} - \ell_{0n}L_{3n} + L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + (-\ell_{0n}\ell_{3n} + \ell_{3n}\ell_{0n}) & (L_{2n}L_{1n} - L_{1n}L_{2n}) - L_{2n}\ell_{1n} + \ell_{1n}L_{2n} \\ (L_{2n}L_{1n} - L_{1n}L_{2n}) + L_{2n}\ell_{1n} + \ell_{2n}L_{1n} - L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + (+\ell_{2n}\ell_{1n} - \ell_{1n}\ell_{2n}) & (L_{0n}L_{3n} - L_{3n}L_{0n}) - L_{0n}\ell_{3n} + \ell_{3n}L_{0n} - \ell_{0n}\ell_{3n} + \ell_{3n}\ell_{0n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (+\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \\ (+\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (+\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{3m} - L_{3n}L_{0m}) - \ell_{0n}L_{3m} - \ell_{3n}L_{0m} + (+L_{0n}\ell_{3m} + L_{3n}\ell_{0m} - \ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (L_{2n}L_{1m} - L_{1n}L_{2m}) - \ell_{2n}L_{1m} + \ell_{1n}L_{2m} \\ (L_{2n}L_{1m} - L_{1n}L_{2m}) + \ell_{2n}L_{1m} - \ell_{1n}L_{2m} + (+L_{2n}\ell_{1m} - L_{1n}\ell_{2m} + \ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (L_{0n}L_{3m} - L_{3n}L_{0m}) + \ell_{0n}L_{3m} - \ell_{3n}L_{0m} - \ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m} \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{2n}^{\leftrightarrow}D_{1m} - D_{1n}^{\leftrightarrow}D_{2m} + D_{0n}^{\uparrow}D_{3m} - D_{3n}D_{0m}^{\uparrow}) &= \begin{pmatrix} 0 & (L_{2n} - \ell_{2n}) \\ (L_{2n} + \ell_{2n}) & 0 \end{pmatrix} \begin{pmatrix} (L_{1m} + \ell_{1m}) & 0 \\ 0 & (L_{1m} - \ell_{1m}) \end{pmatrix} - \begin{pmatrix} 0 & (L_{1n} - \ell_{1n}) \\ (L_{1n} + \ell_{1n}) & 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n} - \ell_{0n})(L_{3m} + \ell_{3m}) - (L_{3n} + \ell_{3n})(L_{0m} - \ell_{0m}) & (L_{2n} - \ell_{2n})(L_{1m} - \ell_{1m}) - (L_{1n} - \ell_{1n})(L_{2m} - \ell_{2m}) \\ (L_{2n} + \ell_{2n})(L_{1m} + \ell_{1m}) - (L_{1n} + \ell_{1n})(L_{2m} + \ell_{2m}) & (L_{0n} + \ell_{0n})(L_{3m} - \ell_{3m}) - (L_{3n} - \ell_{3n})(L_{0m} + \ell_{0m}) \end{pmatrix} \\ &= \begin{pmatrix} L_{0n}L_{3m} + L_{0n}\ell_{3m} - \ell_{0n}L_{3m} - \ell_{0n}\ell_{3m} - L_{3n}L_{0m} + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + \ell_{3n}\ell_{0m} & L_{2n}L_{1m} - L_{2n}\ell_{1m} - \ell_{2n}L_{1m} + \ell_{2n}\ell_{1m} \\ L_{2n}L_{1m} + L_{2n}\ell_{1m} + \ell_{2n}L_{1m} + \ell_{2n}\ell_{1m} - L_{1n}L_{2m} - L_{1n}\ell_{2m} - \ell_{1n}L_{2m} - \ell_{1n}\ell_{2m} & L_{0n}L_{3m} - L_{0n}\ell_{3m} + \ell_{0n}L_{3m} - \ell_{0n}\ell_{3m} \end{pmatrix} \\ &= \begin{pmatrix} (L_{0n}L_{3m} - L_{3n}L_{0m}) + L_{0n}\ell_{3m} - \ell_{0n}L_{3m} + L_{3n}\ell_{0m} - \ell_{3n}L_{0m} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (L_{2n}L_{1m} - L_{1n}L_{2m}) - L_{2n}\ell_{1m} + \ell_{1n}L_{2m} \\ (L_{2n}L_{1m} - L_{1n}L_{2m}) + L_{2n}\ell_{1m} + \ell_{2n}L_{1m} - L_{1n}\ell_{2m} - \ell_{1n}L_{2m} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (L_{0n}L_{3m} - L_{3n}L_{0m}) - L_{0n}\ell_{3m} + \ell_{3n}L_{0m} - \ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m} \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{0n}L_{3n} - L_{3n}L_{0n}) + L_{0n}\ell_{3n} - \ell_{0n}L_{3n} + L_{3n}\ell_{0n} - \ell_{3n}L_{0n} + (-\ell_{0n}\ell_{3n} + \ell_{3n}\ell_{0n}) & (L_{2n}L_{1n} - L_{1n}L_{2n}) - L_{2n}\ell_{1n} + \ell_{1n}L_{2n} \\ (L_{2n}L_{1n} - L_{1n}L_{2n}) + L_{2n}\ell_{1n} + \ell_{2n}L_{1n} - L_{1n}\ell_{2n} - \ell_{1n}L_{2n} + (+\ell_{2n}\ell_{1n} - \ell_{1n}\ell_{2m}) & (L_{0n}L_{3n} - L_{3n}L_{0n}) - L_{0n}\ell_{3n} + \ell_{3n}L_{0n} - \ell_{0n}\ell_{3n} + \ell_{3n}\ell_{0n} \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (+\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \\ (+\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n} + (+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (+\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n} + (-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{0n}L_{3m} - L_{3n}L_{0m}) - \ell_{0n}L_{3m} - \ell_{3n}L_{0m} + (+L_{0n}\ell_{3m} + L_{3n}\ell_{0m} - \ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & (L_{2n}L_{1m} - L_{1n}L_{2m}) - \ell_{2n}L_{1m} \\ (L_{2n}L_{1m} - L_{1n}L_{2m}) + \ell_{2n}L_{1m} - \ell_{1n}L_{2m} + (+L_{2n}\ell_{1m} - L_{1n}\ell_{2m} + \ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & (L_{0n}L_{3m} - L_{3n}L_{0m}) + \ell_{0n}L_{3m} \end{pmatrix}$$

□

\*\*\*\*

**Lemma I.3.4.4-(ABc4r4):** For linear/differential operators  $L_{ij}$  &  $\ell_{ij}$

and:

$$D_{ij}^+ \equiv (L_{ij} + \ell_{ij}) \quad , \quad D_{ij}^- \equiv (L_{ij} - \ell_{ij})$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

then:

$$(D_{1n}^\dagger D_{1m} + D_{2n}^\dagger D_{2m} + D_{3n}^\dagger D_{3m} + D_{0n} D_{0m}^\dagger) = \begin{pmatrix} (L_{1n}L_{1m} + L_{2n}L_{2m} + L_{3n}L_{3m} + L_{0n}L_{0m}) + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} - L_{0n}\ell_{0m} + \ell_{0n}L_{0m} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + L_{1n}\ell_{1m} - \ell_{1n}L_{1n} + L_{2n}\ell_{2m} - \ell_{2n}L_{2n} + L_{3n}\ell_{3m} - \ell_{3n}L_{3n} - L_{0n}\ell_{0m} + \ell_{0n}L_{0n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} - \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) - (\ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n} + \ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n}) \\ 0 \end{pmatrix} \quad (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1n}L_{1m} + L_{2n}L_{2m} + L_{3n}L_{3m} + L_{0n}L_{0m}) - \ell_{1n}L_{1m} - \ell_{2n}L_{2m} - \ell_{3n}L_{3m} + \ell_{0n}L_{0m} + [(+L_{1n}\ell_{1m} + L_{2n}\ell_{2m} + L_{3n}\ell_{3m}) \\ 0 \end{pmatrix}$$

**Proof:**

$$\begin{aligned} (D_{1n}^\dagger D_{1m} + D_{2n}^\dagger D_{2m} + D_{3n}^\dagger D_{3m} + D_{0n} D_{0m}^\dagger) &= \begin{pmatrix} (L_{1n} - \ell_{1n}) & 0 \\ 0 & (L_{1n} + \ell_{1n}) \end{pmatrix} \begin{pmatrix} (L_{1m} + \ell_{1m}) & 0 \\ 0 & (L_{1m} - \ell_{1m}) \end{pmatrix} + \begin{pmatrix} (L_{2n} - \ell_{2n}) & 0 \\ 0 & (L_{2n} + \ell_{2n}) \end{pmatrix} \\ &= \begin{pmatrix} (L_{1n} - \ell_{1n})(L_{1m} + \ell_{1m}) + (L_{2n} - \ell_{2n})(L_{2m} + \ell_{2m}) + (L_{3n} - \ell_{3n})(L_{3m} + \ell_{3m}) + (L_{0n} + \ell_{0n})(L_{0m} - \ell_{0m}) \\ 0 \end{pmatrix} \quad (L_{1n} + \ell_{1n}) \\ &= \begin{pmatrix} L_{1n}L_{1m} + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} - \ell_{1n}\ell_{1m} + L_{2n}L_{2m} + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} - \ell_{2n}\ell_{2m} + L_{3n}L_{3m} + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} - \ell_{3n}\ell_{3m} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} (L_{1n}L_{1m} + L_{2n}L_{2m} + L_{3n}L_{3m} + L_{0n}L_{0m}) + L_{1n}\ell_{1m} - \ell_{1n}L_{1m} + L_{2n}\ell_{2m} - \ell_{2n}L_{2m} + L_{3n}\ell_{3m} - \ell_{3n}L_{3m} - L_{0n}\ell_{0m} + \ell_{0n}L_{0m} \\ 0 \end{pmatrix} \end{aligned}$$

$L_{jm} = L_{jn}$  :

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + L_{1n}\ell_{1m} - \ell_{1n}L_{1n} + L_{2n}\ell_{2m} - \ell_{2n}L_{2n} + L_{3n}\ell_{3m} - \ell_{3n}L_{3n} - L_{0n}\ell_{0m} + \ell_{0n}L_{0n} \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} - \\ 0 \end{pmatrix}$$

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$= \begin{pmatrix} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) - (\ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m} + \ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m}) \\ 0 \end{pmatrix} \quad (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n})$$

$L_{jn}$  are functions or differential operators &  $\ell_{ij}$  are non-constant functions:

$$= \begin{pmatrix} (L_{1n}L_{1m} + L_{2n}L_{2m} + L_{3n}L_{3m} + L_{0n}L_{0m}) - \ell_{1n}L_{1m} - \ell_{2n}L_{2m} - \ell_{3n}L_{3m} + \ell_{0n}L_{0m} + [(+L_{1n}\ell_{1m} + L_{2n}\ell_{2m} + L_{3n}\ell_{3m}) \\ 0 \end{pmatrix}$$

□

**Theorem II.1:** Given: linear/differential operators  $L_{ij}$  &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

and: **B-factors & A-factors:**  $D_{Bn}$  &  $D_{An}$  , under the same index  $n$  ,

**BA-factorizations & AB-factorizations** are equal:  $D_{Bn}D_{An} = D_{An}D_{Bn}$  ;

i.e.: **BA-factorizations & AB-factorizations** under the same index are commutative.

*Proof:*

From Theorem I.1:

$$D_{Bn}D_{An} = \begin{pmatrix} (D_{0n}D_{0n}^\dagger + D_{3n}^\dagger D_{3n}^\dagger + D_{2n}^\dagger D_{2n}^\dagger + D_{1n}D_{1n}^\dagger) & (D_{0n}D_{3n}^\dagger - D_{3n}^\dagger D_{0n}^\dagger - D_{2n}^\dagger D_{1n}^\dagger + D_{1n}D_{2n}^\dagger) & (-D_{0n}D_{2n}^\dagger - D_{3n}^\dagger D_{1n}^\dagger + D_{2n}^\dagger D_{0n}^\dagger + D_{1n}D_{3n}^\dagger) \\ (D_{3n}^\dagger D_{0n}^\dagger - D_{0n}D_{3n}^\dagger - D_{1n}^\dagger D_{2n}^\dagger + D_{2n}D_{1n}^\dagger) & (D_{3n}^\dagger D_{3n}^\dagger + D_{0n}D_{0n}^\dagger + D_{1n}^\dagger D_{1n}^\dagger + D_{2n}D_{2n}^\dagger) & (-D_{3n}^\dagger D_{2n}^\dagger + D_{0n}D_{1n}^\dagger - D_{1n}^\dagger D_{0n}^\dagger + D_{2n}D_{3n}^\dagger) \\ (-D_{2n}^\dagger D_{0n}^\dagger - D_{1n}^\dagger D_{3n}^\dagger + D_{0n}D_{2n}^\dagger + D_{3n}D_{1n}^\dagger) & (-D_{2n}^\dagger D_{3n}^\dagger + D_{1n}^\dagger D_{0n}^\dagger - D_{0n}D_{1n}^\dagger + D_{3n}D_{2n}^\dagger) & (D_{2n}^\dagger D_{2n}^\dagger + D_{1n}^\dagger D_{1n}^\dagger + D_{0n}D_{0n}^\dagger + D_{3n}D_{3n}^\dagger) \\ (D_{1n}^\dagger D_{0n}^\dagger - D_{2n}^\dagger D_{3n}^\dagger + D_{3n}^\dagger D_{2n}^\dagger - D_{0n}^\dagger D_{1n}^\dagger) & (D_{1n}^\dagger D_{3n}^\dagger + D_{2n}^\dagger D_{0n}^\dagger - D_{3n}^\dagger D_{1n}^\dagger - D_{0n}^\dagger D_{2n}^\dagger) & (-D_{1n}^\dagger D_{2n}^\dagger + D_{2n}^\dagger D_{1n}^\dagger + D_{3n}^\dagger D_{0n}^\dagger - D_{0n}^\dagger D_{3n}^\dagger) \end{pmatrix}$$

$$D_{An}D_{Bn} = \begin{pmatrix} (D_{0n}^\dagger D_{0n} + D_{3n}^\dagger D_{3n}^\dagger + D_{2n}^\dagger D_{2n}^\dagger + D_{1n}D_{1n}^\dagger) & (-D_{0n}^\dagger D_{3n}^\dagger + D_{3n}^\dagger D_{0n} - D_{2n}^\dagger D_{1n}^\dagger + D_{1n}D_{2n}^\dagger) & (D_{0n}^\dagger D_{2n}^\dagger - D_{3n}^\dagger D_{1n}^\dagger - D_{2n}^\dagger D_{0n} + D_{1n}D_{3n}^\dagger) \\ (-D_{3n}^\dagger D_{0n} + D_{0n}^\dagger D_{3n}^\dagger - D_{1n}^\dagger D_{2n}^\dagger + D_{2n}D_{1n}^\dagger) & (D_{3n}^\dagger D_{3n}^\dagger + D_{0n}^\dagger D_{0n} + D_{1n}^\dagger D_{1n}^\dagger + D_{2n}D_{2n}^\dagger) & (-D_{3n}^\dagger D_{2n}^\dagger - D_{0n}^\dagger D_{1n}^\dagger + D_{1n}^\dagger D_{0n} + D_{2n}D_{3n}^\dagger) \\ (D_{2n}^\dagger D_{0n} - D_{1n}^\dagger D_{3n}^\dagger - D_{0n}^\dagger D_{2n}^\dagger + D_{3n}D_{1n}^\dagger) & (-D_{2n}^\dagger D_{3n}^\dagger - D_{1n}^\dagger D_{0n} + D_{0n}^\dagger D_{1n}^\dagger + D_{3n}D_{2n}^\dagger) & (D_{2n}^\dagger D_{2n}^\dagger + D_{1n}^\dagger D_{1n}^\dagger + D_{0n}^\dagger D_{0n} + D_{3n}D_{3n}^\dagger) \\ (D_{1n}^\dagger D_{0n} + D_{2n}^\dagger D_{3n}^\dagger - D_{3n}^\dagger D_{1n}^\dagger - D_{0n}^\dagger D_{1n}^\dagger) & (-D_{1n}^\dagger D_{3n}^\dagger + D_{2n}^\dagger D_{0n} + D_{3n}^\dagger D_{1n}^\dagger - D_{0n}^\dagger D_{2n}^\dagger) & (-D_{1n}^\dagger D_{2n}^\dagger + D_{2n}^\dagger D_{1n}^\dagger + D_{3n}^\dagger D_{0n} - D_{0n}^\dagger D_{3n}^\dagger) \end{pmatrix}$$

So, from Lemmas I.2 & I.3, under the headings:  $L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants &  $\ell_{jm} = \ell_{jn}$  :

$$\Rightarrow D_{Bn}D_{An} = \begin{pmatrix} \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} \end{pmatrix}$$

$$\Rightarrow D_{An}D_{Bn} = \begin{pmatrix} \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} [L_n]^2 - |\ell_n|^2 & 0 \\ 0 & [L_n]^2 - |\ell_n|^2 \end{pmatrix} \end{pmatrix}$$

$$= D_{Bn}D_{An}$$

$$= [L_n]^2 - |\ell_n|^2 \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{pmatrix}$$

where:

$$[L_n]^2 \equiv (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) \quad \& \quad |\ell_n|^2 \equiv (\ell_{1n}\ell_{1n} + \ell_{2n}\ell_{2n} + \ell_{3n}\ell_{3n} + \ell_{0n}\ell_{0n})$$

□

**Theorem II.2:** For linear/differential operators  $L_{ij} \equiv \partial_i$  &  $\ell_{ij} \equiv m_i$  (constants)

$D_{Bn}$  &  $D_{An}$  and  $D_{Bn}D_{An}$  &  $D_{An}D_{Bn}$  are Helmholtzian factors and factorizations, respectively;

and:  $D_{Bn}D_{An} = D_{An}D_{Bn}$ .

*Proof:*

$$L_{ij} \equiv \partial_i \quad \& \quad \ell_{ij} \equiv m_i \Rightarrow D_{ij}^+ \equiv (\partial_i + m_i) \quad , \quad D_{ij}^- \equiv (\partial_i - m_i)$$

So:

$$D_{ij}^+ \equiv (\partial_i + m_i) \quad , \quad D_{ij}^- \equiv (\partial_i - m_i)$$

$$D_{ij} \equiv \begin{pmatrix} D_{ij}^+ & 0 \\ 0 & D_{ij}^- \end{pmatrix} \quad , \quad D_{ij}^\dagger \equiv \begin{pmatrix} D_{ij}^- & 0 \\ 0 & D_{ij}^+ \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow} \equiv \begin{pmatrix} 0 & D_{ij}^- \\ D_{ij}^+ & 0 \end{pmatrix} \quad , \quad D_{ij}^{\leftrightarrow\dagger} \equiv \begin{pmatrix} 0 & D_{ij}^+ \\ D_{ij}^- & 0 \end{pmatrix}$$

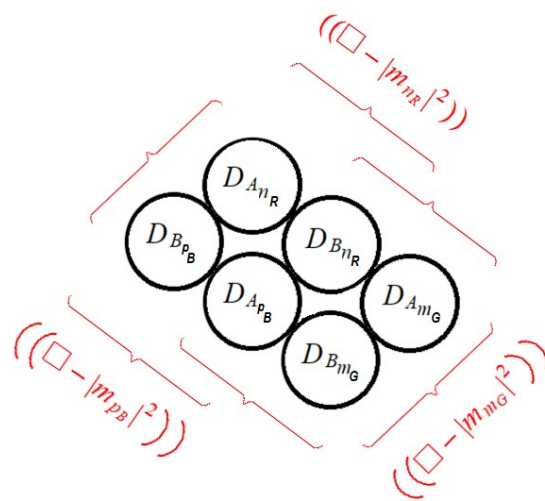
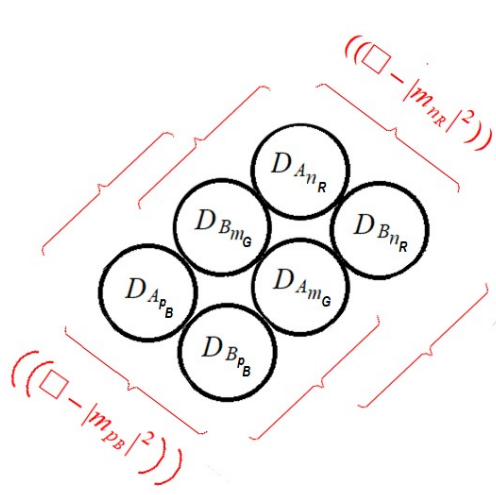
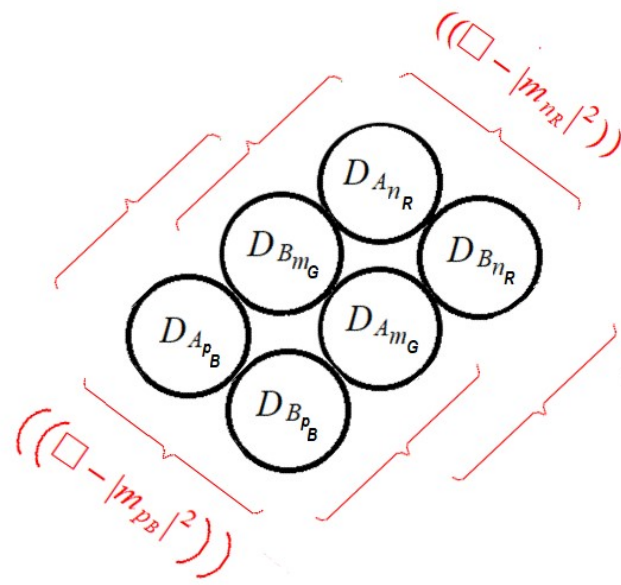
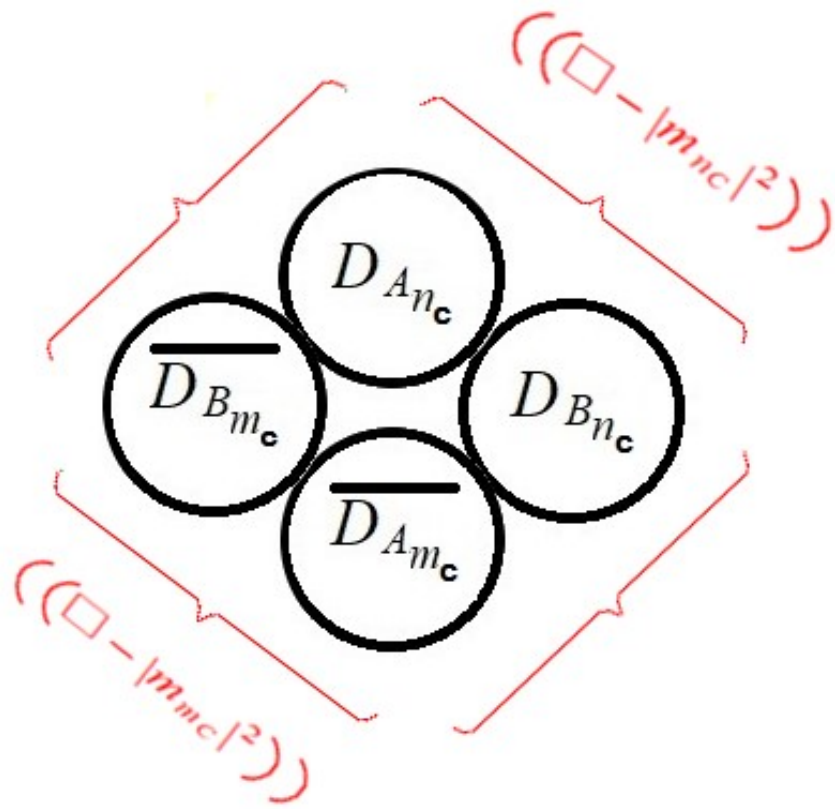
So, from Theorem II.1

$D_{Bn}$  &  $D_{An}$  and  $D_{Bn}D_{An}$  &  $D_{An}D_{Bn}$  are Helmholtzian factors and factorizations, respectively;

and:  $D_{Bn}D_{An} = D_{An}D_{Bn}$ .

□

as follows:



**Theorem III.1:** For linear/differential operators  $D_{Bn}$  &  $D_{Am}$  :  
 where:

$L_{jm} = L_{jn}$  are linear/differential &  $\varrho_{ij}$  are constants:

then:

$$D_{Bn}D_{Am} = \begin{pmatrix} c1 & c2 & c3 & c4 \end{pmatrix} (L_{\mu m} = L_{\mu n})$$

where:



$$\begin{aligned}
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0m} + \ell_{0n})L_{0n} + (-\ell_{3m} + \ell_{3n})L_{3n} + (-\ell_{2m} + \ell_{2n})L_{2n} + (-\ell_{1m} + \ell_{1n})L_{1n} + \\ -(\ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m} + \ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m}) \end{array} \right) \quad \mathbf{0} \\
& \mathbf{0} \quad \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + \\ -(\ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m} + \ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m}) \end{array} \right) \\
& \mathbf{c1=} \left( \begin{array}{l} \left( \begin{array}{l} (-\ell_{2m} + \ell_{2n})L_{1n} + \quad (\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1m} + \ell_{1n})L_{2n} + \quad +(\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \quad +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \end{array} \right) \\ \left( \begin{array}{l} (-\ell_{0m} + \ell_{0n})L_{3n} + \quad (\ell_{2m} - \ell_{2n})L_{1n} + \\ +(-\ell_{3m} + \ell_{3n})L_{0n} + \quad +(-\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \quad +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{1n} + \quad (-\ell_{0m} + \ell_{0n})L_{2n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + \quad +(\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \quad +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + \quad +(-\ell_{1n} + \ell_{1m})L_{3n} + \\ +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \quad +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{1m} - \ell_{1n})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} - \ell_{0m})L_{1n} + \quad +(\ell_{2n} - \ell_{2m})L_{3n} + \\ +(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \quad +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{2n} + \quad (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(+\ell_{2m} - \ell_{2n})L_{3n} + \quad +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \quad +(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2n} - \ell_{2m})L_{1n} + \quad (+\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{2n} + \quad +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \quad +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{0n} + \quad (+\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \quad +(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \end{array} \right) \end{array} \right) \\
& \mathbf{c2=} \left( \begin{array}{l} \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + \\ -(\ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m} + \ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m}) \end{array} \right) \quad \mathbf{0} \\ \mathbf{0} \quad \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + \\ -(\ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m} + \ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{2n} + \quad (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} - \ell_{2m})L_{3n} + \quad +(+\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \quad +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{1n} - \ell_{1m})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} - \ell_{0m})L_{1n} + \quad +(+\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \quad +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + \quad +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \quad +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + \quad +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \quad +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \end{array} \right) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{ll} (+\ell_{3n} - \ell_{3m})L_{1n} + & (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + & +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{3n}\ell_{1m} - \ell_{1m}\ell_{3n}) & +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + & +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{3n} - \ell_{3m})L_{2n} + & (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{2m} - \ell_{3n})L_{2n} + \\ +(\ell_{0m} - \ell_{0n})L_{1n} + & +(\ell_{3m} - \ell_{2n})L_{3n} + \\ +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + \\ -(\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m}) \end{array} \right) \quad 0 \\ \\ 0 \\ \\ \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{0n} + & +(\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + & +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) & +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} +(\ell_{2n} - \ell_{2m})L_{1n} + & +(\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{2n} + & +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(\ell_{0n} - \ell_{0m})L_{1n} + & +(\ell_{2m} - \ell_{2n})L_{3n} + \\ +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) & +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{3n} + & +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) & +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{2m} - \ell_{2n})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + & +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{3n} - \ell_{3m})L_{1n} + & +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) & +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{0n} + & (+\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + & +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) & +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{1n} - \ell_{1m})L_{2n} + & (+\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{1n} + & +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + \\ -(\ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m} + \ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m}) \end{array} \right) \quad 0 \\ \\ 0 \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + \\ -(\ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m} + \ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m}) \end{array} \right) \end{array} \right)
\end{aligned}$$

$$D_{Bm}D_{An} = \left( \begin{array}{cccc} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{array} \right) (L_{\mu n} = L_{\mu m})$$

where:

$$\begin{aligned}
 \mathbf{c1} = & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{2n} + \ell_{2m})L_{2n} + (-\ell_{1n} + \ell_{1m})L_{1n} + \\ -(\ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n} + \ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n}) \end{array} \right) \quad \mathbf{0} \\
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + \\ -(\ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n} + \ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (-\ell_{2n} + \ell_{2m})L_{1n} + & (\ell_{0n} - \ell_{0m})L_{3n} + \\ +(-\ell_{1n} + \ell_{1m})L_{2n} + & +(\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ \\ (-\ell_{0n} + \ell_{0m})L_{3n} + & (\ell_{2n} - \ell_{2m})L_{1n} + \\ +(-\ell_{3n} + \ell_{3m})L_{0n} + & +(-\ell_{1m} + \ell_{1n})L_{2n} + \\ +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{1n} + & (-\ell_{0n} + \ell_{0m})L_{2n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ \\ (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + & +(-\ell_{1m} + \ell_{1n})L_{3n} + \\ +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{1n} - \ell_{1m})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(\ell_{0m} - \ell_{0n})L_{1n} + & +(\ell_{2m} - \ell_{2n})L_{3n} + \\ +(\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) & +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) \\ \\ (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) & +(\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{2m} - \ell_{2n})L_{1n} + & (+\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{2n} + & +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \\ \\ (+\ell_{3n} - \ell_{3m})L_{0n} + & (+\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + & +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & +(\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) \end{array} \right) \\
 \mathbf{c2} = & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + \\ -(\ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n} + \ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n}) \end{array} \right) \quad \mathbf{0} \\
 & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + \\ -(\ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n} + \ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(+\ell_{2m} - \ell_{2n})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \\ \\ (+\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(+\ell_{0m} - \ell_{0n})L_{1n} + & +(+\ell_{2n} - \ell_{2m})L_{3n} + \\ +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \\ \\ (+\ell_{3n} - \ell_{3m})L_{1n} + & (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + \quad +(\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \quad +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + \quad +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \quad +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{2n} + \quad (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{3n} + \quad +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \\ \\ (+\ell_{1n} - \ell_{1m})L_{0n} + \quad (+\ell_{2n} - \ell_{3m})L_{2n} + \\ +(\ell_{0n} - \ell_{0m})L_{1n} + \quad +(\ell_{3n} - \ell_{2m})L_{3n} + \\ +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \quad +(\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + \\ -(\ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n} + \ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n}) \quad \mathbf{0} \\ \\ \mathbf{0} \end{array} \right) \quad \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + \\ -(\ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n} + \ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} \left( \begin{array}{l} +(\ell_{3n} - \ell_{3m})L_{0n} + \quad +(\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \quad +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \\ \\ +(\ell_{2m} - \ell_{2n})L_{1n} + \quad +(\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{2n} + \quad +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \quad +(\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{1n} - \ell_{1m})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{2n} + \\ +(\ell_{0m} - \ell_{0n})L_{1n} + \quad +(\ell_{2n} - \ell_{2m})L_{3n} + \\ +(\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \\ \\ (+\ell_{3n} - \ell_{3m})L_{2n} + \quad (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{3n} + \quad +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \quad +(\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2n} - \ell_{2m})L_{0n} + \quad (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + \quad +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \quad +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \\ \\ (+\ell_{3m} - \ell_{3n})L_{1n} + \quad +(\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + \quad +(\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \quad +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{0n} + \quad (+\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \quad +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\ \\ (+\ell_{1m} - \ell_{1n})L_{2n} + \quad (+\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{1n} + \quad +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \quad +(\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + \\ -(\ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n} + \ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n}) \quad \mathbf{0} \\ \\ \mathbf{0} \end{array} \right) \quad \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + \\ -(\ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n} + \ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} D_{Bn}D_{Am} \pm D_{Bm}D_{An} = \left( \begin{array}{cccc} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{array} \right) (L_{\mu m} = L_{\mu n}) \end{array} \right)
\end{aligned}$$

where:

$$\begin{aligned}
 & \left( \begin{aligned}
 & [(L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n})] + \\
 & + [(-\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} + \ell_{0m})]L_{0n} + [(-\ell_{3m} + \ell_{3n}) \pm (-\ell_{3n} + \ell_{3m})]L_{3n} + \\
 & + [(-\ell_{2m} + \ell_{2n}) \pm (-\ell_{2n} + \ell_{2m})]L_{2n} + [(-\ell_{1m} + \ell_{1n}) \pm (-\ell_{1n} + \ell_{1m})]L_{1n} + \\
 & - [(\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n})]
 \end{aligned} \right) & 0 \\
 & \left( \begin{aligned}
 & [(-\ell_{2m} + \ell_{2n}) \pm (-\ell_{2n} + \ell_{2m})]L_{1n} + & [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + \\
 & + [(-\ell_{1m} + \ell_{1n}) \pm (-\ell_{1n} + \ell_{1m})]L_{2n} + & + [(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{0n} + \\
 & + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})] & + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \pm (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\
 & [(-\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} + \ell_{0m})]L_{3n} + & [(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{1n} + \\
 & + [(-\ell_{3m} + \ell_{3n}) \pm (-\ell_{3n} + \ell_{3m})]L_{0n} + & + [(-\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} + \ell_{1n})]L_{2n} + \\
 & + [(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \pm (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n})] & + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})]
 \end{aligned} \right) & 0 \\
 & \left( \begin{aligned}
 & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + & [(-\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} + \ell_{0m})]L_{2n} + \\
 & + [(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + & + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{0n} + \\
 & + [(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})] & + [(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})] \\
 & [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + & [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{1n} + \\
 & + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{2n} + & + [(-\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} + \ell_{1n})]L_{3n} + \\
 & + [(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})] & + [(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})]
 \end{aligned} \right) & * \\
 & \left( \begin{aligned}
 & [(+\ell_{1m}^- - \ell_{1n}) \pm (+\ell_{1n}^- - \ell_{1m})]L_{0n} + & [(+\ell_{3m}^- - \ell_{3n}) \pm (+\ell_{3n}^- - \ell_{3m})]L_{2n} + \\
 & + [(+\ell_{0n}^- - \ell_{0m}) \pm (+\ell_{0m}^- - \ell_{0n})]L_{1n} + & + [(\ell_{2n}^- - \ell_{2m}) \pm (\ell_{2m}^- - \ell_{2n})]L_{3n} + \\
 & + [(+\ell_{1n}\ell_{0m}^- - \ell_{0n}\ell_{1m}^-) \pm (+\ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-)] & + [(-\ell_{2n}\ell_{3m}^- + \ell_{3n}\ell_{2m}^-) \pm (-\ell_{2m}\ell_{3n}^- + \ell_{3m}\ell_{2n}^-)] \\
 & [(+\ell_{3n}^- - \ell_{3m}) \pm (+\ell_{3m}^- - \ell_{3n})]L_{2n} + & [(+\ell_{1n}^- - \ell_{1m}) \pm (+\ell_{1m}^- - \ell_{1n})]L_{0n} + \\
 & + [(+\ell_{2m}^- - \ell_{2n}) \pm (+\ell_{2n}^- - \ell_{2m})]L_{3n} + & + [(\ell_{0m}^- - \ell_{0n}) \pm (\ell_{0n}^- - \ell_{0m})]L_{1n} + \\
 & + [(-\ell_{2n}\ell_{3m}^- + \ell_{3n}\ell_{2m}^-) \pm (-\ell_{2m}\ell_{3n}^- + \ell_{3m}\ell_{2n}^-)] & + [(+\ell_{1n}\ell_{0m}^- - \ell_{0n}\ell_{1m}^-) \pm (+\ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-)]
 \end{aligned} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{ll}
[(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{1n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{0n} + \\
+[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{2n} + & +[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{3n} + \\
+[(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \pm (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n})] & +[(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \pm (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{0n} + & [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{1n} + \\
+[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{2n} + \\
+[-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \pm (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n})] & +[(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \pm (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n})]
\end{array} \right) \\
& \left( \begin{array}{l}
[(L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n})] + \\
+[(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{3n} + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{0n} + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{2n} + \\
-[(\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n})]
\end{array} \right) 0 \\
& 0 \left( \begin{array}{l}
[(L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n})] + \\
+[(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{3n} + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{0n} + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{2n} + \\
-[(\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{2n} + & [(+\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} - \ell_{1m})]L_{0n} + \\
+[(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{3n} + & +[(+\ell_{0m} - \ell_{0n}) \pm (+\ell_{0n} - \ell_{0m})]L_{1n} + \\
+[(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \pm (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n})] & +[(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \pm (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{0n} + & [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{2n} + \\
+[(+\ell_{0n} - \ell_{0m}) \pm (+\ell_{0m} - \ell_{0n})]L_{1n} + & +[(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{3n} + \\
+[-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \pm (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n})] & +[(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \pm (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{2n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{3n} + \\
+[(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})] & +[(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{1n} + & [(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{0n} + \\
+[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{2n} + \\
+[(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})] & +[(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})]
\end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{ll}
[(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + & [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + \\
+[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{2n} + \\
+[(\ell_{3n}\ell_{1m} - \ell_{1m}\ell_{3n}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})] & +[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{0n} + & [(+\ell_{3m} - \ell_{3n} \pm (+\ell_{3n} - \ell_{3m}))]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{2n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{3n} + \\
+[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})] & +[(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{2n} + & [(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{0n} + \\
+[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{3n} + & +[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{1n} + \\
+[(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \pm (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n})] & +[(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \pm (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} - \ell_{1m})]L_{0n} + & [(+\ell_{2m} - \ell_{3n}) \pm (+\ell_{2n} - \ell_{3m})]L_{2n} + \\
+[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{1n} + & +[(\ell_{3m} - \ell_{2n} \pm (\ell_{3n} - \ell_{2m}))]L_{3n} + \\
+[-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \pm (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n})] & +[(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \pm (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n})]
\end{array} \right) \\
& \left( \begin{array}{l}
[(L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n})] + \\
+[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{2n} + [(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{0n} + [(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{3n} + \\
-[(\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n})]
\end{array} \right) \quad 0 \\
& \left( \begin{array}{l}
0 \\
[(L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n})] + \\
+[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{2n} + [(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{0n} + [(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{3n} + \\
-[(\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
+[(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{0n} + & +[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{3n} + & +[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{2n} + \\
+[(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \pm (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n})] & +[(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
+[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{1n} + & +[(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{0n} + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{2n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + \\
+[-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})] & +[(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \pm (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n})]
\end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l}
\left( \begin{array}{ll}
[(+\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} - \ell_{1m})]L_{0n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{2n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{1n} + & +[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{3n} + \\
+[(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \pm (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n})] & +[(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \pm (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n})]
\end{array} \right) \\
\left( \begin{array}{ll}
[(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{2n} + & [(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{0n} + \\
+[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{3n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{1n} + \\
+[( -\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \pm (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n})] & +[(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \pm (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n})]
\end{array} \right) \\
\left( \begin{array}{ll}
[(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + & [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{2n} + & +[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + \\
+[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})] & +[(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})]
\end{array} \right) \\
\left( \begin{array}{ll}
[(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + & [(+\ell_{0m} - \ell_{0n}) \pm (+\ell_{0n} - \ell_{0m})]L_{2n} + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{3n} + & +[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{0n} + \\
+[(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})] & +[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})]
\end{array} \right) \\
\left( \begin{array}{ll}
[(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{0n} + & [(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{3n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{2n} + \\
+[(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \pm (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n})] & +[(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})]
\end{array} \right) \\
\left( \begin{array}{ll}
[(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{2n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{0n} + \\
+[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{1n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + \\
+[(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})] & +[(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \pm (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n})]
\end{array} \right) \\
\left( \begin{array}{l}
[(L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n})] + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + [(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{2n} + \\
+[(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{3n} + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{0n} + \\
-[(\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n})]
\end{array} \right)
\end{array} \right)
\end{aligned}$$

$$D_{Bp}D_{Am} = \begin{pmatrix} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{pmatrix} (L_{\mu m} = L_{\mu p})$$

where:

$$\begin{aligned}
\mathbf{c1} &= \\
\mathbf{c2} &= \\
\mathbf{c3} &= \\
\mathbf{c4} &=
\end{aligned}$$

$$D_{Bm}D_{Ap} = \begin{pmatrix} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{pmatrix} (L_{\mu p} = L_{\mu m})$$

where:

$$\begin{aligned}
\mathbf{c1} &= \\
\mathbf{c2} &= \\
\mathbf{c3} &= \\
\mathbf{c4} &=
\end{aligned}$$

$$D_{Bp}D_{Am} \pm D_{Bm}D_{Ap} = \begin{pmatrix} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{pmatrix} (L_{\mu m} = L_{\mu p})$$

where:

$$\begin{aligned}
\mathbf{c1} &= \\
\mathbf{c2} &= \\
\mathbf{c3} &= \\
\mathbf{c4} &=
\end{aligned}$$

$$D_{Bn}\overline{D_{Am}} = \begin{pmatrix} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{pmatrix} (L_{\mu m} = L_{\mu n})$$

where:



$$\begin{aligned}
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0m} + \ell_{0n})L_{0n} + (\ell_{3m} + \ell_{3n})L_{3n} + (\ell_{2m} + \ell_{2n})L_{2n} + (\ell_{1m} + \ell_{1n})L_{1n} + \\ -(-\ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m} - \ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + \\ -(-\ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m} - \ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m}) \end{array} \right) \\
c1 = & \left( \begin{array}{l} (+\ell_{2m} + \ell_{2n})L_{1n} + \quad (-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{1m} + \ell_{1n})L_{2n} + \quad +(-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \quad +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \\ \\ (\ell_{0m} + \ell_{0n})L_{3n} + \quad (-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{3m} + \ell_{3n})L_{0n} + \quad +(-\ell_{1n} - \ell_{1m})L_{2n} + \\ +(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \quad +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{1n} + \quad (\ell_{0m} + \ell_{0n})L_{2n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + \quad +(\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \quad +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \\ \\ (-\ell_{2m} - \ell_{2n})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + \quad +(-\ell_{1n} - \ell_{1m})L_{3n} + \\ +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \quad +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{l} (-\ell_{1m}^- - \ell_{1n})L_{0n} + \quad (-\ell_{3m}^- - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} + \ell_{0m}^-)L_{1n} + \quad +(\ell_{2n} + \ell_{2m}^-)L_{3n} + \\ +(-\ell_{1n}\ell_{0m}^- + \ell_{0n}\ell_{1m}^-) \quad +(+\ell_{2n}\ell_{3m}^- - \ell_{3n}\ell_{2m}^-) \\ \\ (+\ell_{3n} + \ell_{3m}^-)L_{2n} + \quad (+\ell_{1n} + \ell_{1m}^-)L_{0n} + \\ +(-\ell_{2m}^- - \ell_{2n})L_{3n} + \quad +(-\ell_{0m}^- - \ell_{0n})L_{1n} + \\ +(+\ell_{2n}\ell_{3m}^- - \ell_{3n}\ell_{2m}^-) \quad +(-\ell_{1n}\ell_{0m}^- + \ell_{0n}\ell_{1m}^-) \end{array} \right) \\
& \left( \begin{array}{l} (+\ell_{2n} + \ell_{2m})L_{1n} + \quad (+\ell_{3n} + \ell_{3m})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{2n} + \quad +(\ell_{0n} + \ell_{0m})L_{3n} + \\ +(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m}) \quad +(+\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \\ \\ (-\ell_{3m} - \ell_{3n})L_{0n} + \quad (-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{3n} + \quad +(-\ell_{1m} - \ell_{1n})L_{2n} + \\ +(+\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \quad +(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m}) \end{array} \right) \\
c2 = & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(-\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + \\ -(-\ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m} - \ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + \\ -(-\ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m} - \ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m}) \end{array} \right) \\
& \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{2n} + \quad (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} + \ell_{2m})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \quad +(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \\ \\ (+\ell_{1n} + \ell_{1m})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} + \ell_{0m})L_{1n} + \quad +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \quad +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \end{array} \right) \\
& \left( \begin{array}{l} (-\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + \quad +(-\ell_{1m} - \ell_{1n})L_{3n} + \\ +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \quad +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \\ \\ (-\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{2n} + \ell_{2m})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \quad +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
\text{c3} = & \left( \begin{array}{l} \left( \begin{array}{ll} (+\ell_{3n} + \ell_{3m})L_{1n} + & (-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + & +(-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{3n}\ell_{1m} + \ell_{1m}\ell_{3n}) & +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{2n} + \ell_{2m})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + & +(-\ell_{1m} - \ell_{1n})L_{3n} + \\ +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{3n} + \ell_{3m})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{3n} + & +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) & +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{1m} - \ell_{1n})L_{0n} + & (-\ell_{2m} - \ell_{3n})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + & +(-\ell_{3m} - \ell_{2n})L_{3n} + \\ +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) & +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \end{array} \right) \end{array} \right) \\ \\ & \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + \\ -(-\ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m} - \ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m}) \end{array} \right) \quad 0 \\ \\ & 0 \quad \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + \\ -(-\ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m} - \ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m}) \end{array} \right) \\ \\ \text{c4} = & \left( \begin{array}{l} \left( \begin{array}{ll} +(-\ell_{3m} - \ell_{3n})L_{0n} + & +(-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + & +(\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +(+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} +(\ell_{2n} + \ell_{2m})L_{1n} + & +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{2n} + & +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{1m} - \ell_{1n})L_{0n} + & +(\ell_{3n} + \ell_{3m})L_{2n} + \\ +(\ell_{0n} + \ell_{0m})L_{1n} + & +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{3n} + & +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{2m} - \ell_{2n})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{3n} + \\ +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} +(\ell_{3n} + \ell_{3m})L_{1n} + & +(-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{0n} + & +(\ell_{2n} + \ell_{2m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + & +(-\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} +(\ell_{1n} + \ell_{1m})L_{2n} + & +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{1n} + & +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \end{array} \right) \end{array} \right) \\ \\ & \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + \\ -(-\ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m} - \ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m}) \end{array} \right) \quad 0 \\ \\ & 0 \quad \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} + \ell_{1m})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + \\ -(-\ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m} - \ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m}) \end{array} \right) \\ \\ \overline{D_{Bm}D_{An}} = \left( \begin{array}{cccc} \text{c1} & \text{c2} & \text{c3} & \text{c4} \end{array} \right) (L_{\mu m} = L_{\mu n})
\end{aligned}$$

where:

$$\begin{aligned}
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0n} - \ell_{0m})L_{0n} + (-\ell_{3n} - \ell_{3m})L_{3n} + (-\ell_{2n} - \ell_{2m})L_{2n} + (-\ell_{1n} - \ell_{1m})L_{1n} + \\ -(-\ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n} - \ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n}) \end{array} \right) \quad 0 \\
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + \\ -(-\ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n} - \ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n}) \end{array} \right) \\
 c1 = & \left( \begin{array}{l} \left( \begin{array}{l} (-\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{0n} + \ell_{0m})L_{3n} + \\ +(-\ell_{1n} - \ell_{1m})L_{2n} + +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{array} \right) \\ \left( \begin{array}{l} (-\ell_{0n} - \ell_{0m})L_{3n} + (\ell_{2n} + \ell_{2m})L_{1n} + \\ +(-\ell_{3n} - \ell_{3m})L_{0n} + +(\ell_{1m} + \ell_{1n})L_{2n} + \\ +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \end{array} \right) \\ \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{1n} + (-\ell_{0n} - \ell_{0m})L_{2n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + +(-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2n} + \ell_{2m})L_{0n} + (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + +(\ell_{1m} + \ell_{1n})L_{3n} + \\ +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{1n} + \ell_{1m})L_{0n} + (+\ell_{3n} + \ell_{3m})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{array} \right) \\ \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} + \ell_{2m})L_{3n} + +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \end{array} \right) \end{array} \right) \\
 & \left( \begin{array}{l} \left( \begin{array}{l} (-\ell_{2m} - \ell_{2n})L_{1n} + (-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{2n} + +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{2m}\ell_{1n} + \ell_{1m}\ell_{2n}) +(+\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{0n} + (+\ell_{2n} + \ell_{2m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + +(\ell_{1n} + \ell_{1m})L_{2n} + \\ +(+\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) +(-\ell_{2m}\ell_{1n} + \ell_{1m}\ell_{2n}) \end{array} \right) \end{array} \right) \\
 c2 = & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + \\ -(-\ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n} - \ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n}) \end{array} \right) \quad 0 \\
 & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(-\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + \\ -(-\ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n} - \ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n}) \end{array} \right) \\
 & \left( \begin{array}{l} \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{2n} + (+\ell_{1n} + \ell_{1m})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) +(+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) \end{array} \right) \\ \left( \begin{array}{l} (-\ell_{1m} - \ell_{1n})L_{0n} + (+\ell_{3n} + \ell_{3m})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + +(\ell_{2n} + \ell_{2m})L_{3n} + \\ +(+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2n} + \ell_{2m})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + +(\ell_{1n} + \ell_{1m})L_{3n} + \\ +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{1n} + (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + +(\ell_{0n} + \ell_{0m})L_{2n} + \\ +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) \end{array} \right) \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l}
(-\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{2n} + \ell_{2m})L_{0n} + \\
+(-\ell_{1m} - \ell_{1n})L_{3n} + \quad +(\ell_{0n} + \ell_{0m})L_{2n} + \\
+(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \\
(-\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{1n} + \\
+(-\ell_{0m} - \ell_{0n})L_{2n} + \quad +(\ell_{1n} + \ell_{1m})L_{3n} + \\
+(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \\
(-\ell_{3m} - \ell_{3n})L_{2n} + \quad (-\ell_{1m} - \ell_{1n})L_{0n} + \\
+(-\ell_{2m} - \ell_{2n})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{1n} + \\
+(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \quad +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \\
(+\ell_{1n} + \ell_{1m})L_{0n} + \quad (+\ell_{2n} + \ell_{3m})L_{2n} + \\
+(\ell_{0n} + \ell_{0m})L_{1n} + \quad +(\ell_{3n} + \ell_{2m})L_{3n} + \\
+ (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n})
\end{array} \right) \\
\text{c3=} & \left( \begin{array}{l}
(L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\
+(\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + \\
-(-\ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n} - \ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n}) \quad 0 \\
0 \\
(L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\
+(-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + \\
-(-\ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n} - \ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n})
\end{array} \right) \\
& \left( \begin{array}{l}
+(\ell_{3n} + \ell_{3m})L_{0n} + \quad +(\ell_{2n} + \ell_{2m})L_{1n} + \\
+(-\ell_{0m} - \ell_{0n})L_{3n} + \quad +(-\ell_{1m} - \ell_{1n})L_{2n} + \\
+(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \quad +(+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\
+(-\ell_{2m} - \ell_{2n})L_{1n} + \quad +(-\ell_{3m} - \ell_{3n})L_{0n} + \\
+(\ell_{1n} + \ell_{1m})L_{2n} + \quad +(\ell_{0n} + \ell_{0m})L_{3n} + \\
+ (+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \quad +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n})
\end{array} \right) \\
\text{c4=} & \left( \begin{array}{l}
(+\ell_{1n} + \ell_{1m})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{2n} + \\
+(-\ell_{0m} - \ell_{0n})L_{1n} + \quad +(\ell_{2n} + \ell_{2m})L_{3n} + \\
+(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \quad +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \\
(+\ell_{3n} + \ell_{3m})L_{2n} + \quad (-\ell_{1m} - \ell_{1n})L_{0n} + \\
+(-\ell_{2m} - \ell_{2n})L_{3n} + \quad +(\ell_{0n} + \ell_{0m})L_{1n} + \\
+ (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \\
(+\ell_{2n} + \ell_{2m})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{1n} + \\
+(-\ell_{0m} - \ell_{0n})L_{2n} + \quad +(-\ell_{1m} - \ell_{1n})L_{3n} + \\
+(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \\
(-\ell_{3m} - \ell_{3n})L_{1n} + \quad +(\ell_{0n} + \ell_{0m})L_{2n} + \\
+(\ell_{1n} + \ell_{1m})L_{3n} + \quad +(-\ell_{2m} - \ell_{2n})L_{0n} + \\
+(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \\
(+\ell_{3n} + \ell_{3m})L_{0n} + \quad (-\ell_{2m} - \ell_{2n})L_{1n} + \\
+(-\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1n} + \ell_{1m})L_{2n} + \\
+(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \quad +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \\
(-\ell_{1m} - \ell_{1n})L_{2n} + \quad (-\ell_{3m} - \ell_{3n})L_{0n} + \\
+(\ell_{2n} + \ell_{2m})L_{1n} + \quad +(\ell_{0n} + \ell_{0m})L_{3n} + \\
+(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \quad +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n})
\end{array} \right) \\
& \left( \begin{array}{l}
(L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\
+(\ell_{1n} + \ell_{1m})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + \\
-(-\ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n} - \ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n}) \quad 0 \\
0 \\
(L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\
+(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + \\
-(-\ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n} - \ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n})
\end{array} \right) \\
& D_{Bn}\overline{D_{Am}} \pm \overline{D_{Bm}}D_{An} = \left( \begin{array}{cccc} \text{c1} & \text{c2} & \text{c3} & \text{c4} \end{array} \right) (L_{\mu m} = L_{\mu n})
\end{aligned}$$

where:

c1=

c2=

c3=

c4=

Proof:

$$D_{Bm}D_{An} = \begin{pmatrix} c1 & c2 & c3 & c4 \end{pmatrix} (L_{\mu m} = L_{\mu n})$$

where:

$$c1 = \begin{pmatrix} \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{2n} + \ell_{2m})L_{2n} + (-\ell_{1n} + \ell_{1m})L_{1n} + \\ -(\ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n} + \ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n}) \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + \\ -(\ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n} + \ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n}) \end{pmatrix} \\ \begin{pmatrix} (-\ell_{2n} + \ell_{2m})L_{1n} + & (\ell_{0n} - \ell_{0m})L_{3n} + \\ +(-\ell_{1n} + \ell_{1m})L_{2n} + & +(\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ \\ (-\ell_{0n} + \ell_{0m})L_{3n} + & (\ell_{2n} - \ell_{2m})L_{1n} + \\ +(-\ell_{3n} + \ell_{3m})L_{0n} + & +(-\ell_{1m} + \ell_{1n})L_{2n} + \\ +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{pmatrix} \\ \begin{pmatrix} (+\ell_{3m} - \ell_{3n})L_{1n} + & (-\ell_{0n} + \ell_{0m})L_{2n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ \\ (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + & +(-\ell_{1m} + \ell_{1n})L_{3n} + \\ +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{pmatrix} \\ \begin{pmatrix} (+\ell_{1n}^- - \ell_{1m})L_{0n} + & (+\ell_{3n}^- - \ell_{3m})L_{2n} + \\ +(+\ell_{0m}^- - \ell_{0n}^-)L_{1n} + & +(\ell_{2m}^- - \ell_{2n}^-)L_{3n} + \\ +(+\ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-) & +(-\ell_{2m}\ell_{3n}^- + \ell_{3m}\ell_{2n}^-) \\ \\ (+\ell_{3m}^- - \ell_{3n}^-)L_{2n} + & (+\ell_{1m}^- - \ell_{1n}^-)L_{0n} + \\ +(+\ell_{2n}^- - \ell_{2m}^-)L_{3n} + & +(\ell_{0n}^- - \ell_{0m}^-)L_{1n} + \\ +(-\ell_{2m}\ell_{3n}^- + \ell_{3m}\ell_{2n}^-) & +(+\ell_{1m}\ell_{0n}^- - \ell_{0m}\ell_{1n}^-) \end{pmatrix} \end{pmatrix}$$

$$\begin{aligned}
& \left( \begin{array}{ll} (+\ell_{2m} - \ell_{2n})L_{1n} + & (+\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{2n} + & +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{3n} - \ell_{3m})L_{0n} + & (+\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + & +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & +(+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) \end{array} \right) \\
& \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + \\ -(\ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n} + \ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} 0 \\ (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + \\ -(\ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n} + \ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n}) \end{array} \right) \\
c2 = & \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(+\ell_{2m} - \ell_{2n})L_{3n} + & +(+\ell_{0n} - \ell_{0m})L_{1n} + + \\ +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(+\ell_{0m} - \ell_{0n})L_{1n} + & +(+\ell_{2n} - \ell_{2m})L_{3n} + \\ +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{3n} - \ell_{3m})L_{1n} + & (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \end{array} \right) \\
c3 = & \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{1n} + & (+\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{2m} - \ell_{2n})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{3n} + & +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{1n} - \ell_{1m})L_{0n} + & (+\ell_{2n} - \ell_{3m})L_{2n} + \\ +(\ell_{0n} - \ell_{0m})L_{1n} + & +(\ell_{3n} - \ell_{2m})L_{3n} + \\ +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + \\ -(\ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n} + \ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} 0 \\ (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + \\ -(\ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n} + \ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{ll} +(\ell_{3n} - \ell_{3m})L_{0n} + & +(\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + & +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{ll} +(\ell_{2m} - \ell_{2n})L_{1n} + & +(\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{2n} + & +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{cc}
(+\ell_{1n} - \ell_{1m})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{2n} + \\
+(\ell_{0m} - \ell_{0n})L_{1n} + & +(\ell_{2n} - \ell_{2m})L_{3n} + \\
+(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \\
(+\ell_{3n} - \ell_{3m})L_{2n} + & (+\ell_{1m} - \ell_{1n})L_{0n} + \\
+(\ell_{2m} - \ell_{2n})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{1n} + \\
+(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n})
\end{array} \right) \\
& \left( \begin{array}{cc}
(+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{1n} + \\
+(\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1m} - \ell_{1n})L_{3n} + \\
+(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \\
(+\ell_{3m} - \ell_{3n})L_{1n} + & +(\ell_{0n} - \ell_{0m})L_{2n} + \\
+(\ell_{1n} - \ell_{1m})L_{3n} + & +(\ell_{2m} - \ell_{2n})L_{0n} + \\
+(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})
\end{array} \right) \\
& \left( \begin{array}{cc}
(+\ell_{3n} - \ell_{3m})L_{0n} + & (+\ell_{2m} - \ell_{2n})L_{1n} + \\
+(\ell_{0m} - \ell_{0n})L_{3n} + & +(\ell_{1n} - \ell_{1m})L_{2n} + \\
+(\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\
(+\ell_{1m} - \ell_{1n})L_{2n} + & (+\ell_{3m} - \ell_{3n})L_{0n} + \\
+(\ell_{2n} - \ell_{2m})L_{1n} + & +(\ell_{0n} - \ell_{0m})L_{3n} + \\
+(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & +(\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n})
\end{array} \right) \\
& \left( \begin{array}{l}
(L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\
+(\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + \quad 0 \\
-(\ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n} + \ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n}) \\
0
\end{array} \right) \quad \begin{array}{l}
(L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\
+(\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + \\
-(\ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n} + \ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n})
\end{array} \\
\Rightarrow D_{Bn}D_{Am} = \left( \begin{array}{cccc}
c1 & c2 & c3 & c4
\end{array} \right) \text{ (flip n's \& m's) } (L_{\mu m} = L_{\mu n}) \\
\text{where:}
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0m} + \ell_{0n})L_{0n} + (-\ell_{3m} + \ell_{3n})L_{3n} + (-\ell_{2m} + \ell_{2n})L_{2n} + (-\ell_{1m} + \ell_{1n})L_{1n} + \\ -(\ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m} + \ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m}) \end{array} \right) \quad \mathbf{0} \\
& \mathbf{0} \quad \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + \\ -(\ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m} + \ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m}) \end{array} \right) \\
& \mathbf{c1=} \left( \begin{array}{l} \left( \begin{array}{l} (-\ell_{2m} + \ell_{2n})L_{1n} + \quad (\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1m} + \ell_{1n})L_{2n} + \quad +(\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \quad +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \end{array} \right) \\ \left( \begin{array}{l} (-\ell_{0m} + \ell_{0n})L_{3n} + \quad (\ell_{2m} - \ell_{2n})L_{1n} + \\ +(-\ell_{3m} + \ell_{3n})L_{0n} + \quad +(-\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \quad +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{1n} + \quad (-\ell_{0m} + \ell_{0n})L_{2n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + \quad +(\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \quad +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + \quad +(-\ell_{1n} + \ell_{1m})L_{3n} + \\ +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \quad +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{1m}^- - \ell_{1n})L_{0n} + \quad (+\ell_{3m}^- - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} - \ell_{0m}^-)L_{1n} + \quad +(\ell_{2n} - \ell_{2m}^-)L_{3n} + \\ +(+\ell_{1n}\ell_{0m}^- - \ell_{0n}\ell_{1m}^-) \quad +(-\ell_{2n}\ell_{3m}^- + \ell_{3n}\ell_{2m}^-) \\ (+\ell_{3n} - \ell_{3m}^-)L_{2n} + \quad (+\ell_{1n} - \ell_{1m}^-)L_{0n} + \\ +(+\ell_{2m}^- - \ell_{2n})L_{3n} + \quad +(\ell_{0m}^- - \ell_{0n})L_{1n} + \\ +(-\ell_{2n}\ell_{3m}^- + \ell_{3n}\ell_{2m}^-) \quad +(+\ell_{1n}\ell_{0m}^- - \ell_{0n}\ell_{1m}^-) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2n} - \ell_{2m})L_{1n} + \quad (+\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{2n} + \quad +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \quad +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \\ (+\ell_{3m} - \ell_{3n})L_{0n} + \quad (+\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \quad +(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \end{array} \right) \end{array} \right) \\
& \mathbf{c2=} \left( \begin{array}{l} \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + \\ -(\ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m} + \ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m}) \end{array} \right) \quad \mathbf{0} \\ \mathbf{0} \quad \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + \\ -(\ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m} + \ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{2n} + \quad (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} - \ell_{2m})L_{3n} + \quad +(+\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \quad +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \\ (+\ell_{1n} - \ell_{1m})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} - \ell_{0m})L_{1n} + \quad +(+\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \quad +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \end{array} \right) \\ \left( \begin{array}{l} (+\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + \quad +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \quad +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \\ (+\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + \quad +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \quad +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \end{array} \right) \end{array} \right)
\end{aligned}$$



$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{1n} + \quad (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + \quad +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{3n}\ell_{1m} - \ell_{1m}\ell_{3n}) \quad +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2n} - \ell_{2m})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + \quad +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \quad +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{2n} + \quad (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{3n} + \quad +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \quad +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \\ \\ (+\ell_{1m} - \ell_{1n})L_{0n} + \quad (+\ell_{2m} - \ell_{3n})L_{2n} + \\ +(\ell_{0m} - \ell_{0n})L_{1n} + \quad +(\ell_{3m} - \ell_{2n})L_{3n} + \\ +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \quad +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + \\ -(\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m}) \quad 0 \\ \\ 0 \end{array} \right) \quad \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + \\ -(\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} \left( \begin{array}{l} +(\ell_{3m} - \ell_{3n})L_{0n} + \quad +(\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + \quad +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \quad +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \\ \\ +(\ell_{2n} - \ell_{2m})L_{1n} + \quad +(\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{2n} + \quad +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \quad +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{1m} - \ell_{1n})L_{0n} + \quad (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(\ell_{0n} - \ell_{0m})L_{1n} + \quad +(\ell_{2m} - \ell_{2n})L_{3n} + \\ +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \quad +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \\ \\ (+\ell_{3m} - \ell_{3n})L_{2n} + \quad (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{3n} + \quad +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \quad +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + \quad +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \quad +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \\ \\ (+\ell_{3n} - \ell_{3m})L_{1n} + \quad +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + \quad +(\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \quad +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{0n} + \quad (+\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + \quad +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \quad +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \\ \\ (+\ell_{1n} - \ell_{1m})L_{2n} + \quad (+\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{1n} + \quad +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \quad +(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + \\ -(\ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m} + \ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m}) \quad 0 \\ \\ 0 \end{array} \right) \quad \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + \\ -(\ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m} + \ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m}) \end{array} \right) \\ \\ \Rightarrow D_{Bn}\overline{D_{Am}} = \left( \begin{array}{cccc} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{array} \right) \text{ (negate m's) } (L_{\mu m} = L_{\mu n})
\end{aligned}$$

where:

$$\begin{aligned}
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0m} + \ell_{0n})L_{0n} + (\ell_{3m} + \ell_{3n})L_{3n} + (\ell_{2m} + \ell_{2n})L_{2n} + (\ell_{1m} + \ell_{1n})L_{1n} + \\ -(-\ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m} - \ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m}) \end{array} \right) \quad 0 \\
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + \\ -(-\ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m} - \ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{l} (+\ell_{2m} + \ell_{2n})L_{1n} + \quad (-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{1m} + \ell_{1n})L_{2n} + \quad +(-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \quad +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \\ \\ (\ell_{0m} + \ell_{0n})L_{3n} + \quad (-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{3m} + \ell_{3n})L_{0n} + \quad +(-\ell_{1n} - \ell_{1m})L_{2n} + \\ +(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \quad +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{1n} + \quad (\ell_{0m} + \ell_{0n})L_{2n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + \quad +(\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \quad +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \\ \\ (-\ell_{2m} - \ell_{2n})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + \quad +(-\ell_{1n} - \ell_{1m})L_{3n} + \\ +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \quad +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{l} (-\ell_{1m} - \ell_{1n})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} + \ell_{0m})L_{1n} + \quad +(\ell_{2n} + \ell_{2m})L_{3n} + \\ +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \quad +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \\ \\ (+\ell_{3n} + \ell_{3m})L_{2n} + \quad +(\ell_{1n} + \ell_{1m})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \quad +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{l} (+\ell_{2n} + \ell_{2m})L_{1n} + \quad (+\ell_{3n} + \ell_{3m})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{2n} + \quad +(\ell_{0n} + \ell_{0m})L_{3n} + \\ +(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m}) \quad +(+\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \\ \\ (-\ell_{3m} - \ell_{3n})L_{0n} + \quad (-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{3n} + \quad +(-\ell_{1m} - \ell_{1n})L_{2n} + \\ +(+\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \quad +(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m}) \end{array} \right) \\
 & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(-\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + \\ -(-\ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m} - \ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m}) \end{array} \right) \quad 0 \\
 & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + \\ -(-\ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m} - \ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m}) \end{array} \right) \\
 & \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{2n} + \quad (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} + \ell_{2m})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \quad +(\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \\ \\ (+\ell_{1n} + \ell_{1m})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} + \ell_{0m})L_{1n} + \quad +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \quad +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \end{array} \right) \\
 & \left( \begin{array}{l} (-\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + \quad +(-\ell_{1m} - \ell_{1n})L_{3n} + \\ +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \quad +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \\ \\ (-\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{2n} + \ell_{2m})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \quad +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{ll} (+\ell_{3n} + \ell_{3m})L_{1n} + & (-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + & +(-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{3n}\ell_{1m} + \ell_{1m}\ell_{3n}) & +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{2n} + \ell_{2m})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + & +(-\ell_{1m} - \ell_{1n})L_{3n} + \\ +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (+\ell_{3n} + \ell_{3m})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{3n} + & +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) & +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{1m} - \ell_{1n})L_{0n} + & (-\ell_{2m} - \ell_{3n})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + & +(-\ell_{3m} - \ell_{2n})L_{3n} + \\ +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) & +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \end{array} \right) \end{array} \right) \\
\text{c3=} & \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + \\ -(-\ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m} - \ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m}) \quad 0 \\ \\ 0 \\ \\ \left( \begin{array}{ll} +(-\ell_{3m} - \ell_{3n})L_{0n} + & +(-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + & +(\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +(+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} +(\ell_{2n} + \ell_{2m})L_{1n} + & +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{2n} + & +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{1m} - \ell_{1n})L_{0n} + & +(\ell_{3n} + \ell_{3m})L_{2n} + \\ +(\ell_{0n} + \ell_{0m})L_{1n} + & +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{3n} + & +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{2m} - \ell_{2n})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{3n} + \\ +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} +(\ell_{3n} + \ell_{3m})L_{1n} + & +(-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{0n} + & +(\ell_{2n} + \ell_{2m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + & +(-\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{ll} +(\ell_{1n} + \ell_{1m})L_{2n} + & +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{1n} + & +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \end{array} \right) \end{array} \right) \\
\text{c4=} & \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + \\ -(-\ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m} - \ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m}) \quad 0 \\ \\ 0 \\ \\ (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} + \ell_{1m})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + \\ -(-\ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m} - \ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m}) \end{array} \right) \\
\Rightarrow \overline{D_{Bm}D_{An}} = & \left( \text{c1} \quad \text{c2} \quad \text{c3} \quad \text{c4} \right) \text{ (negate m's) } (L_{\mu n} = L_{\mu m})
\end{aligned}$$

where:

$$\begin{aligned}
 c1 = & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0n} - \ell_{0m})L_{0n} + (-\ell_{3n} - \ell_{3m})L_{3n} + (-\ell_{2n} - \ell_{2m})L_{2n} + (-\ell_{1n} - \ell_{1m})L_{1n} + \\ -(-\ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n} - \ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n}) \end{array} \right) 0 \\
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + \\ -(-\ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n} - \ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (-\ell_{2n} - \ell_{2m})L_{1n} + & (\ell_{0n} + \ell_{0m})L_{3n} + \\ +(-\ell_{1n} - \ell_{1m})L_{2n} + & +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \\ \\ (-\ell_{0n} - \ell_{0m})L_{3n} + & (\ell_{2n} + \ell_{2m})L_{1n} + \\ +(-\ell_{3n} - \ell_{3m})L_{0n} + & +(+\ell_{1m} + \ell_{1n})L_{2n} + \\ +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{1n} + & (-\ell_{0n} - \ell_{0m})L_{2n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + & +(-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) & +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) \\ \\ (+\ell_{2n} + \ell_{2m})L_{0n} + & (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + & +(+\ell_{1m} + \ell_{1n})L_{3n} + \\ +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) & +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{1n} + \ell_{1m})L_{0n} + & (+\ell_{3n} + \ell_{3m})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + & +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \\ \\ (-\ell_{3m} - \ell_{3n})L_{2n} + & (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} + \ell_{2m})L_{3n} + & +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (-\ell_{2m} - \ell_{2n})L_{1n} + & (-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{2n} + & +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{2m}\ell_{1n} + \ell_{1m}\ell_{2n}) & +(+\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \\ \\ (+\ell_{3n} + \ell_{3m})L_{0n} + & (+\ell_{2n} + \ell_{2m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + & +(\ell_{1n} + \ell_{1m})L_{2n} + \\ +(+\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) & +(-\ell_{2m}\ell_{1n} + \ell_{1m}\ell_{2n}) \end{array} \right) \\
 c2 = & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + \\ -(-\ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n} - \ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n}) \end{array} \right) 0 \\
 & \left( \begin{array}{l} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + \\ +(-\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + \\ -(-\ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n} - \ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1n} + \ell_{1m})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + & +(+\ell_{0n} + \ell_{0m})L_{1n} + \\ +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) & +(\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) \\ \\ (-\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{3n} + \ell_{3m})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + & +(+\ell_{2n} + \ell_{2m})L_{3n} + \\ +(+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) & +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{2n} + \ell_{2m})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{3n} + \\ +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) & +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) \\ \\ (+\ell_{3n} + \ell_{3m})L_{1n} + & (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{0n} + \ell_{0m})L_{2n} + \\ +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) & +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + \quad +(\ell_{0n} + \ell_{0m})L_{2n} + \\ +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (-\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + \quad +(\ell_{1n} + \ell_{1m})L_{3n} + \\ +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{2n} + \quad (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \quad +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \\ \\ (+\ell_{1n} + \ell_{1m})L_{0n} + \quad (+\ell_{2n} + \ell_{3m})L_{2n} + \\ +(\ell_{0n} + \ell_{0m})L_{1n} + \quad +(\ell_{3n} + \ell_{2m})L_{3n} + \\ +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \end{array} \right) \\ \\ 0 \\ \\ \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + \\ -(-\ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n} - \ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n}) \end{array} \right) \quad 0 \\ \\ 0 \\ \\ \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{0n} + \quad +(\ell_{2n} + \ell_{2m})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{3n} + \quad +(-\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \quad +(+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \\ \\ +(-\ell_{2m} - \ell_{2n})L_{1n} + \quad +(-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{2n} + \quad +(\ell_{0n} + \ell_{0m})L_{3n} + \\ +(+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \quad +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{1n} + \ell_{1m})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + \quad +(\ell_{2n} + \ell_{2m})L_{3n} + \\ +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \quad +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \\ \\ (+\ell_{3n} + \ell_{3m})L_{2n} + \quad (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + \quad +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2n} + \ell_{2m})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + \quad +(-\ell_{1m} - \ell_{1n})L_{3n} + \\ +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \\ \\ (-\ell_{3m} - \ell_{3n})L_{1n} + \quad +(\ell_{0n} + \ell_{0m})L_{2n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + \quad +(-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{0n} + \quad (-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \quad +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \\ \\ (-\ell_{1m} - \ell_{1n})L_{2n} + \quad (-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{1n} + \quad +(\ell_{0n} + \ell_{0m})L_{3n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \quad +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} + \ell_{1m})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + \\ -(-\ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n} - \ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n}) \end{array} \right) \quad 0 \\ \\ 0 \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + \\ -(-\ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n} - \ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n}) \end{array} \right) \end{array} \right) \\ \\ \Rightarrow D_{Bn}D_{Am} \pm D_{Bm}D_{An} = \left( \begin{array}{cccc} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{array} \right) (\mathbf{nm}'\mathbf{s} \pm \mathbf{mn}'\mathbf{s}) \quad (L_{\mu m} = L_{\mu n})
\end{aligned}$$

where:

$$\begin{aligned}
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0m} + \ell_{0n})L_{0n} + (-\ell_{3m} + \ell_{3n})L_{3n} + (-\ell_{2m} + \ell_{2n})L_{2n} + (-\ell_{1m} + \ell_{1n})L_{1n} + \\ -(\ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m} + \ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m}) \end{array} \right) \quad 0 \\
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + \\ -(\ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m} + \ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{ll} (-\ell_{2m} + \ell_{2n})L_{1n} + & (\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1m} + \ell_{1n})L_{2n} + & +(\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \\ \\ (-\ell_{0m} + \ell_{0n})L_{3n} + & (\ell_{2m} - \ell_{2n})L_{1n} + \\ +(-\ell_{3m} + \ell_{3n})L_{0n} + & +(-\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{3n} - \ell_{3m})L_{1n} + & (-\ell_{0m} + \ell_{0n})L_{2n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + & +(\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \\ \\ (+\ell_{2m} - \ell_{2n})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + & +(-\ell_{1n} + \ell_{1m})L_{3n} + \\ +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} - \ell_{0m})L_{1n} + & +(\ell_{2n} - \ell_{2m})L_{3n} + \\ +(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) & +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \\ \\ (+\ell_{3n} - \ell_{3m})L_{2n} + & (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(+\ell_{2m} - \ell_{2n})L_{3n} + & +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) & +(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \end{array} \right) \\
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{3n} + \ell_{3m})L_{3n} + (-\ell_{2n} + \ell_{2m})L_{2n} + (-\ell_{1n} + \ell_{1m})L_{1n} + \\ -(\ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n} + \ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n}) \end{array} \right) \quad 0 \\
 & \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + \\ -(\ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n} + \ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (-\ell_{2n} + \ell_{2m})L_{1n} + & (\ell_{0n} - \ell_{0m})L_{3n} + \\ +(-\ell_{1n} + \ell_{1m})L_{2n} + & +(\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \\ \\ (-\ell_{0n} + \ell_{0m})L_{3n} + & (\ell_{2n} - \ell_{2m})L_{1n} + \\ +(-\ell_{3n} + \ell_{3m})L_{0n} + & +(-\ell_{1m} + \ell_{1n})L_{2n} + \\ +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{3m} - \ell_{3n})L_{1n} + & (-\ell_{0n} + \ell_{0m})L_{2n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \\ \\ (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + & +(-\ell_{1m} + \ell_{1n})L_{3n} + \\ +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{array} \right) \\
 & \left( \begin{array}{ll} (+\ell_{1n} - \ell_{1m})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(+\ell_{0m} - \ell_{0n})L_{1n} + & +(\ell_{2m} - \ell_{2n})L_{3n} + \\ +(+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) & +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) \\ \\ (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} - \ell_{2m})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n}) & +(+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n}) \end{array} \right)
 \end{aligned}$$

$$\begin{aligned}
& \left( \begin{aligned} & [(L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n})] + \\ & + [(-\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} + \ell_{0m})]L_{0n} + [(-\ell_{3m} + \ell_{3n}) \pm (-\ell_{3n} + \ell_{3m})]L_{3n} + \\ & + [(-\ell_{2m} + \ell_{2n}) \pm (-\ell_{2n} + \ell_{2m})]L_{2n} + [(-\ell_{1m} + \ell_{1n}) \pm (-\ell_{1n} + \ell_{1m})]L_{1n} + \\ & - [(\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n})] \end{aligned} \right) \quad 0 \\
& \left( \begin{aligned} & 0 \\ & 0 \end{aligned} \right) \quad \begin{aligned} & [(L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) \\ & + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{0n} + [(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{3n} \\ & + [(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{2n} + [(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} \\ & - [(\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n})] \end{aligned} \quad * \\
& \left( \begin{aligned} & [(-\ell_{2m} + \ell_{2n}) \pm (-\ell_{2n} + \ell_{2m})]L_{1n} + & [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + \\ & + [(-\ell_{1m} + \ell_{1n}) \pm (-\ell_{1n} + \ell_{1m})]L_{2n} + & + [(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{0n} + \\ & + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})] & + (-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \pm (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \end{aligned} \right) \\
& \left( \begin{aligned} & [(-\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} + \ell_{0m})]L_{3n} + & [(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{1n} + \\ & + [(-\ell_{3m} + \ell_{3n}) \pm (-\ell_{3n} + \ell_{3m})]L_{0n} + & + [(-\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} + \ell_{1n})]L_{2n} + \\ & + [(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \pm (-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n})] & + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})] \end{aligned} \right) \\
& \left( \begin{aligned} & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + & [(-\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} + \ell_{0m})]L_{2n} + \\ & + [(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + & + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{0n} + \\ & + [(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})] & + [(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})] \end{aligned} \right) \\
& \left( \begin{aligned} & [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + & [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{1n} + \\ & + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{2n} + & + [(-\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} + \ell_{1n})]L_{3n} + \\ & + [(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})] & + [(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})] \end{aligned} \right) \\
& \left( \begin{aligned} & [(+\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} - \ell_{1m})]L_{0n} + & [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{2n} + \\ & + [(+\ell_{0n} - \ell_{0m}) \pm (+\ell_{0m} - \ell_{0n})]L_{1n} + & + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{3n} + \\ & + [(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \pm (+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n})] & + [(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \pm (-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n})] \end{aligned} \right) \\
& \left( \begin{aligned} & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{2n} + & [(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{0n} + \\ & + [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{3n} + & + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{1n} + \\ & + [(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m}) \pm (-\ell_{2m}\ell_{3n} + \ell_{3m}\ell_{2n})] & + [(+\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m}) \pm (+\ell_{1m}\ell_{0n} - \ell_{0m}\ell_{1n})] \end{aligned} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{cc} (+\ell_{2n} - \ell_{2m})L_{1n} + & (+\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{2n} + & +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) & +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3m} - \ell_{3n})L_{0n} + & (+\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + & +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) & +(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \end{array} \right) \\
& \left( \begin{array}{cc} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + & \\ +(\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + & 0 \\ -(\ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m} + \ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m}) & \end{array} \right) \\
& 0 \quad \left( \begin{array}{cc} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + & \\ +(\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + & \\ -(\ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m} + \ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m}) & \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3n} - \ell_{3m})L_{2n} + & (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} - \ell_{2m})L_{3n} + & +(+\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{1n} - \ell_{1m})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} - \ell_{0m})L_{1n} + & +(+\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{2m} - \ell_{2n})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + & +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) & +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3m} - \ell_{3n})L_{1n} + & (+\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + & +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) & +(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{2m} - \ell_{2n})L_{1n} + & (+\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{2n} + & +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) & +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3n} - \ell_{3m})L_{0n} + & (+\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + & +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) & +(+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n}) \end{array} \right) \\
& \left( \begin{array}{cc} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + & \\ +(\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + & 0 \\ -(\ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n} + \ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n}) & \end{array} \right) \\
& 0 \quad \left( \begin{array}{cc} (L_{3n}L_{3n} + L_{0n}L_{0n} + L_{1n}L_{1n} + L_{2n}L_{2n}) + & \\ +(\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + & \\ -(\ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n} + \ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n}) & \end{array} \right) \\
& \pm \left( \begin{array}{cc} (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(+\ell_{2m} - \ell_{2n})L_{3n} + & +(+\ell_{0n} - \ell_{0m})L_{1n} + \\ +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(+\ell_{0m} - \ell_{0n})L_{1n} + & +(+\ell_{2n} - \ell_{2m})L_{3n} + \\ +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) & +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3n} - \ell_{3m})L_{1n} + & (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n}) & +(\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n}) \end{array} \right)
\end{aligned}$$



$$\begin{aligned}
&= \left( \begin{array}{l} \left( \begin{array}{ll} [(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{1n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{0n} + \\ +[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{2n} + & +[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{3n} + \\ +[(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \pm (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n})] & +[-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \pm (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{0n} + & [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{1n} + \\ +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{2n} + \\ +[-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \pm (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n})] & +[(+\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m}) \pm (+\ell_{2m}\ell_{1n} - \ell_{1m}\ell_{2n})] \end{array} \right) \\ \\ \left( \begin{array}{l} [(L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n})] + \\ +[(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{3n} + [(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{0n} + \\ +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{2n} + \\ -[(\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n})] \end{array} \right) \\ \\ 0 \\ \\ \left( \begin{array}{l} [(L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n})] + \\ +[(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{3n} + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{0n} + \\ +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{2n} + \\ -[(\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{2n} + & [(+\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} - \ell_{1m})]L_{0n} + \\ +[(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{3n} + & +[(+\ell_{0m} - \ell_{0n}) \pm (+\ell_{0n} - \ell_{0m})]L_{1n} + \\ +[(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \pm (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n})] & +[-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \pm (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{0n} + & [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{2n} + \\ +[(+\ell_{0n} - \ell_{0m}) \pm (+\ell_{0m} - \ell_{0n})]L_{1n} + & +[(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{3n} + \\ +[-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \pm (-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n})] & +[(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \pm (+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + \\ +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{2n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{3n} + \\ +[(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})] & +[(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{1n} + & [(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{0n} + \\ +[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{2n} + \\ +[(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m}) \pm (\ell_{1m}\ell_{3n} - \ell_{3m}\ell_{1n})] & +[(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m}) \pm (\ell_{2m}\ell_{0n} - \ell_{0m}\ell_{2n})] \end{array} \right) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{cc} (+\ell_{3n} - \ell_{3m})L_{1n} + & (+\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + & +(\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{3n}\ell_{1m} - \ell_{1m}\ell_{3n}) & +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{2n} - \ell_{2m})L_{0n} + & (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + & +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) & +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3n} - \ell_{3m})L_{2n} + & (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) & +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{1m} - \ell_{1n})L_{0n} + & (+\ell_{2m} - \ell_{3n})L_{2n} + \\ +(\ell_{0m} - \ell_{0n})L_{1n} + & +(\ell_{3m} - \ell_{2n})L_{3n} + \\ +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) & +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + \\ -(\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} 0 \\ (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + \\ -(\ell_{2n}\ell_{2m} + \ell_{1n}\ell_{1m} + \ell_{0n}\ell_{0m} + \ell_{3n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{cc} +(\ell_{3m} - \ell_{3n})L_{0n} + & +(\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + & +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) & +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{cc} +(\ell_{2n} - \ell_{2m})L_{1n} + & +(\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{2n} + & +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3m} - \ell_{3n})L_{1n} + & (+\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) & +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{2m} - \ell_{2n})L_{0n} + & (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) & +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3m} - \ell_{3n})L_{2n} + & (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{3n} + & +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) & +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{1n} - \ell_{1m})L_{0n} + & (+\ell_{2n} - \ell_{3m})L_{2n} + \\ +(\ell_{0n} - \ell_{0m})L_{1n} + & +(\ell_{3n} - \ell_{2m})L_{3n} + \\ +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) & +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{0m} - \ell_{0n})L_{0n} + (\ell_{3m} - \ell_{3n})L_{3n} + \\ -(\ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n} + \ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} 0 \\ (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n} + \\ -(\ell_{2m}\ell_{2n} + \ell_{1m}\ell_{1n} + \ell_{0m}\ell_{0n} + \ell_{3m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{cc} +(\ell_{3n} - \ell_{3m})L_{0n} + & +(\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + & +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{cc} +(\ell_{2m} - \ell_{2n})L_{1n} + & +(\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n} - \ell_{1m})L_{2n} + & +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{ll}
[(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + & [(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + \\
+[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{2n} + \\
+[(\ell_{3n}\ell_{1m} - \ell_{1m}\ell_{3n}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})] & +[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})] \\
[(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{0n} + & [(+\ell_{3m} - \ell_{3n} \pm (+\ell_{3n} - \ell_{3m}))]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{2n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{3n} + \\
+[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})] & +[(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{2n} + & [(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{0n} + \\
+[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{3n} + & +[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{1n} + \\
+[(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \pm (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n})] & +[(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \pm (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n})] \\
[(+\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} - \ell_{1m})]L_{0n} + & [(+\ell_{2m} - \ell_{3n}) \pm (+\ell_{2n} - \ell_{3m})]L_{2n} + \\
+[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{1n} + & +[(\ell_{3m} - \ell_{2n} \pm (\ell_{3n} - \ell_{2m}))]L_{3n} + \\
+[-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \pm (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n})] & +[(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \pm (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n})]
\end{array} \right) \\
= & \left( \begin{array}{l}
[(L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n})] + \\
+[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{2n} + [(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{0n} + [(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{3n} + \\
-[(\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n})] \\
0 \\
0 \\
\end{array} \right) \quad \begin{array}{l} 0 \\ \\ \\ \end{array} \\
& \left( \begin{array}{ll}
+[(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{0n} + & +[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{3n} + & +[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{2n} + \\
+[(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \pm (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n})] & +[(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})] \\
+[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{1n} + & +[(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{0n} + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{2n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + \\
+[-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})] & +[(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \pm (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n})]
\end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{l} (+\ell_{1m} - \ell_{1n})L_{0n} + \quad (+\ell_{3n} - \ell_{3m})L_{2n} + \\ +(\ell_{0n} - \ell_{0m})L_{1n} + \quad +(\ell_{2m} - \ell_{2n})L_{3n} + \\ +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \quad +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{2n} + \quad (+\ell_{1n} - \ell_{1m})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{3n} + \quad +(\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \quad +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2m} - \ell_{2n})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{2n} + \quad +(\ell_{1n} - \ell_{1m})L_{3n} + \\ +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \quad +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{1n} + \quad (+\ell_{0m} - \ell_{0n})L_{2n} + \\ +(\ell_{1m} - \ell_{1n})L_{3n} + \quad +(\ell_{2n} - \ell_{2m})L_{0n} + \\ +(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \quad +(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{0n} + \quad (+\ell_{2n} - \ell_{2m})L_{1n} + \\ +(\ell_{0n} - \ell_{0m})L_{3n} + \quad +(\ell_{1m} - \ell_{1n})L_{2n} + \\ +(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \quad +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{1n} - \ell_{1m})L_{2n} + \quad (+\ell_{3n} - \ell_{3m})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{1n} + \quad +(\ell_{0m} - \ell_{0n})L_{3n} + \\ +(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \quad +(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + \quad 0 \\ -(\ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m} + \ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m}) \end{array} \right) \\ \\ 0 \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + \\ -(\ell_{1n}\ell_{1m} + \ell_{2n}\ell_{2m} + \ell_{3n}\ell_{3m} + \ell_{0n}\ell_{0m}) \end{array} \right)
\end{array} \right) \\
\pm \\
& \left( \begin{array}{l} \left( \begin{array}{l} (+\ell_{1n} - \ell_{1m})L_{0n} + \quad (+\ell_{3m} - \ell_{3n})L_{2n} + \\ +(\ell_{0m} - \ell_{0n})L_{1n} + \quad +(\ell_{2n} - \ell_{2m})L_{3n} + \\ +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{2n} + \quad (+\ell_{1m} - \ell_{1n})L_{0n} + \\ +(\ell_{2m} - \ell_{2n})L_{3n} + \quad +(\ell_{0n} - \ell_{0m})L_{1n} + \\ +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \quad +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2n} - \ell_{2m})L_{0n} + \quad (+\ell_{3n} - \ell_{3m})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{2n} + \quad +(\ell_{1m} - \ell_{1n})L_{3n} + \\ +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \quad +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{0n} - \ell_{0m})L_{2n} + \\ +(\ell_{1n} - \ell_{1m})L_{3n} + \quad +(\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n}) \quad +(\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} - \ell_{3m})L_{0n} + \quad (+\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1n} - \ell_{1m})L_{2n} + \\ +(\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \quad +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{1m} - \ell_{1n})L_{2n} + \quad (+\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{2n} - \ell_{2m})L_{1n} + \quad +(\ell_{0n} - \ell_{0m})L_{3n} + \\ +(\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \quad +(\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} - \ell_{1m})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n} + (\ell_{3n} - \ell_{3m})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + \quad 0 \\ -(\ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n} + \ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n}) \end{array} \right) \\ \\ 0 \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n} + \\ -(\ell_{1m}\ell_{1n} + \ell_{2m}\ell_{2n} + \ell_{3m}\ell_{3n} + \ell_{0m}\ell_{0n}) \end{array} \right)
\end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l}
\left[ \begin{array}{ll}
[(+\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} - \ell_{1m})]L_{0n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{2n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{1n} + & +[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{3n} + \\
+[(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \pm (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n})] & +[-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \pm (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n})]
\end{array} \right. \\
\left[ \begin{array}{ll}
[(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{2n} + & [(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{0n} + \\
+[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{3n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{1n} + \\
+[-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \pm (-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n})] & +[(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \pm (+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n})]
\end{array} \right. \\
\left[ \begin{array}{ll}
[(+\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} - \ell_{2m})]L_{0n} + & [(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{2n} + & +[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{3n} + \\
+[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})] & +[(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})]
\end{array} \right. \\
\left[ \begin{array}{ll}
[(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{1n} + & +[(+\ell_{0m} - \ell_{0n}) \pm (+\ell_{0n} - \ell_{0m})]L_{2n} + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{3n} + & +[(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{0n} + \\
+[(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m}) \pm (\ell_{3m}\ell_{1n} - \ell_{1m}\ell_{3n})] & +[(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m}) \pm (\ell_{0m}\ell_{2n} - \ell_{2m}\ell_{0n})]
\end{array} \right. \\
\left[ \begin{array}{ll}
[(+\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} - \ell_{3m})]L_{0n} + & [(+\ell_{2n} - \ell_{2m}) \pm (+\ell_{2m} - \ell_{2n})]L_{1n} + \\
+[(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{3n} + & +[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{2n} + \\
+[(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \pm (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n})] & +[(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})]
\end{array} \right. \\
\left[ \begin{array}{ll}
[(+\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} - \ell_{1n})]L_{2n} + & [(+\ell_{3n} - \ell_{3m}) \pm (+\ell_{3m} - \ell_{3n})]L_{0n} + \\
+[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{1n} + & +[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{3n} + \\
+[(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \pm (\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n})] & +[(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m}) \pm (\ell_{0m}\ell_{3n} - \ell_{3m}\ell_{0n})]
\end{array} \right. \\
\left[ \begin{array}{l}
[(L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n})] + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} - \ell_{1m})]L_{1n} + [(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} - \ell_{2m})]L_{2n} + \\
+[(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} - \ell_{3m})]L_{3n} + [(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} - \ell_{0m})]L_{0n} + \\
-[(\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n})]
\end{array} \right. \\
0 \\
0
\end{array} \right)
\end{aligned}$$

$$\Rightarrow D_{Bn}\overline{D_{Am}} \pm \overline{D_{Bm}}D_{An} = \left( \begin{array}{cccc} \mathbf{c1} & \mathbf{c2} & \mathbf{c3} & \mathbf{c4} \end{array} \right) (\mathbf{n}\overline{\mathbf{m}}\mathbf{i}'\mathbf{s} \pm \overline{\mathbf{m}}\mathbf{n}'\mathbf{s}) \quad (L_{\mu m} = L_{\mu n})$$

where:

$$\begin{aligned}
& [(L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n})] + \\
& +[(\ell_{1n} - \ell_{1m}) \pm (\ell_{1m} - \ell_{1n})]L_{1n} + [(\ell_{2n} - \ell_{2m}) \pm (\ell_{2m} - \ell_{2n})]L_{2n} + \\
& +[(\ell_{3n} - \ell_{3m}) \pm (\ell_{3m} - \ell_{3n})]L_{3n} + [(\ell_{0n} - \ell_{0m}) \pm (\ell_{0m} - \ell_{0n})]L_{0n} + \\
& -[(\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) + (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) + (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) + (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n})]
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0m} + \ell_{0n})L_{0n} + (\ell_{3m} + \ell_{3n})L_{3n} + (\ell_{2m} + \ell_{2n})L_{2n} + (\ell_{1m} + \ell_{1n})L_{1n} + \\ -(-\ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m} - \ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + \\ -(-\ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m} - \ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{2m} + \ell_{2n})L_{1n} + & (-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{1m} + \ell_{1n})L_{2n} + & +(-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) & +(+\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{ll} (\ell_{0m} + \ell_{0n})L_{3n} + & (-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{3m} + \ell_{3n})L_{0n} + & +(-\ell_{1n} - \ell_{1m})L_{2n} + \\ +(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) & +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{3n} + \ell_{3m})L_{1n} + & (\ell_{0m} + \ell_{0n})L_{2n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + & +(\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) & +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \end{array} \right) \\
& \left( \begin{array}{ll} (-\ell_{2m} - \ell_{2n})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + & +(-\ell_{1n} - \ell_{1m})L_{3n} + \\ +(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) & +(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{ll} (-\ell_{1m} - \ell_{1n})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{2n} + \\ +(+\ell_{0n} + \ell_{0m})L_{1n} + & +(\ell_{2n} + \ell_{2m})L_{3n} + \\ +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) & +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{3n} + \ell_{3m})L_{2n} + & (+\ell_{1n} + \ell_{1m})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + & +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m}) & +(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(-\ell_{0n} - \ell_{0m})L_{0n} + (-\ell_{3n} - \ell_{3m})L_{3n} + (-\ell_{2n} - \ell_{2m})L_{2n} + (-\ell_{1n} - \ell_{1m})L_{1n} + \\ -(-\ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n} - \ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} (L_{0n}L_{0n} + L_{3n}L_{3n} + L_{2n}L_{2n} + L_{1n}L_{1n}) + \\ +(\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + \\ -(-\ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n} - \ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{ll} (-\ell_{2n} - \ell_{2m})L_{1n} + & (\ell_{0n} + \ell_{0m})L_{3n} + \\ +(-\ell_{1n} - \ell_{1m})L_{2n} + & +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) & +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{ll} (-\ell_{0n} - \ell_{0m})L_{3n} + & (\ell_{2n} + \ell_{2m})L_{1n} + \\ +(-\ell_{3n} - \ell_{3m})L_{0n} + & +(+\ell_{1m} + \ell_{1n})L_{2n} + \\ +(+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n}) & +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{1n} + & (-\ell_{0n} - \ell_{0m})L_{2n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + & +(-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) & +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{2n} + \ell_{2m})L_{0n} + & (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + & +(+\ell_{1m} + \ell_{1n})L_{3n} + \\ +(-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n}) & +(-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{ll} (+\ell_{1n} + \ell_{1m})L_{0n} + & (+\ell_{3n} + \ell_{3m})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + & +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) & +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) \end{array} \right) \\
& \left( \begin{array}{ll} (-\ell_{3m} - \ell_{3n})L_{2n} + & (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(+\ell_{2n} + \ell_{2m})L_{3n} + & +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(+\ell_{2m}\ell_{3n} - \ell_{3m}\ell_{2n}) & +(-\ell_{1m}\ell_{0n} + \ell_{0m}\ell_{1n}) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left[ (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) \right] + \\
& + [(\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} - \ell_{0m})]L_{0n} + [(\ell_{3m} + \ell_{3n}) \pm (-\ell_{3n} - \ell_{3m})]L_{3n} + \\
& + [(\ell_{2m} + \ell_{2n}) \pm (-\ell_{2n} - \ell_{2m})]L_{2n} + [(\ell_{1m} + \ell_{1n}) \pm (-\ell_{1n} - \ell_{1m})]L_{1n} + \\
& - [(-\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n}) - (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) - (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) - (\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n})] \quad 0 \\
& 0 \\
& \left[ (+\ell_{2m} + \ell_{2n}) \pm (-\ell_{2n} - \ell_{2m}) \right]L_{1n} + \quad [(-\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} + \ell_{0m})]L_{3n} + \\
& + [(+\ell_{1m} + \ell_{1n}) \pm (-\ell_{1n} - \ell_{1m})]L_{2n} + \quad + [(-\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} + \ell_{3m})]L_{0n} + \\
& + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})] \quad + [(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \pm (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n})] \\
& [(\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} - \ell_{0m})]L_{3n} + \quad [(-\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} + \ell_{2m})]L_{1n} + \\
& + [(\ell_{3m} + \ell_{3n}) \pm (-\ell_{3n} - \ell_{3m})]L_{0n} + \quad + [(-\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} + \ell_{1n})]L_{2n} + \\
& + [(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m}) \pm (+\ell_{3m}\ell_{0n} - \ell_{0m}\ell_{3n})] \quad + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})] \\
& \left[ (+\ell_{3n} + \ell_{3m}) \pm (-\ell_{3m} - \ell_{3n}) \right]L_{1n} + \quad [(\ell_{0m} + \ell_{0n}) \pm (-\ell_{0n} - \ell_{0m})]L_{2n} + \\
& + [(\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} - \ell_{1n})]L_{3n} + \quad + [(\ell_{2n} + \ell_{2m}) \pm (-\ell_{2m} - \ell_{2n})]L_{0n} + \\
& + [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \pm (-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n})] \quad + [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \pm (-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n})] \\
& [(-\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} + \ell_{2m})]L_{0n} + \quad [(-\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} + \ell_{3m})]L_{1n} + \\
& + [(-\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} + \ell_{0m})]L_{2n} + \quad + [(-\ell_{1n} - \ell_{1m}) \pm (+\ell_{1m} + \ell_{1n})]L_{3n} + \\
& + [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m}) \pm (-\ell_{2m}\ell_{0n} + \ell_{0m}\ell_{2n})] \quad + [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m}) \pm (-\ell_{1m}\ell_{3n} + \ell_{3m}\ell_{1n})] \\
& \left[ (-\ell_{1m}^- - \ell_{1n}) \pm (+\ell_{1n}^- + \ell_{1m}) \right] L_{0n} + \quad [(-\ell_{3m}^- - \ell_{3n}) \pm (+\ell_{3n}^- + \ell_{3m})] L_{2n} + \\
& + [(\ell_{0n}^- + \ell_{0m}^-) \pm (-\ell_{0m}^- - \ell_{0n}^-)] L_{1n} + \quad + [(\ell_{2n}^- + \ell_{2m}^-) \pm (-\ell_{2m}^- - \ell_{2n}^-)] L_{3n} + \\
& + [(-\ell_{1n}\ell_{0m}^- + \ell_{0n}\ell_{1m}^-) \pm (-\ell_{1m}\ell_{0n}^- + \ell_{0m}\ell_{1n}^-)] \quad + [(+\ell_{2n}\ell_{3m}^- - \ell_{3n}\ell_{2m}^-) \pm (+\ell_{2m}\ell_{3n}^- - \ell_{3m}\ell_{2n}^-)] \\
& [(\ell_{3n}^- + \ell_{3m}^-) \pm (-\ell_{3m}^- - \ell_{3n}^-)] L_{2n} + \quad [(+\ell_{1n}^- + \ell_{1m}^-) \pm (-\ell_{1m}^- - \ell_{1n}^-)] L_{0n} + \\
& + [(-\ell_{2m}^- - \ell_{2n}^-) \pm (+\ell_{2n}^- + \ell_{2m}^-)] L_{3n} + \quad + [(-\ell_{0m}^- - \ell_{0n}^-) \pm (\ell_{0n}^- + \ell_{0m}^-)] L_{1n} + \\
& + [(+\ell_{2n}\ell_{3m}^- - \ell_{3n}\ell_{2m}^-) \pm (+\ell_{2m}\ell_{3n}^- - \ell_{3m}\ell_{2n}^-)] \quad + [(-\ell_{1n}\ell_{0m}^- + \ell_{0n}\ell_{1m}^-) \pm (-\ell_{1m}\ell_{0n}^- + \ell_{0m}\ell_{1n}^-)]
\end{aligned}$$





$$\begin{aligned}
&= \left( \begin{array}{l} \left( \begin{array}{ll} [(+l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{1n} + & [(+l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{0n} + \\ +[(l_{1n} + l_{1m}) \pm (-l_{1m} - l_{1n})]L_{2n} + & +[(l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{3n} + \\ +[(-l_{2n}l_{1m} + l_{1n}l_{2m}) \pm (-l_{2m}l_{1n} + l_{1m}l_{2n})] & +[(+l_{0n}l_{3m} - l_{3n}l_{0m}) \pm (+l_{0m}l_{3n} - l_{3m}l_{0n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(-l_{3m} - l_{3n}) \pm (+l_{3n} + l_{3m})]L_{0n} + & [(-l_{2m} - l_{2n}) \pm (+l_{2n} + l_{2m})]L_{1n} + \\ +[(-l_{0m} - l_{0n}) \pm (l_{0n} + l_{0m})]L_{3n} + & +[(-l_{1m} - l_{1n}) \pm (l_{1n} + l_{1m})]L_{2n} + \\ +[(+l_{0n}l_{3m} - l_{3n}l_{0m}) \pm (+l_{0m}l_{3n} - l_{3m}l_{0n})] & +[(-l_{2n}l_{1m} + l_{1n}l_{2m}) \pm (-l_{2m}l_{1n} + l_{1m}l_{2n})] \end{array} \right) \\ \\ \left( \begin{array}{l} [(L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n})] + \\ +[(-l_{3m} - l_{3n}) \pm (l_{3n} + l_{3m})]L_{3n} + [(l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{0n} + \\ +[(-l_{1m} - l_{1n}) \pm (l_{1n} + l_{1m})]L_{1n} + [(l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{2n} + \\ -[-(l_{3n}l_{3m} \pm l_{3m}l_{3n}) - (l_{0n}l_{0m} \pm l_{0m}l_{0n}) - (l_{1n}l_{1m} \pm l_{1m}l_{1n}) - (l_{2n}l_{2m} \pm l_{2m}l_{2n})] \end{array} \right) \\ \\ 0 \\ \\ \left( \begin{array}{l} [(L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n})] + \\ +[(l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{3n} + [(-l_{0m} - l_{0n}) \pm (l_{0n} + l_{0m})]L_{0n} + \\ +[(-l_{1m} - l_{1n}) \pm (l_{1n} + l_{1m})]L_{1n} + [(l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{2n} + \\ -[-(l_{3n}l_{3m} \pm l_{3m}l_{3n}) - (l_{0n}l_{0m} \pm l_{0m}l_{0n}) - (l_{1n}l_{1m} \pm l_{1m}l_{1n}) - (l_{2n}l_{2m} \pm l_{2m}l_{2n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(+l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{2n} + & [(-l_{1m} - l_{1n}) \pm (+l_{1n} + l_{1m})]L_{0n} + \\ +[(+l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{3n} + & +[(-l_{0m} - l_{0n}) \pm (+l_{0n} + l_{0m})]L_{1n} + \\ +[(-l_{2n}l_{3m} + l_{3n}l_{2m}) \pm (-l_{2m}l_{3n} + l_{3m}l_{2n})] & +[(+l_{1n}l_{0m} - l_{0n}l_{1m}) \pm (+l_{1m}l_{0n} - l_{0m}l_{1n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(+l_{1n} + l_{1m}) \pm (-l_{1m} - l_{1n})]L_{0n} + & [(-l_{3m} - l_{3n}) \pm (+l_{3n} + l_{3m})]L_{2n} + \\ +[(+l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{1n} + & +[(-l_{2m} - l_{2n}) \pm (+l_{2n} + l_{2m})]L_{3n} + \\ +[(+l_{1n}l_{0m} - l_{0n}l_{1m}) \pm (+l_{1m}l_{0n} - l_{0m}l_{1n})] & +[(-l_{2n}l_{3m} + l_{3n}l_{2m}) \pm (-l_{2m}l_{3n} + l_{3m}l_{2n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(-l_{2m} - l_{2n}) \pm (+l_{2n} + l_{2m})]L_{0n} + & [(+l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{1n} + \\ +[(l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{2n} + & +[(-l_{1m} - l_{1n}) \pm (l_{1n} + l_{1m})]L_{3n} + \\ +[(-l_{2n}l_{0m} + l_{0n}l_{2m}) \pm (-l_{2m}l_{0n} + l_{0m}l_{2n})] & +[(-l_{1n}l_{3m} + l_{3n}l_{1m}) \pm (-l_{1m}l_{3n} + l_{3m}l_{1n})] \end{array} \right) \\ \\ \left( \begin{array}{ll} [(-l_{3m} - l_{3n}) \pm (+l_{3n} + l_{3m})]L_{1n} + & [(+l_{2n} + l_{2m}) \pm (+l_{2m} - l_{2n})]L_{0n} + \\ +[(l_{1n} + l_{1m}) \pm (-l_{1m} - l_{1n})]L_{3n} + & +[(-l_{0m} - l_{0n}) \pm (l_{0n} + l_{0m})]L_{2n} + \\ +[(-l_{1n}l_{3m} + l_{3n}l_{1m}) \pm (-l_{1m}l_{3n} + l_{3m}l_{1n})] & +[(-l_{2n}l_{0m} + l_{0n}l_{2m}) \pm (-l_{2m}l_{0n} + l_{0m}l_{2n})] \end{array} \right) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{cc} (+\ell_{3n} + \ell_{3m})L_{1n} + & (-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + & +(-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) & +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{2n} + \ell_{2m})L_{0n} + & (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + & +(-\ell_{1m} - \ell_{1n})L_{3n} + \\ +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) & +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{cc} (+\ell_{3n} + \ell_{3m})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{3n} + & +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) & +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) \end{array} \right) \\
& \left( \begin{array}{cc} (-\ell_{1m} - \ell_{1n})L_{0n} + & (-\ell_{2m} - \ell_{3n})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + & +(-\ell_{3m} - \ell_{2n})L_{3n} + \\ +(+\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m}) & +(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + \\ -(-\ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m} - \ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} 0 \\ (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + \\ -(-\ell_{2n}\ell_{2m} - \ell_{1n}\ell_{1m} - \ell_{0n}\ell_{0m} - \ell_{3n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{cc} +(-\ell_{3m} - \ell_{3n})L_{0n} + & +(-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + & +(\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) & +(+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) \end{array} \right) \\
& \left( \begin{array}{cc} +(\ell_{2n} + \ell_{2m})L_{1n} + & +(\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{2n} + & +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(+\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m}) & +(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m}) \end{array} \right) \\
& \left( \begin{array}{cc} (-\ell_{3m} - \ell_{3n})L_{1n} + & +(\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + & +(\ell_{0n} + \ell_{0m})L_{2n} + \\ +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) & +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{cc} (-\ell_{2m} - \ell_{2n})L_{0n} + & +(\ell_{3n} + \ell_{3m})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + & +(\ell_{1n} + \ell_{1m})L_{3n} + \\ +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) & +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{cc} (-\ell_{3m} - \ell_{3n})L_{2n} + & +(-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + & +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) & +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) \end{array} \right) \\
& \left( \begin{array}{cc} +(\ell_{1n} + \ell_{1m})L_{0n} + & +(\ell_{2n} + \ell_{3m})L_{2n} + \\ +(\ell_{0n} + \ell_{0m})L_{1n} + & +(\ell_{3n} + \ell_{2m})L_{3n} + \\ +(+\ell_{0m}\ell_{1n} - \ell_{1m}\ell_{0n}) & +(-\ell_{3m}\ell_{2n} + \ell_{2m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{l} (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{1n} + \ell_{1m})L_{1n} + (-\ell_{0m} - \ell_{0n})L_{0n} + (-\ell_{3m} - \ell_{3n})L_{3n} + \\ -(-\ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n} - \ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n}) \end{array} \right) \quad 0 \\
& \left( \begin{array}{l} 0 \\ (L_{2n}L_{2n} + L_{1n}L_{1n} + L_{0n}L_{0n} + L_{3n}L_{3n}) + \\ +(-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} + \\ -(-\ell_{2m}\ell_{2n} - \ell_{1m}\ell_{1n} - \ell_{0m}\ell_{0n} - \ell_{3m}\ell_{3n}) \end{array} \right) \\
& \left( \begin{array}{cc} +(\ell_{3n} + \ell_{3m})L_{0n} + & +(\ell_{2n} + \ell_{2m})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{3n} + & +(-\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) & +(+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) \end{array} \right) \\
& \left( \begin{array}{cc} +(-\ell_{2m} - \ell_{2n})L_{1n} + & +(-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{1n} + \ell_{1m})L_{2n} + & +(\ell_{0n} + \ell_{0m})L_{3n} + \\ +(+\ell_{1m}\ell_{2n} - \ell_{2m}\ell_{1n}) & +(-\ell_{3m}\ell_{0n} + \ell_{0m}\ell_{3n}) \end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{ll}
[(+l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{1n} + & [(-l_{2m} - l_{2n}) \pm (+l_{2n} + l_{2m})]L_{0n} + \\
+[(l_{1n} + l_{1m}) \pm (-l_{1m} - l_{1n})]L_{3n} + & +[(-l_{0m} - l_{0n}) \pm (l_{0n} + l_{0m})]L_{2n} + \\
+[-(l_{3n}l_{1m} + l_{1m}l_{3n}) \pm (-l_{3m}l_{1n} + l_{1m}l_{3n})] & +[(-l_{0n}l_{2m} + l_{2n}l_{0m}) \pm (-l_{0m}l_{2n} + l_{2m}l_{0n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{0n} + & [(-l_{3m} - l_{3n}) \pm (+l_{3n} + l_{3m})]L_{1n} + \\
+[(l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{2n} + & +[(-l_{1m} - l_{1n}) \pm (l_{1n} + l_{1m})]L_{3n} + \\
+[-(l_{0n}l_{2m} + l_{2n}l_{0m}) \pm (-l_{0m}l_{2n} + l_{2m}l_{0n})] & +[-(l_{3n}l_{1m} + l_{1n}l_{3m}) \pm (-l_{3m}l_{1n} + l_{1m}l_{3n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(+l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{2n} + & [(+l_{1n} + l_{1m}) \pm (-l_{1m} - l_{1n})]L_{0n} + \\
+[(l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{3n} + & +[(l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{1n} + \\
+[-(l_{3n}l_{2m} + l_{2n}l_{3m}) \pm (-l_{3m}l_{2n} + l_{2m}l_{3n})] & +[(+l_{0n}l_{1m} - l_{1n}l_{0m}) \pm (+l_{0m}l_{1n} - l_{1m}l_{0n})]
\end{array} \right) \\
& \left( \begin{array}{ll}
[(-l_{1m} - l_{1n}) \pm (+l_{1n} + l_{1m})]L_{0n} + & [(-l_{2m} - l_{3n}) \pm (+l_{2n} + l_{3m})]L_{2n} + \\
+[-(l_{0m} - l_{0n}) \pm (l_{0n} + l_{0m})]L_{1n} + & +[(-l_{3m} - l_{2n}) \pm (l_{3n} + l_{2m})]L_{3n} + \\
+[(+l_{0n}l_{1m} - l_{1n}l_{0m}) \pm (+l_{0m}l_{1n} - l_{1m}l_{0n})] & +[-(l_{3n}l_{2m} + l_{2n}l_{3m}) \pm (-l_{3m}l_{2n} + l_{2m}l_{3n})]
\end{array} \right) \\
& = \left( \begin{array}{l}
[(L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n})] + \\
+ [(-l_{2m} - l_{2n}) \pm (l_{2n} + l_{2m})]L_{2n} + [(-l_{1m} - l_{1n}) \pm (l_{1n} + l_{1m})]L_{1n} + \\
+ [(l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{0n} + [(l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{3n} + \\
- [-(l_{2n}l_{2m} \pm l_{2m}l_{2n}) - (l_{1n}l_{1m} \pm l_{1m}l_{1n}) - (l_{0n}l_{0m} \pm l_{0m}l_{0n}) - (l_{3n}l_{3m} \pm l_{3m}l_{3n})] \\
0 \\
0 \\
\left( \begin{array}{ll}
[(-l_{3m} - l_{3n}) \pm (l_{3n} + l_{3m})]L_{0n} + & [(-l_{2m} - l_{2n}) \pm (l_{2n} + l_{2m})]L_{1n} + \\
+[(l_{0n} + l_{0m}) \pm (-l_{0m} - l_{0n})]L_{3n} + & +[(l_{1n} + l_{1m}) \pm (-l_{1m} - l_{1n})]L_{2n} + \\
+[-(l_{3n}l_{0m} + l_{0n}l_{3m}) \pm (-l_{3m}l_{0n} + l_{0m}l_{3n})] & +[(+l_{1n}l_{2m} - l_{2n}l_{1m}) \pm (+l_{1m}l_{2n} - l_{2m}l_{1n})]
\end{array} \right) \\
\left( \begin{array}{ll}
[(l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{1n} + & [(l_{3n} + l_{3m}) \pm (-l_{3m} - l_{3n})]L_{0n} + \\
+ [(-l_{1m} - l_{1n}) \pm (l_{1n} + l_{1m})]L_{2n} + & + [(-l_{0m} - l_{0n}) \pm (l_{0n} + l_{0m})]L_{3n} + \\
+ [(+l_{1n}l_{2m} - l_{2n}l_{1m}) \pm (+l_{1m}l_{2n} - l_{2m}l_{1n})] & + [(-l_{3n}l_{0m} + l_{0n}l_{3m} \pm (-l_{3m}l_{0n} + l_{0m}l_{3n}))]
\end{array} \right)
\end{array} \right) \\
& \left( \begin{array}{l}
[(L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n})] + \\
+ [(l_{2n} + l_{2m}) \pm (-l_{2m} - l_{2n})]L_{2n} + [(l_{1n} + l_{1m}) \pm (-l_{1m} - l_{1n})]L_{1n} + \\
+ [(-l_{0m} - l_{0n}) \pm (l_{0n} + l_{0m})]L_{0n} + [(-l_{3m} - l_{3n}) \pm (l_{3n} + l_{2m})]L_{3n} + \\
- [-(l_{2n}l_{2m} \pm l_{2m}l_{2n}) - (l_{1n}l_{1m} \pm l_{1m}l_{1n}) - (l_{0n}l_{0m} \pm l_{0m}l_{0n}) - (l_{3n}l_{3m} \pm l_{3m}l_{3n})]
\end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{l} (-\ell_{1m} - \ell_{1n})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{2n} + \\ +(\ell_{0n} + \ell_{0m})L_{1n} + \quad +(-\ell_{2m} - \ell_{2n})L_{3n} + \\ +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \quad +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{2n} + \quad (+\ell_{1n} + \ell_{1m})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{3n} + \quad +(-\ell_{0m} - \ell_{0n})L_{1n} + \\ +(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \quad +(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (-\ell_{2m} - \ell_{2n})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{2n} + \quad +(\ell_{1n} + \ell_{1m})L_{3n} + \\ +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \quad +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{1n} + \quad (-\ell_{0m} - \ell_{0n})L_{2n} + \\ +(-\ell_{1m} - \ell_{1n})L_{3n} + \quad +(\ell_{2n} + \ell_{2m})L_{0n} + \\ +(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \quad +(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{0n} + \quad (+\ell_{2n} + \ell_{2m})L_{1n} + \\ +(\ell_{0n} + \ell_{0m})L_{3n} + \quad +(-\ell_{1m} - \ell_{1n})L_{2n} + \\ +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \quad +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{1n} + \ell_{1m})L_{2n} + \quad (+\ell_{3n} + \ell_{3m})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{1n} + \quad +(-\ell_{0m} - \ell_{0n})L_{3n} + \\ +(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \quad +(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + \quad 0 \\ -(-\ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m} - \ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m}) \end{array} \right) \\ \\ 0 \end{array} \right) \quad \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} + \ell_{1m})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + \\ -(-\ell_{1n}\ell_{1m} - \ell_{2n}\ell_{2m} - \ell_{3n}\ell_{3m} - \ell_{0n}\ell_{0m}) \end{array} \\ \\ \pm \left( \begin{array}{l} \left( \begin{array}{l} (+\ell_{1n} + \ell_{1m})L_{0n} + \quad (-\ell_{3m} - \ell_{3n})L_{2n} + \\ +(-\ell_{0m} - \ell_{0n})L_{1n} + \quad +(\ell_{2n} + \ell_{2m})L_{3n} + \\ +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \quad +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{2n} + \quad (-\ell_{1m} - \ell_{1n})L_{0n} + \\ +(-\ell_{2m} - \ell_{2n})L_{3n} + \quad +(\ell_{0n} + \ell_{0m})L_{1n} + \\ +(+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{2n} + \ell_{2m})L_{0n} + \quad (+\ell_{3n} + \ell_{3m})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{2n} + \quad +(-\ell_{1m} - \ell_{1n})L_{3n} + \\ +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \quad +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (-\ell_{3m} - \ell_{3n})L_{1n} + \quad (+\ell_{0n} + \ell_{0m})L_{2n} + \\ +(\ell_{1n} + \ell_{1m})L_{3n} + \quad +(-\ell_{2m} - \ell_{2n})L_{0n} + \\ +(-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n}) \quad +(-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (+\ell_{3n} + \ell_{3m})L_{0n} + \quad (-\ell_{2m} - \ell_{2n})L_{1n} + \\ +(-\ell_{0m} - \ell_{0n})L_{3n} + \quad +(\ell_{1n} + \ell_{1m})L_{2n} + \\ +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \quad +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (-\ell_{1m} - \ell_{1n})L_{2n} + \quad (-\ell_{3m} - \ell_{3n})L_{0n} + \\ +(\ell_{2n} + \ell_{2m})L_{1n} + \quad +(\ell_{0n} + \ell_{0m})L_{3n} + \\ +(-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n}) \quad +(-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n}) \end{array} \right) \\ \\ \left( \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(\ell_{1n} + \ell_{1m})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} + (\ell_{3n} + \ell_{3m})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + \quad 0 \\ -(-\ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n} - \ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n}) \end{array} \right) \\ \\ 0 \end{array} \right) \quad \begin{array}{l} (L_{1n}L_{1n} + L_{2n}L_{2n} + L_{3n}L_{3n} + L_{0n}L_{0n}) + \\ +(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n} + \\ -(-\ell_{1m}\ell_{1n} - \ell_{2m}\ell_{2n} - \ell_{3m}\ell_{3n} - \ell_{0m}\ell_{0n}) \end{array}
\end{aligned}$$

$$\begin{aligned}
& \left( \begin{array}{l}
\left( \begin{array}{l}
[(-\ell_{1m} - \ell_{1n}) \pm (+\ell_{1n} + \ell_{1m})]L_{0n} + \\
+[(\ell_{0n} + \ell_{0m}) \pm (-\ell_{0m} - \ell_{0n})]L_{1n} + \\
+[(\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \pm (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n})]
\end{array} \right) \quad \left( \begin{array}{l}
[(+\ell_{3n} + \ell_{3m}) \pm (-\ell_{3m} - \ell_{3n})]L_{2n} + \\
+[(-\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} + \ell_{2m})]L_{3n} + \\
+[(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \pm (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n})]
\end{array} \right) \\
\left( \begin{array}{l}
[(-\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} + \ell_{3m})]L_{2n} + \\
+[(\ell_{2n} + \ell_{2m}) \pm (-\ell_{2m} - \ell_{2n})]L_{3n} + \\
+[(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m}) \pm (+\ell_{3m}\ell_{2n} - \ell_{2m}\ell_{3n})]
\end{array} \right) \quad \left( \begin{array}{l}
[(+\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} - \ell_{1n})]L_{0n} + \\
+[(-\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} + \ell_{0m})]L_{1n} + \\
+[(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m}) \pm (-\ell_{0m}\ell_{1n} + \ell_{1m}\ell_{0n})]
\end{array} \right) \\
\left( \begin{array}{l}
[(-\ell_{2m} - \ell_{2n}) \pm (+\ell_{2n} + \ell_{2m})]L_{0n} + \\
+[(\ell_{0n} + \ell_{0m}) \pm (-\ell_{0m} - \ell_{0n})]L_{2n} + \\
+[(\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \pm (-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n})]
\end{array} \right) \quad \left( \begin{array}{l}
[(-\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} + \ell_{3m})]L_{1n} + \\
+[(\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} - \ell_{1n})]L_{3n} + \\
+[(\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \pm (-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n})]
\end{array} \right) \\
\left( \begin{array}{l}
[(+\ell_{3n} + \ell_{3m}) \pm (-\ell_{3m} - \ell_{3n})]L_{1n} + \\
+[(\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} - \ell_{1n})]L_{3n} + \\
+[(\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m}) \pm (-\ell_{3m}\ell_{1n} + \ell_{1m}\ell_{3n})]
\end{array} \right) \quad \left( \begin{array}{l}
[(-\ell_{0m} - \ell_{0n}) \pm (+\ell_{0n} + \ell_{0m})]L_{2n} + \\
+[(\ell_{2n} + \ell_{2m}) \pm (-\ell_{2m} - \ell_{2n})]L_{0n} + \\
+[(\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m}) \pm (-\ell_{0m}\ell_{2n} + \ell_{2m}\ell_{0n})]
\end{array} \right) \\
\left( \begin{array}{l}
[(-\ell_{3m} - \ell_{3n}) \pm (+\ell_{3n} + \ell_{3m})]L_{0n} + \\
+[(\ell_{0n} + \ell_{0m}) \pm (-\ell_{0m} - \ell_{0n})]L_{3n} + \\
+[(\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m}) \pm (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n})]
\end{array} \right) \quad \left( \begin{array}{l}
[(+\ell_{2n} + \ell_{2m}) \pm (-\ell_{2m} - \ell_{2n})]L_{1n} + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} + \ell_{1m})]L_{2n} + \\
+[(\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})]
\end{array} \right) \\
\left( \begin{array}{l}
[(+\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} - \ell_{1n})]L_{2n} + \\
+[(\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} + \ell_{2m})]L_{1n} + \\
+[(\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m}) \pm (-\ell_{1m}\ell_{2n} + \ell_{2m}\ell_{1n})]
\end{array} \right) \quad \left( \begin{array}{l}
[(+\ell_{3n} + \ell_{3m}) \pm (-\ell_{3m} - \ell_{3n})]L_{0n} + \\
+[(\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} + \ell_{0m})]L_{3n} + \\
+[(\ell_{0n}\ell_{3m} + \ell_{3m}\ell_{0n}) \pm (-\ell_{0m}\ell_{3n} + \ell_{3m}\ell_{0n})]
\end{array} \right) \\
\left( \begin{array}{l}
[(L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) + (L_{0n}L_{0n} \pm L_{0n}L_{0n})] + \\
+[(\ell_{1m} - \ell_{1n}) \pm (\ell_{1n} + \ell_{1m})]L_{1n} + [(-\ell_{2m} - \ell_{2n}) \pm (\ell_{2n} + \ell_{2m})]L_{2n} + \\
+[(\ell_{3m} - \ell_{3n}) \pm (\ell_{3n} + \ell_{3m})]L_{3n} + [(-\ell_{0m} - \ell_{0n}) \pm (\ell_{0n} + \ell_{0m})]L_{0n} + \\
-[(\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) - (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) - (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) - (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n})]
\end{array} \right) \quad 0 \\
0
\end{array} \right) \quad \left( \begin{array}{l}
[(L_{1n}L_{1n} \pm L_{1n}L_{1n}) + (L_{2n}L_{2n} \pm L_{2n}L_{2n}) + (L_{3n}L_{3n} \pm L_{3n}L_{3n}) \\
+[(\ell_{1n} + \ell_{1m}) \pm (-\ell_{1m} - \ell_{1n})]L_{1n} + [(\ell_{2n} + \ell_{2m}) \pm (-\ell_{2m} - \ell_{2n})]L_{2n} + \\
+[(\ell_{3n} + \ell_{3m}) \pm (-\ell_{3m} - \ell_{3n})]L_{3n} + [(\ell_{0n} + \ell_{0m}) \pm (-\ell_{0m} - \ell_{0n})]L_{0n} + \\
-[(\ell_{1n}\ell_{1m} \pm \ell_{1m}\ell_{1n}) - (\ell_{2n}\ell_{2m} \pm \ell_{2m}\ell_{2n}) - (\ell_{3n}\ell_{3m} \pm \ell_{3m}\ell_{3n}) - (\ell_{0n}\ell_{0m} \pm \ell_{0m}\ell_{0n})]
\end{array} \right)
\end{aligned}$$

□

**Corollary III.2:** For linear/differential operators  $D_{Bn}$  &  $D_{Am}$  :

where:

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

then:

$$D_{Bn}D_{Am} + D_{Bm}D_{An} = 2 \left[ \sum_{\mu=0}^3 L_{\mu n} L_{\mu n} - \sum_{\mu=0}^3 \ell_{\mu n} \ell_{\mu n} \right] \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{pmatrix}$$

$$D_{Bn}D_{Am} - D_{Bm}D_{An} = \begin{pmatrix} c1 & c2 & c3 & c4 \end{pmatrix}$$

where:

$$\begin{aligned}
c1=2 & \left( \begin{array}{l}
[(\ell_{0n} - \ell_{0n})L_{0n} + (-\ell_{3m} + \ell_{3n})L_{3n} + (-\ell_{2m} + \ell_{2n})L_{2n} + (-\ell_{1m} + \ell_{1n})L_{1n}] \sigma^3 \\
\left( \begin{array}{l}
[(-\ell_{2m} + \ell_{2n})L_{1n} + (-\ell_{1m} + \ell_{1n})L_{2n}] \sigma^3 + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m})] \mathbf{I}_2 + \\
+[(\ell_{0m} - \ell_{0n})L_{3n} + (\ell_{3m} - \ell_{3n})L_{0n}] i \sigma^2 + [(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m})] \sigma^1
\end{array} \right) \\
\left( \begin{array}{l}
[(+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n}] \sigma^3 + [(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m})] \mathbf{I}_2 + \\
+[(\ell_{0m} + \ell_{0n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{0n}] i \sigma^2 + [(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m})] \sigma^1
\end{array} \right) \\
\left( \begin{array}{l}
[(+\ell_{1m} - \ell_{1n})L_{0n} + (+\ell_{0n} - \ell_{0m})L_{1n}] \sigma^3 + [(\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m})] \mathbf{I}_2 + \\
+[(+\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n}] i \sigma^2 + [(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m})] \sigma^1
\end{array} \right)
\end{array} \right) \\
c2=2 & \left( \begin{array}{l}
\left( \begin{array}{l}
[(+\ell_{2n} - \ell_{2m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n}] \sigma^3 + [(\ell_{2n}\ell_{1m} - \ell_{1n}\ell_{2m})] \mathbf{I}_2 + \\
+[(+\ell_{3n} - \ell_{3m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n}] i \sigma^2 + [(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m})] \sigma^1
\end{array} \right) \\
[(\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} - \ell_{2m})L_{2n}] \sigma^3 \\
[(+\ell_{3n} - \ell_{3m})L_{2n} + (+\ell_{2n} - \ell_{2m})L_{3n}] \sigma^3 + [(\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m})] \mathbf{I}_2 + \\
+[(+\ell_{1m} - \ell_{1n})L_{0n} + (+\ell_{0m} - \ell_{0n})L_{1n}] i \sigma^2 + [(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m})] \sigma^1 \\
\left( \begin{array}{l}
[(+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n}] \sigma^3 + [(\ell_{2n}\ell_{0m} - \ell_{0n}\ell_{2m})] \mathbf{I}_2 + \\
+[(+\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1m} - \ell_{1n})L_{3n}] i \sigma^2 + [(\ell_{1n}\ell_{3m} - \ell_{3n}\ell_{1m})] \sigma^1
\end{array} \right)
\end{array} \right)
\end{aligned}$$

$$\begin{aligned}
& \left. \begin{aligned}
& \text{c3=2} \left( \begin{aligned}
& \left( [(\ell_{3n} - \ell_{3m})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n}]\sigma^3 + [(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0m} - \ell_{0n})L_{2n}]i\sigma^2 + [(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m})]\sigma^1 \right) \\
& \left( [(\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2n} - \ell_{2m})L_{3n}]\sigma^3 + [(\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{1n} - \ell_{1m})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n}]i\sigma^2 + [(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m})]\sigma^1 \right) \\
& [(\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} - \ell_{0m})L_{0n} + (\ell_{3n} - \ell_{3m})L_{3n}]\sigma^3 \\
& \left( [(\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n}]\sigma^3 + [(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{2n}]i\sigma^2 + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})]\sigma^1 \right)
\end{aligned} \right) \\
& \text{c4=2} \left( \begin{aligned}
& \left( [(\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{1n}]\sigma^3 + [(\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{3n} - \ell_{3m})L_{2n} + (\ell_{2m} - \ell_{2n})L_{3n}]i\sigma^2 + [(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m})]\sigma^1 \right) \\
& \left( [(\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n}]\sigma^3 + [(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n}]i\sigma^2 + [(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m})]\sigma^1 \right) \\
& \left( [(\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{3n}]\sigma^3 + [(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1m} - \ell_{1n})L_{2n}]i\sigma^2 + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m})]\sigma^1 \right) \\
& [(\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2m} - \ell_{2n})L_{2n} + (\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0m} - \ell_{0n})L_{0n}]\sigma^1
\end{aligned} \right) \\
& D_{Bn}\overline{D_{Am}} + \overline{D_{Bm}}D_{An} = 2 \left[ \sum_{\mu=0}^3 L_{\mu n} L_{\mu n} - \sum_{\mu=0}^3 \ell_{\mu n} \ell_{\mu n} \right] \begin{pmatrix} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{0}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{I}_2 \end{pmatrix}
\end{aligned}$$

$$D_{Bn}\overline{D_{Am}} - \overline{D_{Bm}}D_{An} = \begin{pmatrix} \text{c1} & \text{c2} & \text{c3} & \text{c4} \end{pmatrix}$$

where:

$$\begin{aligned}
& \left. \begin{aligned}
& \text{c1=2} \left( \begin{aligned}
& [(\ell_{0m} + \ell_{0n})L_{0n} + (\ell_{3m} + \ell_{3n})L_{3n} + (\ell_{2m} + \ell_{2n})L_{2n} + (\ell_{1m} + \ell_{1n})L_{1n}] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
& \left( [(\ell_{2m} + \ell_{2n})L_{1n} + (\ell_{1m} + \ell_{1n})L_{2n}]\sigma^3 + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})]\mathbf{I}_2 + \right. \\
& \left. + [(-\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3m} - \ell_{3n})L_{0n}]i\sigma^2 + [(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m})]\sigma^1 \right) \\
& \left( [(\ell_{3n} + \ell_{3m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}]\sigma^3 + [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{0m} + \ell_{0n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{0n}]i\sigma^2 + [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})]\sigma^1 \right) \\
& \left( [(-\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{1n}]\sigma^3 + [(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m})]\mathbf{I}_2 + \right. \\
& \left. + [(-\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}]i\sigma^2 + [(\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m})]\sigma^1 \right) \\
& \left( [(\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n}]\sigma^3 + [(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n}]i\sigma^2 + [(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m})]\sigma^1 \right)
\end{aligned} \right) \\
& \text{c2=2} \left( \begin{aligned}
& \left( [(\ell_{2n} + \ell_{2m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{2n}]\sigma^3 + [(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{3n} + \ell_{3m})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}]i\sigma^2 + [(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m})]\sigma^1 \right) \\
& [(-\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n}]\sigma^3 \\
& \left( [(\ell_{3n} + \ell_{3m})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}]\sigma^3 + [(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m})]\mathbf{I}_2 + \right. \\
& \left. + [(-\ell_{1m} - \ell_{1n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{1n}]i\sigma^2 + [(\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m})]\sigma^1 \right) \\
& \left( [(-\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{2n}]\sigma^3 + [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{3n} + \ell_{3m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{3n}]i\sigma^2 + [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})]\sigma^1 \right)
\end{aligned} \right) \\
& \text{c3=2} \left( \begin{aligned}
& \left( [(\ell_{3n} + \ell_{3m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}]\sigma^3 + [(-\ell_{3n}\ell_{1m} + \ell_{1m}\ell_{3n})]\mathbf{I}_2 + \right. \\
& \left. + [(-\ell_{2m} - \ell_{2n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{2n}]i\sigma^2 + [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})]\sigma^1 \right) \\
& \left( [(\ell_{3n} + \ell_{3m})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}]\sigma^3 + [(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m})]\mathbf{I}_2 + \right. \\
& \left. + [(\ell_{1n} + \ell_{1m})L_{0n} + (\ell_{0n} + \ell_{0m})L_{1n}]i\sigma^2 + [(\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m})]\sigma^1 \right) \\
& [(-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n}]\sigma^3 \\
& \left( [(-\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}]\sigma^3 + [(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m})]\mathbf{I}_2 + \right. \\
& \left. + [(-\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} + \ell_{1m})L_{2n}]i\sigma^2 + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m})]\sigma^1 \right)
\end{aligned} \right)
\end{aligned}$$

$$c4=2 \left( \begin{array}{l} \left( \begin{array}{l} [(-\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{1n}]\sigma^3 + [(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m})]\mathbf{I}_2 + \\ +[(+\ell_{3n} + \ell_{3m})L_{2n} + (-\ell_{2m} - \ell_{2n})L_{3n}]i\sigma^2 + [(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m})]\sigma^1 \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{2n}]\sigma^3 + [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})]\mathbf{I}_2 + \\ +[(-\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}]i\sigma^2 + [(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m})]\sigma^1 \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}]\sigma^3 + [(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m})]\mathbf{I}_2 + \\ +[(+\ell_{2n} + \ell_{2m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{2n}]i\sigma^2 + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})]\sigma^1 \end{array} \right) \\ [(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n}]\sigma^3 \end{array} \right)$$

*Proof:*

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2$$

↓

$$c1=2 \left( \begin{array}{l} [(\ell_{0m} + \ell_{0n})L_{0n} + (\ell_{3m} + \ell_{3n})L_{3n} + (\ell_{2m} + \ell_{2n})L_{2n} + (\ell_{1m} + \ell_{1n})L_{1n}] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \left( \begin{array}{l} [(+\ell_{2m} + \ell_{2n})L_{1n} + (+\ell_{1m} + \ell_{1n})L_{2n}] + \quad [(-\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3m} - \ell_{3m})L_{0n}] + \\ +[(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})] \quad +[(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m})] \end{array} \right) \\ \left( \begin{array}{l} [(\ell_{0m} + \ell_{0n})L_{3n} + (\ell_{3m} + \ell_{3n})L_{0n}] + \quad [(-\ell_{2m} - \ell_{2n})L_{1n} + (-\ell_{1n} - \ell_{1m})L_{2n}] + \\ +[(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m})] \quad +[(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})] \end{array} \right) \\ \left( \begin{array}{l} [(+\ell_{3n} + \ell_{3m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}] + \quad [(\ell_{0m} + \ell_{0n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{0n}] + \\ +[(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})] \quad +[(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})] \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{2m} - \ell_{2n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{2n}] + \quad [(-\ell_{3m} - \ell_{3n})L_{1n} + (-\ell_{1n} - \ell_{1m})L_{3n}] + \\ +[(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})] \quad +[(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})] \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{1m} - \ell_{1n})L_{0n} + (+\ell_{0n} + \ell_{0m})L_{1n}] + \quad [(-\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}] + \\ +[(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m})] \quad +[(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m})] \end{array} \right) \\ \left( \begin{array}{l} [(+\ell_{3n} + \ell_{3m})L_{2n} + (-\ell_{2m} - \ell_{2n})L_{3n}] + \quad [(+\ell_{1n} + \ell_{1m})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{1n}] + \\ +[(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m})] \quad +[(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m})] \end{array} \right) \end{array} \right)$$

$$= 2 \left( \begin{array}{l} [(\ell_{0m} + \ell_{0n})L_{0n} + (\ell_{3m} + \ell_{3n})L_{3n} + (\ell_{2m} + \ell_{2n})L_{2n} + (\ell_{1m} + \ell_{1n})L_{1n}] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ \left( \begin{array}{l} [(+\ell_{2m} + \ell_{2n})L_{1n} + (+\ell_{1m} + \ell_{1n})L_{2n}]\sigma^3 + (-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})\mathbf{I}_2 + \\ +[(-\ell_{0m} - \ell_{0n})L_{3n} + (-\ell_{3m} - \ell_{3m})L_{0n}]i\sigma^2 + [(\ell_{3n}\ell_{0m} - \ell_{0n}\ell_{3m})]\sigma^1 \end{array} \right) \\ \left( \begin{array}{l} [(+\ell_{3n} + \ell_{3m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}]\sigma^3 + [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})]\mathbf{I}_2 + \\ +[(\ell_{0m} + \ell_{0n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{0n}]i\sigma^2 + [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})]\sigma^1 \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{1m} - \ell_{1n})L_{0n} + (+\ell_{0n} + \ell_{0m})L_{1n}]\sigma^3 + [(-\ell_{1n}\ell_{0m} + \ell_{0n}\ell_{1m})]\mathbf{I}_2 + \\ +[(-\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}]i\sigma^2 + [(+\ell_{2n}\ell_{3m} - \ell_{3n}\ell_{2m})]\sigma^1 \end{array} \right) \\ \left( \begin{array}{l} [(+\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} - \ell_{0m})L_{2n}]\sigma^3 + [(\ell_{0n}\ell_{2m} - \ell_{2n}\ell_{0m})]\mathbf{I}_2 + \\ +[(+\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} - \ell_{1m})L_{3n}]i\sigma^2 + [(\ell_{3n}\ell_{1m} - \ell_{1n}\ell_{3m})]\sigma^1 \end{array} \right) \end{array} \right)$$

$$\begin{aligned}
& \left( \begin{array}{cc}
\left( \begin{array}{cc}
[(+\ell_{2n} + \ell_{2m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{2n}] + & [(+\ell_{3n} + \ell_{3m})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}] + \\
+ [(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m})] & + [(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m})] \\
[(-\ell_{3m} - \ell_{3n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{3n}] + & [(-\ell_{2m} - \ell_{2n})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{2n}] + \\
+ [(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m})] & + [(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m}) \pm (-\ell_{2m}\ell_{1n} + \ell_{1m}\ell_{2n})]
\end{array} \right) \\
\left[ (-\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\left( \begin{array}{cc}
[(+\ell_{3n} + \ell_{3m})L_{2n} + (+\ell_{2n} + \ell_{2m})L_{3n}] + & [(-\ell_{1m} - \ell_{1n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{1n}] + \\
+ [(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m})] & + [(\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m})] \\
[(+\ell_{1n} + \ell_{1m})L_{0n} + (+\ell_{0n} + \ell_{0m})L_{1n}] + & [(-\ell_{3m} - \ell_{3n})L_{2n} + (-\ell_{2m} - \ell_{2n})L_{3n}] + \\
+ [(\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m})] & + [(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m})]
\end{array} \right) \\
\left( \begin{array}{cc}
[(-\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{2n}] + & [(+\ell_{3n} + \ell_{3m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{3n}] + \\
+ [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})] & + [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})] \\
[(-\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}] + & [(+\ell_{2n} + \ell_{2m})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{2n}] + \\
+ [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})] & + [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})]
\end{array} \right)
\end{array} \right) \\
= 2 & \left( \begin{array}{c}
\left( \begin{array}{c}
[(+\ell_{2n} + \ell_{2m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{2n}]\sigma^3 + [(-\ell_{2n}\ell_{1m} + \ell_{1n}\ell_{2m})]\mathbf{I}_2 + \\
+ [(\ell_{0n} + \ell_{0m})L_{3n}]i\sigma^2 + [(\ell_{0n}\ell_{3m} - \ell_{3n}\ell_{0m})]\sigma^1
\end{array} \right) \\
\left[ (-\ell_{3m} - \ell_{3n})L_{3n} + (\ell_{0n} + \ell_{0m})L_{0n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{2n} + \ell_{2m})L_{2n} \right] \sigma^3 \\
\left( \begin{array}{c}
[(+\ell_{3n} + \ell_{3m})L_{2n} + (+\ell_{2n} + \ell_{2m})L_{3n}]\sigma^3 + [(-\ell_{2n}\ell_{3m} + \ell_{3n}\ell_{2m})]\mathbf{I}_2 + \\
+ [(-\ell_{1m} - \ell_{1n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{1n}]i\sigma^2 + [(\ell_{1n}\ell_{0m} - \ell_{0n}\ell_{1m})]\sigma^1
\end{array} \right) \\
\left( \begin{array}{c}
[(-\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{2n}]\sigma^3 + [(-\ell_{2n}\ell_{0m} + \ell_{0n}\ell_{2m})]\mathbf{I}_2 + \\
+ [(\ell_{0n} + \ell_{0m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{3n}]i\sigma^2 + [(-\ell_{1n}\ell_{3m} + \ell_{3n}\ell_{1m})]\sigma^1
\end{array} \right)
\end{array} \right) \\
= 2 & \left( \begin{array}{cc}
\left( \begin{array}{cc}
[(+\ell_{3n} + \ell_{3m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}] + & [(-\ell_{2m} - \ell_{2n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{2n}] + \\
+ [(-\ell_{3n}\ell_{1m} + \ell_{1m}\ell_{3n})] & + [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})] \\
[(+\ell_{2n} + \ell_{2m})L_{0n} + (\ell_{0n} + \ell_{0m})L_{2n}] + & [(-\ell_{3m} - \ell_{3n})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{3n}] + \\
+ [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})] & + [(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m})]
\end{array} \right) \\
\left( \begin{array}{cc}
[(+\ell_{3n} + \ell_{3m})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}] + & [(+\ell_{1n} + \ell_{1m})L_{0n} + (\ell_{0n} + \ell_{0m})L_{1n}] + \\
+ [(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m})] & + [(\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m})] \\
[(-\ell_{1m} - \ell_{1n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{1n}] + & [(-\ell_{2m} - \ell_{3n})L_{2n} + (-\ell_{3m} - \ell_{2n})L_{3n}] + \\
+ [(\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m})] & + [(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m})]
\end{array} \right) \\
\left[ (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} \right] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\left( \begin{array}{cc}
[(-\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}] + & [(-\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} + \ell_{1m})L_{2n}] + \\
+ [(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m})] & + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m})] \\
[(\ell_{2n} + \ell_{2m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{2n}] + & [(\ell_{3n} + \ell_{3m})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{3n}] + \\
+ [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m})] & + [(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m})]
\end{array} \right) \\
\left( \begin{array}{c}
[(+\ell_{3n} + \ell_{3m})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}]\sigma^3 + [(-\ell_{3n}\ell_{1m} + \ell_{1m}\ell_{3n})]\mathbf{I}_2 + \\
+ [(-\ell_{2m} - \ell_{2n})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{2n}]i\sigma^2 + [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})]\sigma^1
\end{array} \right) \\
\left( \begin{array}{c}
[(+\ell_{3n} + \ell_{3m})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}]\sigma^3 + [(-\ell_{3n}\ell_{2m} + \ell_{2n}\ell_{3m})]\mathbf{I}_2 + \\
+ [(\ell_{1n} + \ell_{1m})L_{0n} + (\ell_{0n} + \ell_{0m})L_{1n}]i\sigma^2 + [(\ell_{0n}\ell_{1m} - \ell_{1n}\ell_{0m})]\sigma^1
\end{array} \right) \\
\left[ (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{1m} - \ell_{1n})L_{1n} + (\ell_{0n} + \ell_{0m})L_{0n} + (\ell_{3n} + \ell_{3m})L_{3n} \right] \sigma^3 \\
\left( \begin{array}{c}
[(-\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}]\sigma^3 + [(-\ell_{3n}\ell_{0m} + \ell_{0n}\ell_{3m})]\mathbf{I}_2 + \\
+ [(-\ell_{2m} - \ell_{2n})L_{1n} + (\ell_{1n} + \ell_{1m})L_{2n}]i\sigma^2 + [(\ell_{1n}\ell_{2m} - \ell_{2n}\ell_{1m})]\sigma^1
\end{array} \right)
\end{array} \right)
\end{aligned}$$



$$\begin{aligned}
& \left( \begin{array}{l} \left( \begin{array}{l} [(-\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{1n}] + \\ + [(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m})] \end{array} \right) \quad \left( \begin{array}{l} [(+\ell_{3n} + \ell_{3m})L_{2n} + (-\ell_{2m} - \ell_{2n})L_{3n}] + \\ + [(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m})] \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{3m} - \ell_{3n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{3n}] + \\ + [(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m})] \end{array} \right) \quad \left( \begin{array}{l} [(+\ell_{1n} + \ell_{1m})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{1n}] + \\ + [(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m})] \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{2n}] + \\ + [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})] \end{array} \right) \quad \left( \begin{array}{l} [(-\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}] + \\ + [(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m})] \end{array} \right) \\ \left( \begin{array}{l} [(+\ell_{3n} + \ell_{3m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{3n}] + \\ + [(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m})] \end{array} \right) \quad \left( \begin{array}{l} [(-\ell_{0m} - \ell_{0n})L_{2n} + (\ell_{2n} + \ell_{2m})L_{0n}] + \\ + [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})] \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}] + \\ + [(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m})] \end{array} \right) \quad \left( \begin{array}{l} [(+\ell_{2n} + \ell_{2m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{2n}] + \\ + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})] \end{array} \right) \\ \left( \begin{array}{l} [(+\ell_{1n} + \ell_{1m})L_{2n} + (-\ell_{2m} - \ell_{2n})L_{1n}] + \\ + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})] \end{array} \right) \quad \left( \begin{array}{l} [(+\ell_{3n} + \ell_{3m})L_{0n} + (-\ell_{0m} - \ell_{0n})L_{3n}] + \\ + [(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m})] \end{array} \right) \end{array} \right) \\
& [(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n}] \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
& = 2 \left( \begin{array}{l} \left( \begin{array}{l} [(-\ell_{1m} - \ell_{1n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{1n}]\sigma^3 + [(-\ell_{0n}\ell_{1m} + \ell_{1n}\ell_{0m})]\mathbf{I}_2 + \\ + [(+\ell_{3n} + \ell_{3m})L_{2n} + (-\ell_{2m} - \ell_{2n})L_{3n}]i\sigma^2 + [(+\ell_{3n}\ell_{2m} - \ell_{2n}\ell_{3m})]\sigma^1 \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{2m} - \ell_{2n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{2n}]\sigma^3 + [(-\ell_{0n}\ell_{2m} + \ell_{2n}\ell_{0m})]\mathbf{I}_2 + \\ + [(-\ell_{3m} - \ell_{3n})L_{1n} + (\ell_{1n} + \ell_{1m})L_{3n}]i\sigma^2 + [(-\ell_{3n}\ell_{1m} + \ell_{1n}\ell_{3m})]\sigma^1 \end{array} \right) \\ \left( \begin{array}{l} [(-\ell_{3m} - \ell_{3n})L_{0n} + (\ell_{0n} + \ell_{0m})L_{3n}]\sigma^3 + [(-\ell_{0n}\ell_{3m} + \ell_{3n}\ell_{0m})]\mathbf{I}_2 + \\ + [(+\ell_{2n} + \ell_{2m})L_{1n} + (-\ell_{1m} - \ell_{1n})L_{2n}]i\sigma^2 + [(-\ell_{1n}\ell_{2m} + \ell_{2n}\ell_{1m})]\sigma^1 \end{array} \right) \\ [(-\ell_{1m} - \ell_{1n})L_{1n} + (-\ell_{2m} - \ell_{2n})L_{2n} + (-\ell_{3m} - \ell_{3n})L_{3n} + (-\ell_{0m} - \ell_{0n})L_{0n}]\sigma^3 \end{array} \right)
\end{aligned}$$

□

**Corollary III.3:** For linear/differential operators  $D_{Bn}$  &  $D_{An}$  :

where:

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

then:

$$D_{Bn}\overline{D_{An}} + \overline{D_{Bn}}D_{An} = D_{Bn}D_{An} + D_{Bm}D_{An} = 2 \left[ \sum_{\mu=0}^3 L_{\mu n} L_{\mu n} - \sum_{\mu=0}^3 \ell_{\mu n} \ell_{\mu n} \right] \mathbf{I}_8$$

*Proof:*

Immediate from theorems and corollaries I-III

□

**Corollary III.4:** For linear/differential operators  $D_{Bn}$  &  $D_{An}$  :

where:

$L_{jm} = L_{jn}$  are linear/differential &  $\ell_{ij}$  are constants:

then:

$$D_{Bn}D_{An} + D_{Bn}\overline{D_{An}} + \overline{D_{Bn}}D_{An} + D_{Bm}D_{Am} = 4 \left[ \sum_{\mu=0}^3 L_{\mu n} L_{\mu n} - \sum_{\mu=0}^3 \ell_{\mu n} \ell_{\mu n} \right] \mathbf{I}_8$$

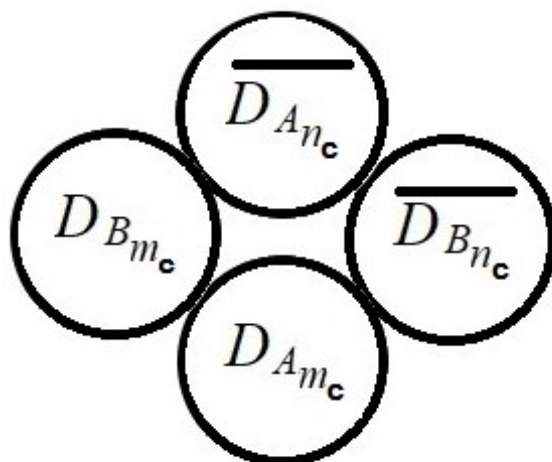
$$D_{Bn}D_{An} + D_{Bn}D_{Am} + D_{Bm}D_{An} + D_{Bm}D_{Am} = 4 \left[ \sum_{\mu=0}^3 L_{\mu n} L_{\mu n} - \sum_{\mu=0}^3 \ell_{\mu n} \ell_{\mu n} \right] \mathbf{I}_8$$

*Proof:*

Immediate from theorems and corollaries I-III

□

depicted as follows:



**Corollary III.5:** For linear/differential operators  $D_{Bn}, D_{An}, D_{Bm}, D_{Am}, D_{Bp}, D_{Ap}$  :

where:

$L_{jm} = L_{jn}$  &  $L_{jp} = L_{jm}$  are linear/differential &  $\ell_{ij}$  are constants:

then:

$$D_{Bn}D_{Am} + D_{Bm}D_{An} = 2 \left[ \sum_{\mu=0}^3 L_{\mu m} L_{\mu m} - \sum_{\mu=0}^3 \ell_{\mu n} \ell_{\mu m} \right] \mathbf{I}_8$$

$$D_{Bp}D_{Am} + D_{Bm}D_{Ap} = 2 \left[ \sum_{\mu=0}^3 L_{\mu m} L_{\mu m} - \sum_{\mu=0}^3 \ell_{\mu p} \ell_{\mu m} \right] \mathbf{I}_8$$

**Proof:**

$$D_{Bn}D_{Am} + D_{Bm}D_{An} = 2 \left[ \sum_{\mu=0}^3 L_{\mu m} L_{\mu m} - \sum_{\mu=0}^3 \ell_{\mu n} \ell_{\mu m} \right] \mathbf{I}_8$$

Is immediate from corollary III.2

and, substituting  $p$  for  $n$  :

$$D_{Bp}D_{Am} + D_{Bm}D_{Ap} = 2 \left[ \sum_{\mu=0}^3 L_{\mu m} L_{\mu m} - \sum_{\mu=0}^3 \ell_{\mu p} \ell_{\mu m} \right] \mathbf{I}_8$$

□

**Corollary III.6:** For commutative-associative linear operators  $D_{Bn}, D_{An}, D_{Bm}, D_{Am}, D_{Bp}, D_{Ap}$  :

where:

$L_{jm} = L_{jn}$  &  $L_{jp} = L_{jm}$  are constants or differential operators &  $\ell_{ij}$  are constants:

then:

$$D_{An_1}D_{Bn_1} + D_{Bn_1}D_{Am_2} + D_{Am_2}D_{Bm_2} + D_{Bm_2}D_{Ap_3} + D_{Ap_3}D_{Bp_3} + D_{Bp_3}D_{An_1} + D_{Ap_3}D_{Bn_1} =$$

$$= \sum_{\mu=0}^3 [(L_{\mu n_1} L_{\mu n_1} + L_{\mu m_2} L_{\mu m_2} + L_{\mu p_3} L_{\mu p_3} +$$

$$+ L_{\mu m_2} L_{\mu m_2} + L_{\mu p_3} L_{\mu p_3} + 2L_{\mu n_1} L_{\mu n_1}) +$$

$$- (\ell_{\mu n_1} \ell_{\mu n_1} + \ell_{\mu m_2} \ell_{\mu m_2} + \ell_{\mu p_3} \ell_{\mu p_3} +$$

$$+ \ell_{\mu n_1} \ell_{\mu m_2} + \ell_{\mu m_2} \ell_{\mu p_3} + 2\ell_{\mu p_3} \ell_{\mu n_1})] \mathbf{I}_8$$

$$= \sum_{\mu=0}^3 [(3L_{\mu n_1} L_{\mu n_1} + 2L_{\mu m_2} L_{\mu m_2} + 2L_{\mu p_3} L_{\mu p_3}) +$$

$$- (\ell_{\mu n_1} (\ell_{\mu n_1} + \ell_{\mu m_2}) + \ell_{\mu m_2} (\ell_{\mu p_3} + \ell_{\mu m_2}) + \ell_{\mu p_3} (\ell_{\mu p_3} + 2\ell_{\mu n_1}))] \mathbf{I}_8$$

**Proof:**

$$D_{An_1}D_{Bn_1} + D_{Bn_1}D_{Am_2} + D_{Am_2}D_{Bm_2} + D_{Bm_2}D_{Ap_3} + D_{Ap_3}D_{Bp_3} + D_{Bp_3}D_{An_1} + D_{Ap_3}D_{Bn_1} =$$

$$= D_{An_1}D_{Bn_1} + D_{Am_2}D_{Bm_2} + D_{Ap_3}D_{Bp_3} + D_{Bn_1}D_{Am_2} + D_{Bm_2}D_{Ap_3} + D_{Bp_3}D_{An_1} + D_{Ap_3}D_{Bn_1} =$$

$$= \left[ \sum_{\mu=0}^3 L_{\mu n_1} L_{\mu n_1} - \sum_{\mu=0}^3 \ell_{\mu n_1} \ell_{\mu n_1} \right] \mathbf{I}_8 + \left[ \sum_{\mu=0}^3 L_{\mu m_2} L_{\mu m_2} - \sum_{\mu=0}^3 \ell_{\mu m_2} \ell_{\mu m_2} \right] \mathbf{I}_8 + \left[ \sum_{\mu=0}^3 L_{\mu p_3} L_{\mu p_3} - \sum_{\mu=0}^3 \ell_{\mu p_3} \ell_{\mu p_3} \right] \mathbf{I}_8 +$$

$$+ \left[ \sum_{\mu=0}^3 L_{\mu m_2} L_{\mu m_2} - \sum_{\mu=0}^3 \ell_{\mu n_1} \ell_{\mu m_2} \right] \mathbf{I}_8 + \left[ \sum_{\mu=0}^3 L_{\mu p_3} L_{\mu p_3} - \sum_{\mu=0}^3 \ell_{\mu m_2} \ell_{\mu p_3} \right] \mathbf{I}_8 + 2 \left[ \sum_{\mu=0}^3 L_{\mu n_1} L_{\mu n_1} - \sum_{\mu=0}^3 \ell_{\mu p_3} \ell_{\mu n_1} \right] \mathbf{I}_8$$

$$= \sum_{\mu=0}^3 [(L_{\mu n_1} L_{\mu n_1} + L_{\mu m_2} L_{\mu m_2} + L_{\mu p_3} L_{\mu p_3} +$$

$$+ L_{\mu m_2} L_{\mu m_2} + L_{\mu p_3} L_{\mu p_3} + 2L_{\mu n_1} L_{\mu n_1}) +$$

$$- (\ell_{\mu n_1} \ell_{\mu n_1} + \ell_{\mu m_2} \ell_{\mu m_2} + \ell_{\mu p_3} \ell_{\mu p_3} +$$

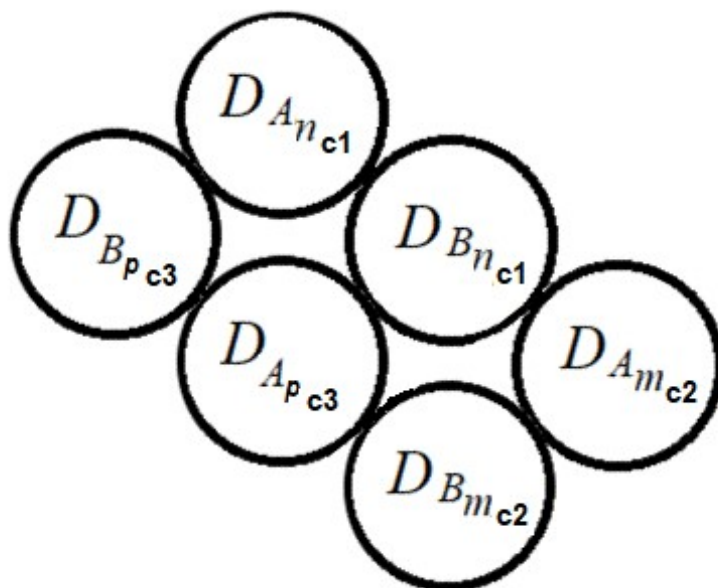
$$+ \ell_{\mu n_1} \ell_{\mu m_2} + \ell_{\mu m_2} \ell_{\mu p_3} + 2\ell_{\mu p_3} \ell_{\mu n_1})] \mathbf{I}_8$$

$$= \sum_{\mu=0}^3 [(3L_{\mu n_1} L_{\mu n_1} + 2L_{\mu m_2} L_{\mu m_2} + 2L_{\mu p_3} L_{\mu p_3}) +$$

$$- (\ell_{\mu n_1} (\ell_{\mu n_1} + \ell_{\mu m_2}) + \ell_{\mu m_2} (\ell_{\mu p_3} + \ell_{\mu m_2}) + \ell_{\mu p_3} (\ell_{\mu p_3} + 2\ell_{\mu n_1}))] \mathbf{I}_8$$

□

depicted as follows:



Now:

From corollaries III.3,4 & 6 and the empirical evidence indicate the actual Yukawa color force between quarks:

Just as the electromanetic force is given by:

$$F_e = \lambda_e \frac{e_1 e_2}{r^2}$$

and the gravitational force is given by: (at least to first approximation):

$$F_g = \lambda_g \frac{m_1 m_2}{r^2}$$

The color force between quarks may be given by:

$$F_q = \lambda_q \frac{q_1 \circ q_2}{r^2} e^{\mu r}$$

as follows:

Let:  $q_c \equiv \sigma_c$ , where:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$q_R \equiv k_R \sigma^1 \quad q_G \equiv k_G \sigma^2 \quad q_B \equiv k_B \sigma^3 \quad k_R, k_G, k_B \in \mathbb{R}$$

$$\det(\sigma^j) = 1 \quad \det(\sigma^0) = \det(\mathbf{I}_2) = 1$$

$$A \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow kA = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

$$\det A = ad - bc \Rightarrow \det(kA) = kakd - kbkc = k^2(ad - bc) = k^2 \det A$$

↓

$\sigma^1 \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2$
$\sigma^1 \sigma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i \sigma^3$
$\sigma^1 \sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -i \sigma^2$
$\sigma^2 \sigma^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -i \sigma^3$
$\sigma^2 \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2$
$\sigma^2 \sigma^3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i \sigma^1$
$\sigma^3 \sigma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i \sigma^2$
$\sigma^3 \sigma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -i \sigma^1$
$\sigma^3 \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}_2$

and, define:  $q_{C_1} \circ q_{C_2} = \det(q_{C_1} q_{C_2})$

↓

$q_R \circ q_R = \det(q_R q_R) = \det(k_R \sigma^1 k_R \sigma^1) = \det(k_R k_R \sigma^1 \sigma^1) = k_R^2 \det(\mathbf{I}_2) = k_R^2 > 0$
$q_R \circ q_G = \det(q_R q_G) = \det(k_R \sigma^1 k_G \sigma^2) = \det(k_R k_G \sigma^1 \sigma^2) =$ $= \det(k_R k_G [i \sigma^3]) = (i k_R k_G)^2 \det(\sigma^3) = -(k_R k_G)^2 < 0$
$q_R \circ q_B = \det(q_R q_B) = \det(k_R \sigma^1 k_B \sigma^3) = \det(k_R k_B \sigma^1 \sigma^3) =$ $= \det(k_R k_B [-i \sigma^2]) = (-i k_R k_B)^2 \det(\sigma^2) = -(k_R k_B)^2 < 0$
$q_G \circ q_R = \det(q_G q_R) = \det(k_G \sigma^2 k_R \sigma^1) = \det(k_G k_R \sigma^2 \sigma^1) =$ $= \det(k_G k_R [-i \sigma^3]) = (-i k_G k_R)^2 \det(\sigma^3) = -(k_G k_R)^2 < 0$
$q_G \circ q_G = \det(q_G q_G) = \det(k_G \sigma^2 k_G \sigma^2) = \det(k_G k_G \sigma^2 \sigma^2) = \det(k_G^2 \mathbf{I}_2) = k_G^2 > 0$
$q_G \circ q_B = \det(q_G q_B) = \det(k_G \sigma^2 k_B \sigma^3) = \det(k_G k_B \sigma^2 \sigma^3) =$ $= \det(k_G k_B [i \sigma^1]) = (i k_G k_B)^2 \det(\sigma^1) = -(k_G k_B)^2 < 0$
$q_B \circ q_R = \det(q_B q_R) = \det(k_B \sigma^3 k_R \sigma^1) = \det(k_B k_R \sigma^3 \sigma^1) =$ $= \det(k_B k_R [i \sigma^2]) = (i k_B k_R)^2 \det(\sigma^2) = -(k_B k_R)^2 < 0$
$q_B \circ q_G = \det(q_B q_G) = \det(k_B \sigma^3 k_G \sigma^2) = \det(k_B k_G \sigma^3 \sigma^2) =$ $= \det(k_B k_G [-i \sigma^1]) = (-i k_B k_G)^2 \det(\sigma^1) = -(k_B k_G)^2 < 0$
$q_B \circ q_B = \det(q_B q_B) = \det(k_B \sigma^3 k_B \sigma^3) = \det(k_B k_B \sigma^3 \sigma^3) = k_B^2 \det(\mathbf{I}_2) = k_B^2 > 0$

↓

$$q_{C_1} \circ q_{C_2} = \begin{cases} k_{C_1}^2 > 0 & , C_1 = C_2 \\ -(k_{C_1} k_{C_2})^2 < 0 & , C_1 \neq C_2 \end{cases}$$

Thus, differing color quarks attract, alike color quarks repel.

And, define:  $\overline{q_{C_1}} = iq_{C_1} \Rightarrow \overline{q_{C_1}} \circ q_{C_1} = q_{C_1} \circ \overline{q_{C_1}} = \det(iq_{C_1}q_{C_1}) = -k_{C_1}^2 < 0$   
 $\Rightarrow$  quark/anti-quark attraction

vis.:

$q_R \circ \overline{q_R} = \det(q_R \overline{q_R}) = \det(k_R \sigma^1 i k_R \sigma^1) = \det(i k_R k_R \sigma^1 \sigma^1) = (i k_R)^2 \det(\mathbf{I}_2) = -k_R^2 < 0$
$q_R \circ \overline{q_G} = \det(q_R \overline{q_G}) = \det(k_R \sigma^1 i k_G \sigma^2) = \det(i k_R k_G \sigma^1 \sigma^2) =$
$= \det(i k_R k_G [i \sigma^3]) = (i^2 k_R k_G)^2 \det(\sigma^3) = (-k_R k_G)^2 = (k_R k_G)^2 > 0$

In this way, meson & baryon color force attraction/repulsion is manifested with a force field similar to that of the electric and gravitational.

(In fact, since:  $\alpha \in R$  is in a same equivalence class as  $\alpha \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ , is indistinguishable from it.)

Thus, the above forms may equivalently written:

$$F_e = (e_1 \circ e_2) \left( \frac{\lambda_e}{r^2} \right), \quad F_g = (m_1 \circ m_2) \left( \frac{\lambda_g}{r^2} \right), \quad F_q = (q_1 \circ q_2) \left( \lambda_q \frac{e^{i\mu r}}{r^2} \right)$$

( the potentials being solutions of  $(\square - \lambda_\mu)\phi_\mu = J_\mu$  : (using:  $L_j = \partial_j$ )

( the first two being of the d'Alembert, and the third being of the

( Klein-Gordon/ Helmholtzian operator, respectively.

( actually all the same, understanding that the first two have zero constant term;

( and that the space-time of the first and third are flat euclidean, the gravitational

( is curvature of Swarzschild/Eddington/Kerr-Newman/Kerr-Schild/Gibbons-Hawking

( metric and coordinates, as appropriate

$$\Rightarrow F_\xi = (\rho_1 \circ \rho_2) \lambda_\xi \phi_\xi$$

where:

$$(\xi, \rho_{\xi j}, \lambda_\xi, \phi_\xi, J_\xi) \in \left\{ \left( e, e_j, \lambda_e, \frac{1}{r^2}, J_e \right), \left( g, m_j, \lambda_g, \frac{1}{r^2}, J_g \right), \left( q, q_j, \lambda_q, \frac{e^{i\mu r}}{r^2}, J_q \right) \right\}$$

NOTE:

( Using the Klein-Gordon/ Helmholtzian with space-time coorinates with metric and

( curvature as appropriate or using the Covariant Helmholtzian rather than:

(  $G^{\alpha\gamma} = R^{\alpha\gamma} - \frac{1}{2} g^{\alpha\gamma} R = -\frac{8\pi\kappa}{c^2} T^{\alpha\gamma}$ , in effect separates the field from the space curvature -

( even with the same result.

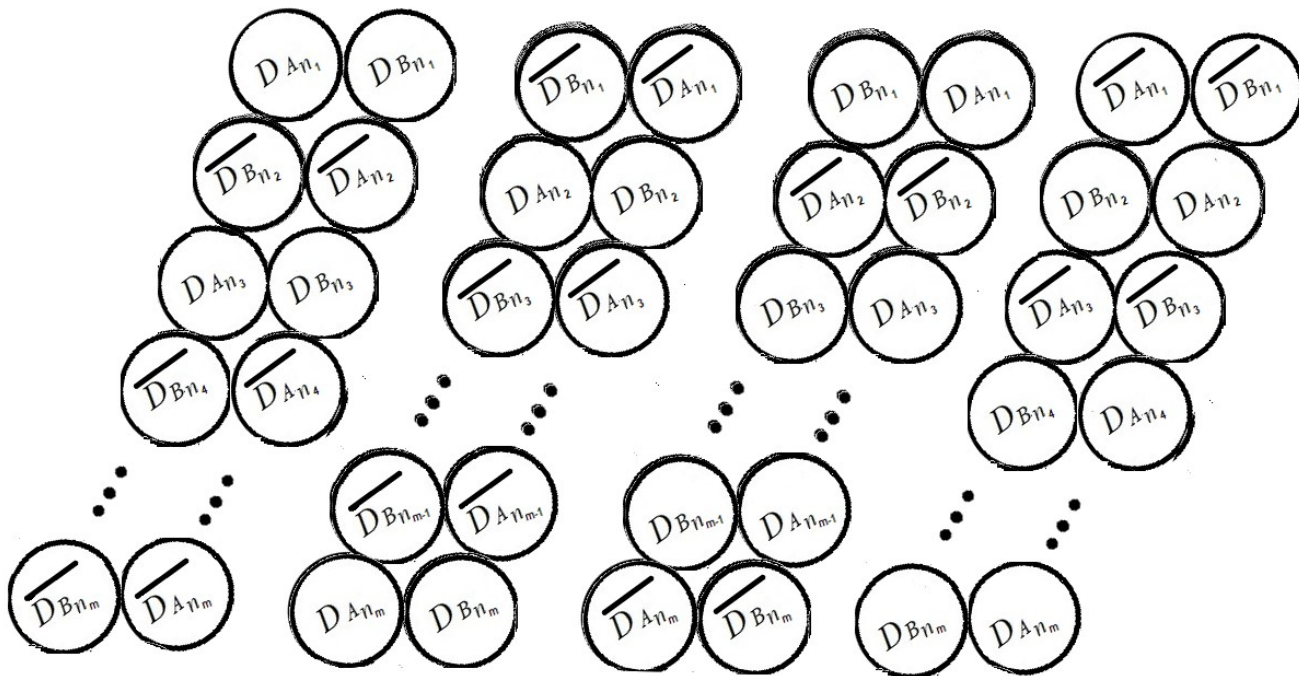
( So, the field is expressed using fortuitive coordinates as is often done

( (such as using polar coordinates in analysis of a pendulum simple harmonic oscillator)

( Thus, notions on wormholes and such become fictional speculations.

But, now for the reason the Helmholtzian operator factorization was generalized to to Multiple Associative Commutative Linear Operator Factor(s). (MACLOFs)

First, the above figures for meson factors may be chained as follows:



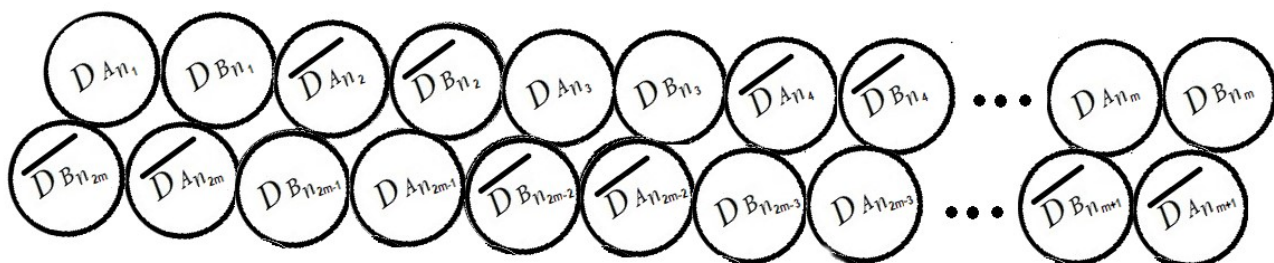
Note:

$$D_{A_1} D_{B_1} \overline{D_{A_2}} \overline{D_{B_2}} D_{A_3} D_{B_3} \overline{D_{A_4}} \overline{D_{B_4}} \cdots D_{A_{m-1}} D_{B_{m-1}} \overline{D_{A_m}} \overline{D_{B_m}}$$

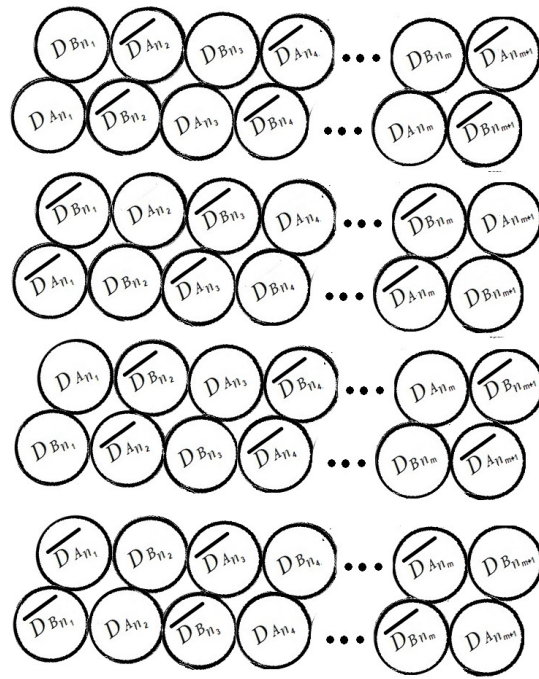
$\Downarrow$

$$D_{A_1} D_{B_1} + D_{B_1} \overline{D_{A_2}} + \overline{D_{A_2}} \overline{D_{B_2}} + \overline{D_{B_2}} D_{A_3} + D_{A_3} D_{B_3} + D_{B_3} \overline{D_{A_4}} + \overline{D_{A_4}} \overline{D_{B_4}} + \cdots + D_{B_{m-1}} \overline{D_{A_m}} + \overline{D_{A_m}} \overline{D_{B_m}}$$

Similarly:

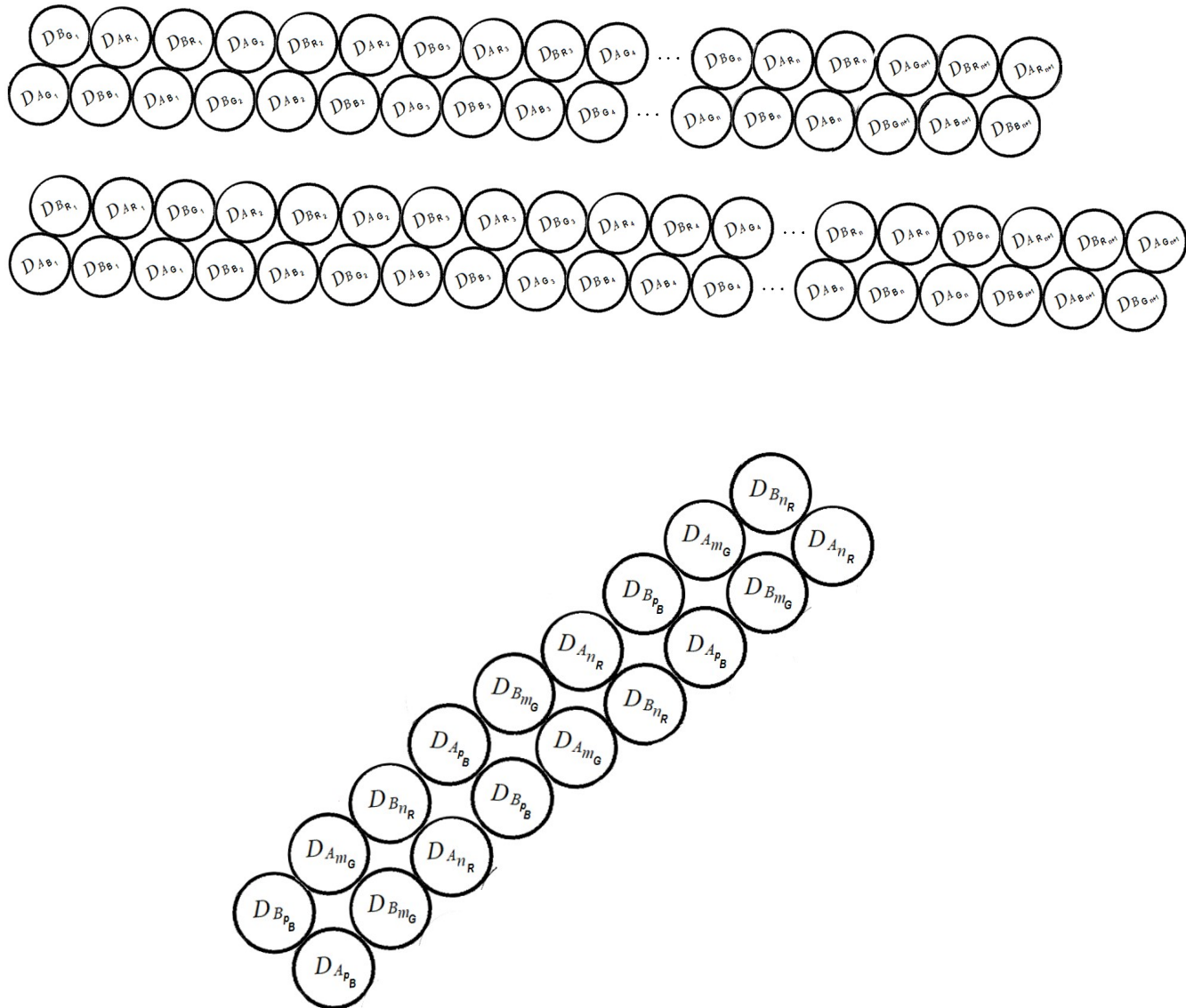


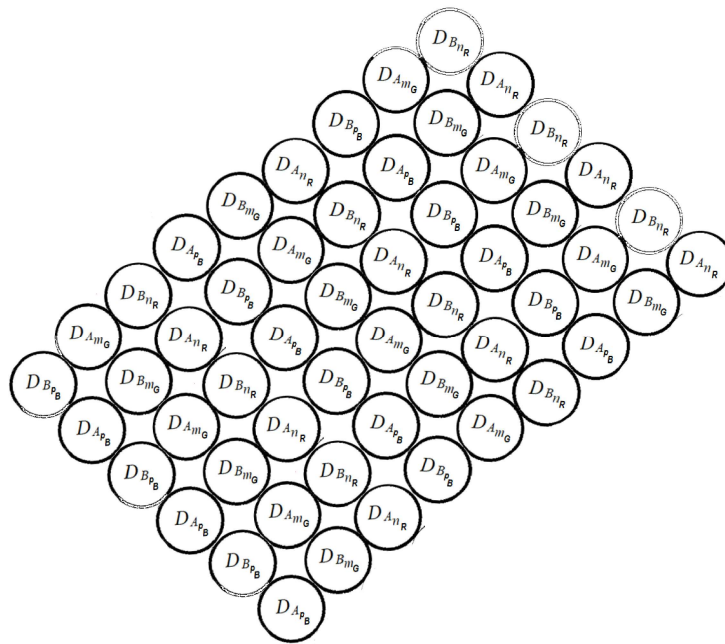
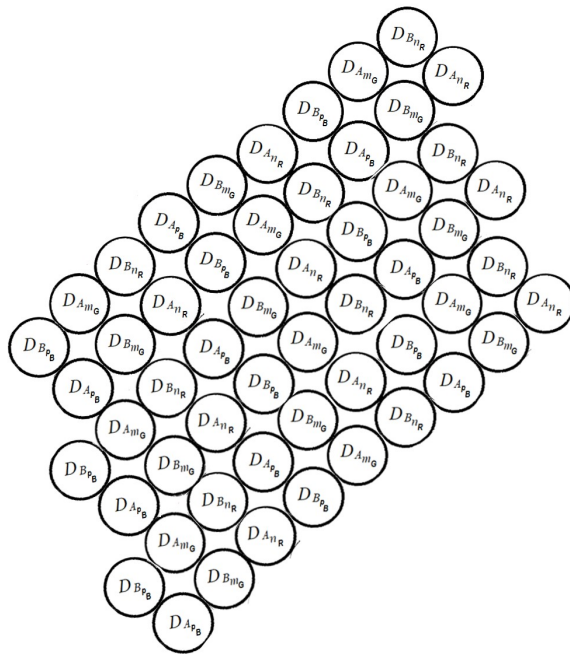
likewise:



So, note that the mesons may link end-to-end of infinite length, and that they may link side-to-side of infinite width. Further, they may link above/below each other infinitely, thus forming chains, rings, sheets, meshes, bricks and blocks. Even though mesons have rather short lifespans, that is relative and increased by the already observed naturally occurring high-speed time dilation effect. Meson speeds may be even faster and in higher density in high-energy plasmas in regions such as star formation knots.

Next, the above figures for baryon factors may be linked as follows:

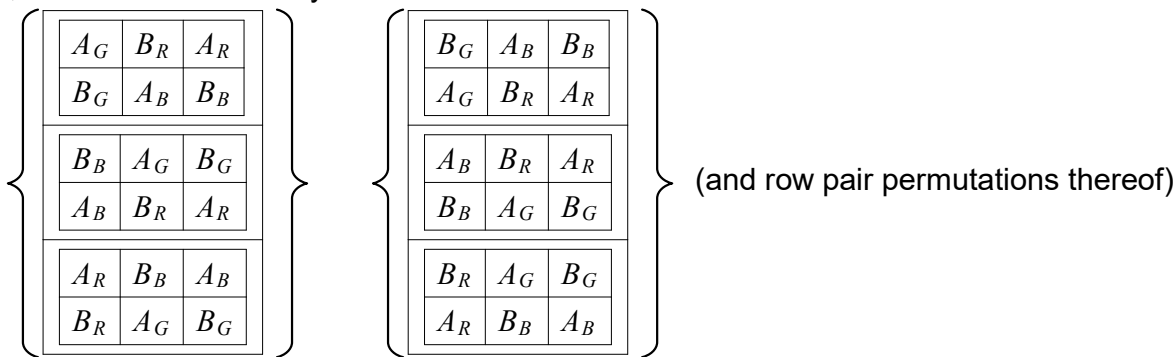




These graphic depictions of baryon chains and sheets may be sketched schematically, dropping the  $D$  :  
in table form:

$A_G$	$B_R$	$A_R$	$A_G$	$B_B$	$A_B$	$B_G$	$A_B$	$B_B$	$B_G$	$A_R$	$B_R$
$B_G$	$A_B$	$B_B$	$B_G$	$A_R$	$B_R$	$A_G$	$B_R$	$A_R$	$A_G$	$B_B$	$A_B$
$A_R$	$B_G$	$A_G$	$A_R$	$B_B$	$A_B$	$B_R$	$A_B$	$B_B$	$B_R$	$A_G$	$B_G$
$B_R$	$A_B$	$B_B$	$B_R$	$A_G$	$B_G$	$A_R$	$B_G$	$A_G$	$A_R$	$B_B$	$A_B$
$A_B$	$B_G$	$A_G$	$A_B$	$B_R$	$A_R$	$B_B$	$A_R$	$B_R$	$B_B$	$A_G$	$B_G$
$B_B$	$A_R$	$B_R$	$B_B$	$A_G$	$B_G$	$A_B$	$B_G$	$A_G$	$A_B$	$B_R$	$A_R$

thus, the above chains may be stacked such as:



and chained accordingly, to any width and to any depth.

Similarly, these stacks may extend side-to-side, as well.

Thus, baryons form into chains, rings, sheets, meshes, and clumped into bricks, blocks, etc. (nucleon clumps are known as atomic nuclei)

Note that the baryons are made up of triple  $B_{C_1}A_{C_2}B_{C_3}/A_{C_1}B_{C_2}A_{C_3}$  and mesons are made up of pair  $B_{C_1}A_{C_2}/A_{C_1}B_{C_2}$  which indicates that  $B_C/A_C$  or  $B_C\phi/A_C\phi$  represent a quark, so a matrix pair must represent a  $B_C$  or  $A_C$ . And, as noted previously, a matrix pair may even represent mesons.

Thus, the idea of chaining Helmholtzian operator factorizations arose.

And so, envisioning chaining Helmholtzian operator factorizations leads to envisioning Multiple Associative Commutative Linear Operator Factor(s) (MACLOFs); using the operator commutivity where the binding force may be generalized beyond the Yukawa force, to effect chaining between objects of a general nature.

Thus, just as the Yukawa force is the binder for the elementary particles using the Helmholtzian partial derivatives/constants operators; the operators for other chainable objects include:

ionic bond Coulomb force:

$$F_C = -k \frac{q_1 q_2}{r^2}$$

$q_1, q_2$  : the ion charges

$k$ :  $8.98 \times 10^9 Nm^2/C^2$

$r$ : the distance between the ions

Van der Waals forces:

$$F_{vw} = -\frac{1}{6} \left( \frac{R_1 R_2}{R_1 + R_2} \right) \frac{A}{r^2}$$

$R_1, R_2$  : spherical bodies radii

$A$ : Hamaker constant

$r$ : the distance between the surfaces

Using this MACLOF methodology, to chemical compound chaining - amino acid chains, proteins, nucleic acids - RNA & DNA - and beyond - beyond chemistry as well - beyond even science ; opening a grand new vista in understanding!