On Topological Defects and High-Energy Physics

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Abstract

The Kibble-Zurek mechanism (KZM) describes the universal formation of *topological defects* in systems undergoing continuous phase transitions. KZM is traditionally applied to the study of defects in the early Universe and condensed matter phenomena. The goal of this brief report is to uncover the remarkable analogy between KZM and the flavor composition of particle physics. Our findings suggest that defect formation in particle physics and cosmology is rooted in the *multifractal topology* of the early Universe.

Key words: Kibble-Zurek mechanism, topological defects, Standard Model of particle physics, multifractals, cosmology, galaxy clusters.

According to [1], there is a tentative path leading from Dimensional Regularization of Quantum Field Theory to *fractal spacetime*, on the one hand, and to *fractional dynamics*, on the other. The connection between the correlation length of critical phenomena (ξ) and the continuous dimensional deviation from four spacetime dimensions ($|\varepsilon = 4 - D| << 1$) is given by,

$$\xi(|\varepsilon|) \propto |\varepsilon|^{-1/2} \tag{1}$$

Relation (1) underlies the idea of *criticality in continuous dimensions*, as the divergence of ξ necessarily ties in with a vanishing fractality at our observation scale ($\varepsilon = 0$). In line with [2], an alternative expression of (1) may be presented as,

$$\xi(\varepsilon) = \frac{\xi_0}{|\varepsilon|^{\nu}} \tag{2}$$

Likewise, the relaxation time associated with (2) takes the form,

$$\tau(\varepsilon) = \frac{\tau_0}{|\varepsilon|^{z\nu}} \tag{3}$$

where ν, z denote the critical and dynamic exponents, respectively. The dimensional deviation can be further mapped to a reduced distance parameter λ,

$$\varepsilon = 1 - \frac{\lambda}{\lambda_c} \tag{4}$$

such that $\varepsilon = 0$ when $\lambda = \lambda_c$. Under the linear quench assumption, the parameter λ evolves according to,

$$\lambda(t) = \lambda_c[1 - \varepsilon(t)] \tag{5}$$

in which the time-dependent dimensional deviation is measured relative to the quench time τ_Q as in [2],

$$\varepsilon(t) = \frac{t}{\tau_Q} \tag{6}$$

Fig. 1 shows the schematic diagram of KZM, where $t \in [-\tau_Q, \tau_Q]$ with the critical point being reached at $t = 0$. The dynamics of the KZ phase transition remains adiabatic outside the critical region $[-\hat{t}, \hat{t}]$ and enters the "frozen" stage once $t \in [-\hat{t}, \hat{t}]$. In this stage KZM is no longer adiabatic and $\tau(\varepsilon) \to \infty$. As a result, $\hat{t}$ plays the role of a "freeze out" time and marks the *crossover* boundary between the adiabatic and the critical regime of KZM.

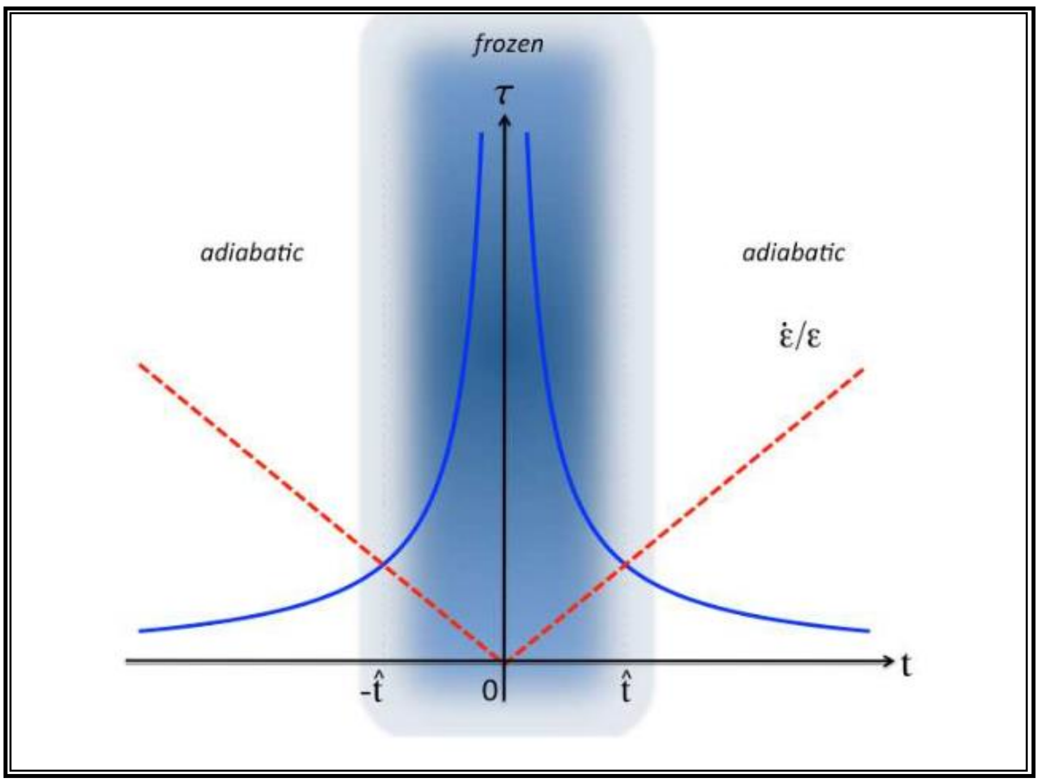

Fig1: Schematic diagram of KZM [2].

By (2)-(6), the density of topological defects in the KZM is,

$$n = \frac{1}{\xi_0^{D-d}} \left(\frac{\tau_0}{\tau_Q}\right)^{\kappa} \tag{7}$$

Here, D and d are the dimensions of the underlying space and of the defects, respectively, and

$$\kappa = (D-d)\frac{\nu}{1+z\nu} \tag{8}$$

The number of defects N follows by multiplying (7) with the *effective space volume* $V = L^{D-d}$, which yields,

$$N = \frac{L^{D-d}}{\xi_0^{D-d}} \left(\frac{\tau_0}{\tau_Q}\right)^{\kappa} = \left(\frac{L}{\xi_0}\right)^{D-d} \left(\frac{\tau_0}{\tau_Q}\right)^{\kappa} \qquad (8)$$

It is convenient to cast (8) in the following form,

$$N \Rightarrow N\left(\frac{a}{r}\right) \tag{9}$$

where,

$$a = \frac{\xi_0}{L} \tag{10}$$

$$r = (\frac{\tau_Q}{\tau_0})^{\kappa} \tag{11}$$

With reference to Appendix B, a one-to-one correspondence may be established between (9) - (11) and the iterative construction of fractal sets. Let us assume for simplicity that,

$$\frac{\nu}{1+z\nu} = 1 \Rightarrow \nu = \frac{1}{1-z} \tag{12}$$

By (B1) - (B4), (9) can be generalized to the case when there are $j = 1, 2, ..., S$ independent species of topological defects, namely,

$$N \Rightarrow N(\frac{a}{r_1}) + N(\frac{a}{r_2}) + ... + N(\frac{a}{r_S}) \tag{13}$$

In this interpretation, topological defects are considered *analogues* of the independent scales acting on the fractal set defined in Appendix B.

Taking $\Delta = D - d$ to represent the fractal dimension of the set, the replica of (B3) – (B4) amounts to

$$(\frac{\tau_Q}{\tau_{0,1}})^{\Delta} + (\frac{\tau_Q}{\tau_{0,2}})^{\Delta} + ...(\frac{\tau_Q}{\tau_{0,S}})^{\Delta} = 1 \tag{14}$$

With reference to Appendix A and B, (14) recovers the "sum-of-squares" relationship of the Standard Model upon recalling that "mass" is the inverse of a time parameter *or* a time scale. Specifically, under the identifications,

$$t_Q \propto v^{-1} \tag{15a}$$

$$\tau_0 \propto m^{-1} \tag{15b}$$

$$\Delta = 2 \tag{15c}$$

(14) turns into

$$\boxed{\sum_j (\frac{m_j}{v})^2 = 1} \tag{16}$$

with v denoting the Fermi scale of electroweak interactions.

We close the report with the following observations, which may turn out to be relevant for future developments of the topic:

1) The choice $\Delta = 2$ matches the fractal dimension of random walks and Brownian motion [11]. It also matches the KZ scenario for the genesis of vortex lines in three dimensional superfluidity ($D = 3$ and $d = 1$). [2, 7].

2) Relation (16), along with (B4) and (B5), matches the geometry of the cosmic web, in particular, the multifractal clustering of galaxy masses [5].

3) It follows from previous observations that the coalescence of topological defects in early Universe cosmology shares commonalities with particle physics, insofar the generation of cosmic structures and the hierarchical composition of the Standard Model are concerned [2 – 4].

4) It is known that the Gross-Pitaevskii equation of superfluidity is a particular embodiment of the Landau-Ginzburg theory of critical phenomena. Along with 1), 4) lends further support to the bifurcation mechanism of structure formation in the Standard Model of particle physics [12 - 13].

5) At least in principle, observations 1) and 4) open the door for an unexplored connection between the spin/chirality of elementary particles and vorticity of filaments in superfluidity.

APPENDIX A: On the "Sum of Squares" Relationship

The "sum-of-square" relationship of the Standard Model links the square of elementary particle masses to the square of the Fermi scale v viz.

$$m_W^2 + m_Z^2 + m_H^2 + \sum_f m_f^2 = v^2 \tag{A1}$$

where W, Z and H stand for the electroweak and the Higgs bosons, respectively, and the sum in the left-hand side is taken over the whole spectrum of SM fermions [10 - 11]. The contribution of bosons and fermions in (A1) is split in nearly equal shares, that is,

$$\sum_b m_b^2 \approx \sum_f m_f^2 \approx \frac{v^2}{2} \tag{A2}$$

(A1) can be cast in a form that highlights its analogy with (16) and (B4), that is,

$$\sum_j (\frac{m_j}{v})^2 = 1 \quad \text{(A3)}$$

APPENDIX B: Construction of a two-scale Cantor set

Fractals are typically created starting from an elementary geometric object (the generator) and allowing its components ($j = 1, 2, ..., S$) to be independently scaled by a factor r_j, where $\sum_j r_j < 1$ [6].

With reference to Fig. 2, consider the simplest case of a Cantor set with two scales, r_1 and r_2. The recursive construction of the Cantor set consists of taking the segment of unit mass length, dividing it into segments of lengths $r_1, 1-(r_1+r_2), r_2$ and removing the middle segment. The division of segments continues indefinitely, generating a scale-reduced replica of the original construction. Aside from a scale factor, the subsets lying in the disjoint intervals $\left[0, r_1\right]$ and $\left[1-r_2, 1\right]$ are images of the whole set. Let the whole set be covered with unit segments (or dimensional boxes) of size a. The number of boxes needed to cover the set is given by,

$$N(a) \propto a^{-\Delta} \tag{B1}$$

in which Δ stands for the *fractal (Hausdorff) dimension*. Upon magnification with scales $1/r_1$ and $1/r_2$, the number of boxes covering the interval $\left[0, r_1\right]$ and $\left[1-r_2, 1\right]$ are $N(a/r_1)$ and $N(a/r_2)$, respectively, which leads to

$$N(a) = N(\frac{a}{r_1}) + N(\frac{a}{r_2}) \tag{B2}$$

or

$$r_1^{\Delta} + r_2^{\Delta} = 1 \tag{B3}$$

The generalization of (B3) when the generator is composed by S elements is straightforward, namely,

$$\sum_{j=1}^{S} r_j^{\Delta} = 1 \tag{B4}$$

Finally, (B4) can be further extended to generic multifractal distributions,

$$\sum_{j=1}^{S} p_j^q r_j^{\tau(q)} = 1 \qquad \text{(B5)}$$

where p_j are probabilities associated with the scales r_j and $q, \tau(q)$ are the characteristic critical exponents [8 – 9]. It is apparent that (B4) represents the case when $q = 0,\ \tau(0) = \Delta$.

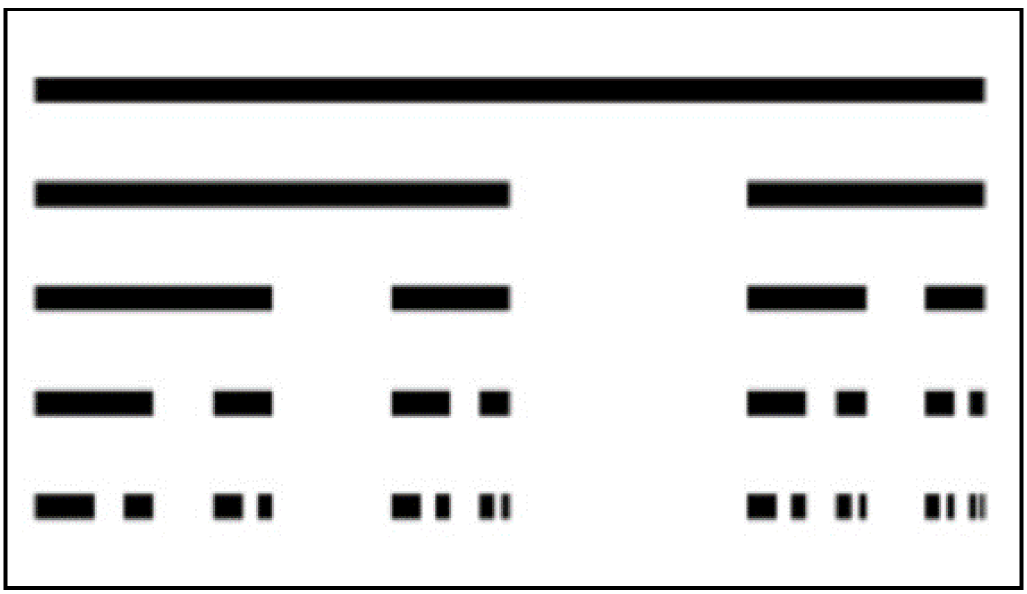

Fig. 2: Iterative construction of a two-scale Cantor set

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