THE EXPLICIT FORMULA OF BERNOULLI NUMBERS

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Abstract : The aim of this paper is to give an elementary proof of a well-known explicit formula for Bernoulli numbers, with some remarks.

Keywords : Stirling numbers of the second kind, Bernoulli numbers, Bernoulli polynomials.

1 Introduction

The numbers :

$$b_0 = 1, \qquad b_2 = \frac{1}{6}, \qquad b_4 = -\frac{1}{30}, \qquad b_6 = \frac{1}{42},$$
$$b_8 = -\frac{1}{30} \qquad \ldots, \qquad b_1 = -\frac{1}{2}, \qquad b_3 = b_5 = b_7 = b_9 = \cdots = 0$$

are called the Bernoulli numbers. They can be defined by the following exponential generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} b_n \frac{t^n}{n!}$$

For a long time, mathematicians computed b_n using recursive relations like the following one:

$$\begin{cases} \text{For } n = 0, & b_n = 1 \\ \forall n \geq 1, & \displaystyle\sum_{k=0}^{n} \binom{n+1}{k} b_k = 0 \end{cases}$$

In 1883, Worpitzky gave the following explicit formula for b_n [1]:

$$b_n = \sum_{k=0}^{n} \frac{1}{k+1} \sum_{i=0}^{k} \binom{k}{i} (-1)^i i^n \qquad (1)$$

We can also find other mathematicians from the 19th century who proved formula (1), such as Cesaro in 1885 [2] and D'Ocagne in 1889 [3].

For our part, we present an elementary proof of the formula (1).

2 Stirling numbers of the second kind

Let Y be an arbitrary function of x, and set :

$$D^nY = \underbrace{x(\dots x(x(x}_{n\ times} Y \overbrace{{}')')' \dots)'}^{n\ times}$$

If we develop D^nY for $n = 1, 2, 3, 4$, we find :

$D^1Y = xY'$

$D^2Y = xY' + x^2Y''$

$D^3Y = xY' + 3x^2Y'' + x^3Y^{(3)}$

$D^4Y = xY' + 7x^2Y'' + 6x^3Y^{(3)} + x^4Y^{(4)}$
...

We conjecture that :

$$D^nY = S_n^0Y + S_n^1xY' + S_n^2x^2Y'' + \dots + S_n^nx^nY^{(n)} \qquad (2)$$

The coefficients S_n^k are called Stirling numbers of the second kind. They can be represented in a triangle similar to Pascal's triangle. The triangle of the numbers S_n^k is the following :

	k=0	k=1	k=2	k=3	k=4	k=5	…
n=0	1						
n=1	0	1					
n=2	0	1	1				
n=3	0	1	3	1			
n=4	0	1	7	6	1		
n=5	0	1	15	25	10	1	
…	…	…	…	…	…	…	…

Table 1 : The triangle of Stirling numbers of the second kind S_n^k

We observe that :

$$\begin{cases} & S_0^0 = 1 \\ \forall n \geq 1, & S_n^0 = 0 \end{cases}$$

The law for forming the numbers S_n^k in the above table is given by :

$$S_n^k = S_{n-1}^{k-1} + kS_{n-1}^k$$

3 The explicit formula of Stirling numbers of the second kind

If we put $Y = e^x$ in the formula (2) we obtain :

$$D^n e^x = e^x \sum_{k=0}^{n} S_n^k x^k$$

$$\Longrightarrow \quad e^{-x}. D^n e^x = \sum_{k=0}^{n} S_n^k x^k$$

$$\Longrightarrow \quad \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right). D^n \left(\sum_{i=0}^{\infty} \frac{x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k x^k$$

$$\Longrightarrow \quad \left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{D^n x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k x^k$$

One can easily prove that $D^n x^i = i^n x^i$, so :

$$\left(\sum_{j=0}^{\infty} \frac{(-1)^j x^j}{j!}\right)\left(\sum_{i=0}^{\infty} \frac{i^n x^i}{i!}\right) = \sum_{k=0}^{n} S_n^k\, x^k$$

If we develop the left-hand side we obtain :

$$\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} \frac{(-1)^{k-i}\binom{k}{i} i^n}{k!}\right) x^k = \sum_{k=0}^{n} S_n^k\, x^k$$

Comparing coefficients in both summations we conclude that :

$$S_n^k = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n \qquad (3)$$

4 Relation between Stirling numbers of the second kind and Bernoulli numbers

Putting $Y = x^y$ in the formula (2), we get :

$$D^n x^y = \sum_{k=0}^{n} S_n^k \, x^k (x^y)^{(k)}$$

We know that $(x^y)^{(k)} = y(y-1)\dots(y-k+1)x^{y-k}$ and $D^n x^y = y^n x^y$so we get :

$$y^n = \sum_{k=0}^{n} S_n^k y(y-1)\dots(y-k+1) \qquad (4)$$

The polynomial $y(y-1)\dots(y-k+1)$ is called the falling factorial of order k of y. Pochhammer used the symbol $(y)_k$ to denote it, so the formula (4) becomes using Pochhammer symbol :

$$y^n = \sum_{k=0}^{n} S_n^k (y)_k \qquad (4')$$

One interesting property of the falling factorial function is the following :

Proposition 1

Let n and y be non-negative integers, then :

$$(y+1)_{n+1} - (y)_{n+1} = (n+1)(y)_n$$

Proof

$$\begin{aligned} (y+1)_{n+1} - (y)_{n+1} &= (y+1)y(y-1)\dots(y-n+1) - y(y-1)\dots(y-n+1)(y-n) \\ &= [(y+1)-(y-n)]y(y-1)\dots(y-n+1) \\ &= (n+1)(y)_n \end{aligned}$$

We are going to use this property in the proof of the following proposition.

Proposition 2

Let $n \in \mathbb{N}$ and $m \in \mathbb{N}^*$. We have :

$$\sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k \frac{(m)_{k+1}}{k+1} \qquad (5)$$

Proof

If we sum for y in the formula (4') we find :

$$\sum_{y=0}^{m-1} y^n = \sum_{y=0}^{m-1}\left(\sum_{k=0}^{n} S_n^k\,(y)_k\right)$$

$$\Longrightarrow \quad \sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k\left(\sum_{y=0}^{m-1}(y)_k\right)$$

$$\Longrightarrow \quad \sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k\left(\sum_{y=0}^{m-1}\frac{(y+1)_{k+1}-(y)_{k+1}}{k+1}\right)$$

$$\Longrightarrow \quad \sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k\left(\frac{(m)_{k+1}-(0)_{k+1}}{k+1}\right)$$

Therefore :

$$\sum_{y=0}^{m-1} y^n = \sum_{k=0}^{n} S_n^k\,\frac{(m)_{k+1}}{k+1}$$

Definition

Let $n \in \mathbb{N}$
The Bernoulli polynomials $B_n(x)$ are defined by the following exponential generating function :

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}$$

One interesting observation to make about Bernoulli polynomials is that if we put $x = 0$ we get :

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n(0)\frac{t^n}{n!}$$

This generating function corresponds to the generating function of Bernoulli numbers b_n. Hence for all $n \in \mathbb{N}$, we have :

$$B_n(0) = b_n$$

Another interesting property of the Bernoulli polynomials is the following :

Proposition 3

Let $n \in \mathbb{N}$

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

Proof

On the one hand :

$$\begin{aligned}
\sum_{n=0}^{\infty}\{B_n(x+1) - B_n(x)\}\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} B_n(x+1)\frac{t^n}{n!}\right) - \left(\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}\right) \\
&= \frac{te^{t(x+1)}}{e^t - 1} - \frac{te^{tx}}{e^t - 1} \\
&= \frac{te^{tx}.e^t - te^{tx}}{e^t - 1} \\
&= \frac{te^{tx}(e^t - 1)}{e^t - 1} \\
&= te^{tx}
\end{aligned}$$

On the other hand :

$$\begin{aligned}
\sum_{n=0}^{\infty} nx^{n-1}\frac{t^n}{n!} &= \sum_{n=1}^{\infty} t\frac{(xt)^{n-1}}{(n-1)!} \\
&= t\sum_{n=0}^{\infty}\frac{(xt)^n}{n!} \\
&= te^{xt}
\end{aligned}$$

Comparing coefficients of both summations we conclude that for all $n \in \mathbb{N}$:

$$B_n(x+1) - B_n(x) = nx^{n-1}$$

Proposition 4

Let $n \in \mathbb{N}$

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k} b_{n-k}x^k$$

Proof

$$\begin{aligned}
\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} &= \frac{te^{tx}}{e^t-1} \\
&= \frac{t}{e^t-1}.e^{tx} \\
&= \left(\sum_{n=0}^{\infty} b_n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{(xt)^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k}\frac{t^{n-k}}{(n-k)!}.\frac{(xt)^k}{k!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} b_{n-k}\binom{n}{k}x^k\right)\frac{t^n}{n!}
\end{aligned}$$

Therefore :

$$B_n(x) = \sum_{k=0}^{n} b_{n-k}\binom{n}{k}x^k$$

Summing for y in the relation $B_{n+1}(y+1) - B_{n+1}(y) = (n+1)y^n$ we obtain :

$$\begin{aligned}
(n+1)\sum_{y=0}^{m-1} y^n &= \sum_{y=0}^{m-1}\left\{B_{n+1}(y+1) - B_{n+1}(y)\right\} \\
&= B_{n+1}(m) - B_{n+1}(0) \\
&= B_{n+1}(m) - b_{n+1}
\end{aligned}$$

Thus :

$$(n+1)\sum_{y=0}^{m-1} y^n = B_{n+1}(m) - b_{n+1} \qquad (6)$$

Comparing formula (5) with formula (6) we conclude that :

$$B_{n+1}(m) - b_{n+1} = (n+1)\sum_{k=0}^{n} S_n^k \frac{(m)_{k+1}}{k+1} \qquad (7)$$

If we develop the expression of $(X)_{k+1}$ in terms of the powers of X we find :

$$\begin{aligned}
(X)_{k+1} &= X(X-1)\dots(X-k) \\
&= X\left(X^k - \frac{k(k+1)}{2}X^{k-1} + \cdots + (-1)^k k!\right) \\
&= X\sum_{j=0}^{k} c_j\, X^j \\
&= \sum_{j=0}^{k} c_j\, X^{j+1}
\end{aligned}$$

Therefore :

$$(X)_{k+1} = \sum_{j=0}^{k} c_j\, X^{j+1}$$

If we apply the above formula for $(m)_{k+1}$ in the formula (7) we find:

$$B_{n+1}(m) - b_{n+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} \sum_{j=0}^{k} c_j\, m^{j+1}$$

Substituting also $B_{n+1}(m)$ by its explicit expression, we finally get :

$$\begin{aligned}
&\left(\sum_{k=0}^{n+1} \binom{n+1}{k} b_{n+1-k} m^k\right) - b_{n+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} \sum_{j=0}^{k} c_j\, m^{j+1} \\
\Rightarrow\quad & \sum_{k=1}^{n+1} \binom{n+1}{k} b_{n+1-k} m^k = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} \sum_{j=0}^{k} c_j\, m^{j+1} \\
\Rightarrow\quad & \sum_{j=0}^{n} \binom{n+1}{j+1} b_{n-j} m^{j+1} = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} \sum_{j=0}^{k} c_j\, m^{j+1} \\
\Rightarrow\quad & \sum_{j=0}^{n} \left(\binom{n+1}{j+1} b_{n-j}\right) m^j = \sum_{j=0}^{n} \left(\sum_{k=j}^{n} S_n^k \frac{n+1}{k+1} c_j\right) m^j
\end{aligned}$$

We have equality between two polynomials in m, both of degree n, so the coefficients of the terms of the same degree are equal. In particular for $j = 0$ we have :

$$\binom{n+1}{1} b_n = \sum_{k=0}^{n} S_n^k \frac{n+1}{k+1} c_0$$

$$\Rightarrow \qquad b_n = \sum_{k=0}^{n} S_n^k \frac{(-1)^k k!}{k+1} \qquad (8)$$

To get the explicit expression of b_n in terms of n, we substitute S_n^k in the above identity by its explicit expression, and after simplification we obtain the remarkable formula (1) for the Bernoulli numbers.

5 Some observations

From formula (6) we can deduce Bernoulli's formula, we have :

$$\begin{aligned}
\sum_{y=0}^{m-1} y^n &= \frac{1}{n+1}\{B_{n+1}(m) - b_{n+1}\} \\
&= \frac{1}{n+1}\left\{\left(\sum_{k=0}^{n+1}\binom{n+1}{k} b_{n+1-k} m^k\right) - b_{n+1}\right\} \\
&= \frac{1}{n+1}\sum_{k=1}^{n+1}\binom{n+1}{k} b_{n+1-k} m^k \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j+1} b_{n-j} m^{j+1} \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\binom{n+1}{j} b_j m^{n-j+1}
\end{aligned}$$

The proof given by Cesaro and D'Ocagne use the Bernoulli formula directly instead of introducing Bernoulli's polynomials like we did.

Formula (2) is called Grunert's formula [4], it furnishes an original definition for the Stirling numbers of the second kind. Stirling in his book "Methodus Differentialis" define the Stirling numbers of the second kind using formula (4') result of formula (2), so we conclude that Stirling definition is but a special case of Grünert's.

We can deduce identity (8) from the explicit formula of Stirling numbers of the second kind. We know form formula (3) that for all $0 \le k \le n$:

$$k!\,S_n^k = \sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i} i^n$$

If we invert the above formula we find :

$$\begin{aligned} k^n &= \sum_{i=0}^{k} \binom{k}{i} i!\, S_n^i \\ \Longrightarrow \quad k^n &= \sum_{i=0}^{k} S_n^i (k)_i \end{aligned}$$

This formula is similar to formula (4') with the exception that the sum is taken here from 0 to k, and this is valid only for $k \in \{0, 1, \dots, n\}$, while in formula (4') the sum was taken from 0 to n, and that was valid for all real number y.

Now summing for k in the last formula we obtain :

$$\begin{aligned} \sum_{k=0}^{n} k^n &= \sum_{k=0}^{n} \left(\sum_{i=0}^{k} S_n^i (k)_i \right) \\ &= \sum_{i=0}^{n} S_n^i \sum_{k=i}^{n} (k)_i \\ &= \sum_{i=0}^{n} S_n^i \left(\frac{(n+1)_{i+1} - (i)_{i+1}}{i+1} \right) \\ &= \sum_{i=0}^{n} S_n^i \frac{(n+1)_{i+1}}{i+1} \\ &= \sum_{i=0}^{n} S_n^i \frac{1}{i+1} \sum_{j=0}^{i} c_j \, (n+1)^{i+1} \\ &= \sum_{j=0}^{n} \left(\sum_{i=j}^{n} S_n^i \frac{c_j}{i+1} \right) (n+1)^{j+1} \end{aligned}$$

Thus we have :

$$\sum_{k=0}^{n} k^n = \sum_{j=0}^{n} \left(\sum_{i=j}^{n} S_n^i \frac{c_j}{i+1} \right) (n+1)^{j+1}$$

Using Bernoulli's formula we conclude that :

$$\sum_{j=0}^{n} \left(\frac{\binom{n+1}{j+1}}{n+1} b_{n-j} \right) (n+1)^{j+1} = \sum_{j=0}^{n} \left(\sum_{i=j}^{n} S_n^i \frac{c_j}{i+1} \right) (n+1)^{j+1}$$

The coefficients of $n+1$ in both representations are equal so :

$$\frac{\binom{n+1}{1}}{n+1} b_n = \sum_{i=0}^{n} S_n^i \frac{c_0}{i+1} \quad \Rightarrow \quad b_n = \sum_{i=0}^{n} S_n^i \frac{(-1)^i i!}{i+1}$$

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