# A THEOREM IN THE TRAPEZOID

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**Abstract** : In this note, we prove a remarkable theorem in the trapezoid.

**Theorem.**

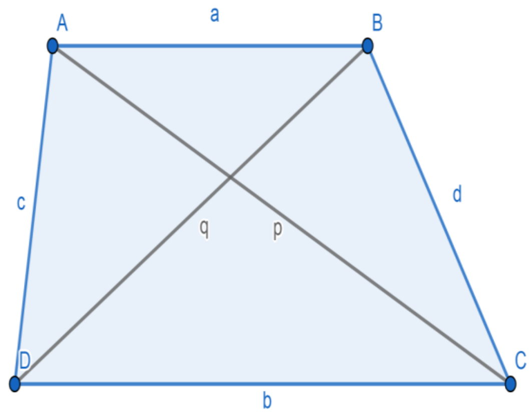
Let $ABCD$ be a trapezoid with bases $AB = a$, $DC = b$, and legs $CB = d$, $AD = c$, and diagonals $AC = p$ and $BD = q$, then we have :

$$\begin{cases} a = \sqrt{\dfrac{(d^2 - p^2)^2 - (c^2 - q^2)^2}{2(d^2 + p^2) - 2(c^2 + q^2)}} & (1) \\ \\ b = \sqrt{\dfrac{(c^2 - p^2)^2 - (d^2 - q^2)^2}{2(c^2 + p^2) - 2(d^2 + q^2)}} & (2) \end{cases}$$

*Proof.*

We are going to prove formula (2).

We construct the trapezoid $ABCD$.




Heron's formula for the area of a triangle with lengths $a, b, c$ is given by :

$$\frac{1}{4}\sqrt{(a+b+c)(-a+b+c)(a-b++c)(a+b-c)} \quad (3)$$

One can easily prove that the expression (3) is equivalent to the following expression :

$$\frac{1}{4}\sqrt{4a^2b^2-(a^2+b^2-c^2)^2} \quad (4)$$

Now, using formula (4), the area $K_1$ of the triangle $ACD$ in the above trapezoid is:

$$K_1 = \frac{1}{4}\sqrt{4b^2c^2-(b^2+c^2-p^2)^2}$$

Similarly the area $K_2$ of the triangle $BCD$ is:

$$K_2 = \frac{1}{4}\sqrt{4b^2d^2-(b^2+d^2-q^2)^2}$$

Since the two triangles $ACD$ and $BCD$ have the same base and altitude, their areas are equal, therefore :

$$K_1 = K_2 \quad \Leftrightarrow \quad \frac{1}{4}\sqrt{4b^2c^2 - (b^2 + c^2 - p^2)^2} = \frac{1}{4}\sqrt{4b^2d^2 - (b^2 + d^2 - q^2)^2}$$

$$\Leftrightarrow \quad 4b^2c^2 - (b^2 + c^2 - p^2)^2 = 4b^2d^2 - (b^2 + d^2 - q^2)^2$$

$$\Leftrightarrow \quad 4b^2c^2 - 4b^2d^2 = -(b^2 + d^2 - q^2)^2 + (b^2 + c^2 - p^2)^2$$

$$\Leftrightarrow \quad 4b^2c^2 - 4b^2d^2 = (b^2 + c^2 - p^2 + b^2 + d^2 - q^2)(b^2 + c^2 - p^2 - b^2 - d^2 + q^2)$$

$$\Leftrightarrow \quad 4b^2c^2 - 4b^2d^2 = (2b^2 + c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2)$$

$$\Leftrightarrow \quad 4b^2c^2 - 4b^2d^2 = 2b^2(c^2 - p^2 - d^2 + q^2) + (c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2)$$

$$\Leftrightarrow \quad 4b^2c^2 - 4b^2d^2 - 2b^2(c^2 - p^2 - d^2 + q^2) = (c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2)$$

$$\Leftrightarrow \quad b^2(4c^2 - 4d^2 - 2c^2 + 2p^2 + 2d^2 - 2q^2) = (c^2 - p^2 + d^2 - q^2)(c^2 - p^2 - d^2 + q^2)$$

$$\Leftrightarrow \quad b^2(2c^2 + 2p^2 - 2d^2 - 2q^2) = (c^2 - p^2)^2 - (d^2 - q^2)^2$$

$$\Leftrightarrow \quad b^2 = \frac{(c^2 - p^2)^2 - (d^2 - q^2)^2}{2c^2 + 2p^2 - 2d^2 - 2q^2}$$

$$\Leftrightarrow \quad b = \sqrt{\frac{(c^2 - p^2)^2 - (d^2 - q^2)^2}{2(c^2 + p^2) - 2(d^2 + q^2)}}$$

Similarly we prove formula (1) by considering the triangles $ABD$ and $ABC$. ■