Optimization Problem

Edgar Valdebenito

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ABSTRACT

What is the maximum value of $f(x)=\tan^{-1}\left(e^{-e^{x-2}}\right)-\tan^{-1}\left(e^{-e^{x+2}}\right)$?

I. Introduction

1. Let

$$f(x)=\tan^{-1}\left(e^{-e^{x-2}}\right)-\tan^{-1}\left(e^{-e^{x+2}}\right) \quad , \ x\in(-\infty,\infty) \tag{1}$$

2. Derivative

$$\frac{df}{dx}=\frac{e^{2-e^{2+x}+x}}{1+e^{-2e^{2+x}}}-\frac{e^{-2-e^{-2+x}+x}}{1+e^{-2e^{-2+x}}} \tag{2}$$

3. Critical points

$$\frac{df}{dx}=0\Longrightarrow \cosh\left(e^{x+2}\right)-e^{4}\cosh\left(e^{x-2}\right)=0\Longrightarrow x=s=-0.453127\ldots \tag{3}$$

4. Maximum

$$\frac{df}{dx}>0 \text{ for all } x<s \text{ and } \frac{df}{dx}<0 \text{ for all } x>s \Longrightarrow f(s) \text{ is the absolute maximum value of } f \tag{4}$$

$$f(s)=0.733314\ldots \tag{5}$$

5. The number s

$$s=\ln r \quad , \quad r=e^{-2}\cosh^{-1}\left(e^{4}\cosh\left(e^{-4}\left(\cosh^{-1}\left(e^{4}\cosh\left(e^{-4}\ldots\right)\right)\right)\right)\right) \tag{6}$$

$$\tanh(e^{s}\sinh(2))\tanh(e^{s}\cosh(2))=\tanh(2) \tag{7}$$

$$u_{n+1}=e^{-4}\cosh^{-1}\left(e^{4}\cosh(u_n)\right) \ , \ u_0=0 \ \Longrightarrow u_n\longrightarrow u=e^{s-2}\Longrightarrow s=2+\ln u \tag{8}$$

II. Twenty-five Integrals

$$\pi=\int_{-\infty}^{\infty}\left(\tan^{-1}\left(e^{-e^{x-2}}\right)-\tan^{-1}\left(e^{-e^{x+2}}\right)\right)dx \tag{9}$$

$$\pi=\int_{-\infty}^{\infty}\left(\tan^{-1}\left(e^{e^{x+2}}\right)-\tan^{-1}\left(e^{e^{x-2}}\right)\right)dx \tag{10}$$

$$\pi=\int_{-\infty}^{\infty}\left(\tan^{-1}\left(\tanh\left(\frac{e^{x+2}}{2}\right)\right)-\tan^{-1}\left(\tanh\left(\frac{e^{x-2}}{2}\right)\right)\right)dx \tag{11}$$

$$\pi=\int_{0}^{\infty}\frac{\tan^{-1}(\tanh(x))-\tan^{-1}\left(\tanh\left(x\,e^{-4}\right)\right)}{x}dx \tag{12}$$

$$\pi=\int_{0}^{\infty}\frac{\tan^{-1}\left(\tanh\left(x\,e^{4}\right)\right)-\tan^{-1}(\tanh(x))}{x}dx \tag{13}$$

$$\pi=\int_{0}^{\infty}\frac{\tan^{-1}\left(e^{-x\,e^{-2}}\right)-\tan^{-1}\left(e^{-x\,e^{2}}\right)}{x}dx \tag{14}$$

$$\pi = \int_0^\infty \frac{1}{x} \tan^{-1}\left(\frac{e^{-x e^{-2}} - e^{-x e^2}}{1 + e^{-2x\cosh(2)}} \right) dx \tag{15}$$

$$\pi = \int_0^\infty \frac{1}{x} \tan^{-1}\left(\frac{\sinh(x\sinh(2))}{\cosh(x\cosh(2))} \right) dx \tag{16}$$

$$\pi = \int_0^\infty \frac{1}{x} \tan^{-1}\left(\frac{\sinh(x\tanh(2))}{\cosh(x)} \right) dx \tag{17}$$

$$\pi = \int_0^\infty \frac{1}{x} \tan^{-1}\left(\frac{\sinh(x)}{\cosh(x\coth(2))} \right) dx \tag{18}$$

$$\pi = \int_{-\infty}^\infty \tan^{-1}\left(\frac{\sinh(e^x\sinh(2))}{\cosh(e^x\cosh(2))} \right) dx \tag{19}$$

$$\pi = -\int_0^1 \frac{1}{x\ln x} \tan^{-1}\left(\frac{x^{e^{-2}} - x^{e^2}}{1 + x^{2\cosh(2)}} \right) dx \tag{20}$$

$$\pi = -\int_0^1 \frac{\tan^{-1}(x) - \tan^{-1}\left(x^{e^4}\right)}{x\ln x} dx \tag{21}$$

$$\pi = -\int_0^1 \frac{\tan^{-1}\left(x^{e^{-4}}\right) - \tan^{-1}(x)}{x\ln x} dx \tag{22}$$

$$\pi = \int_1^\infty \frac{\tan^{-1}\left(x^{e^4}\right) - \tan^{-1}(x)}{x\ln x} dx \tag{23}$$

$$\pi = \int_1^\infty \frac{\tan^{-1}(x) - \tan^{-1}\left(x^{e^{-4}}\right)}{x\ln x} dx \tag{24}$$

$$\pi = -\int_0^{\pi/4} \frac{x - \tan^{-1}\left((\tan x)^{e^4}\right)}{\sin x\cos x\ln(\tan x)} dx \tag{25}$$

$$\pi = -\int_0^{\pi/4} \frac{\tan^{-1}\left((\tan x)^{e^{-4}}\right) - x}{\sin x\cos x\ln(\tan x)} dx \tag{26}$$

$$\pi = \int_{\pi/4}^{\pi/2} \frac{\tan^{-1}\left((\tan x)^{e^4}\right) - x}{\sin x\cos x\ln(\tan x)} dx \tag{27}$$

$$\pi = \int_{\pi/4}^{\pi/2} \frac{x - \tan^{-1}\left((\tan x)^{e^{-4}}\right)}{\sin x\cos x\ln(\tan x)} dx \tag{28}$$

$$\pi = \int_0^1 \frac{\tan^{-1}\left(e^{-x e^{-2}}\right) - \tan^{-1}\left(e^{-x e^2}\right) + \tan^{-1}\left(e^{-e^{-2}/x}\right) - \tan^{-1}\left(e^{-e^2/x}\right)}{x} dx \tag{29}$$

$$\pi = \int_1^\infty \frac{\tan^{-1}\left(e^{-x e^{-2}}\right) - \tan^{-1}\left(e^{-x e^2}\right) + \tan^{-1}\left(e^{-e^{-2}/x}\right) - \tan^{-1}\left(e^{-e^2/x}\right)}{x} dx \tag{30}$$

$$\pi = \int_0^\infty \left(\tan^{-1}\left(e^{-e^{-x-2}}\right) - \tan^{-1}\left(e^{-e^{-x+2}}\right)\right) dx + \int_0^4 \tan^{-1}\left(e^{-e^{x-2}}\right) dx \tag{31}$$

$$\pi = \int_0^\infty \left(\tan^{-1}\left(e^{-e^{-x-2}}\right) - \tan^{-1}\left(e^{-e^{-x+2}}\right)\right) dx + \int_0^{\tan^{-1}\left(e^{-e^{-2}}\right)} \ln\left(\ln(\cot x)^{e^2}\right) dx - \int_0^{\tan^{-1}\left(e^{-e^2}\right)} \ln\left(\ln(\cot x)^{e^{-2}}\right) dx \tag{32}$$

$$\pi = \int_0^\infty \left(\tan^{-1}\left(e^{-e^{-x-2}}\right) - \tan^{-1}\left(e^{-e^{-x+2}}\right)\right) dx + \int_{\tan^{-1}\left(e^{-e^2}\right)}^{\tan^{-1}\left(e^{-e^{-2}}\right)} \ln(\ln(\cot x))\, dx + 2\left(\tan^{-1}\left(e^{-e^{-2}}\right) + \tan^{-1}\left(e^{-e^2}\right)\right) \tag{33}$$

III. Endnote

$$\pi = \int_0^a \frac{\tan^{-1}\left(e^{-x\,e^{-2}}\right) - \tan^{-1}\left(e^{-x\,e^{2}}\right)}{x}\,dx + \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1}\,\Gamma\left(0,\,(2\,n+1)\,a\,e^{-2},\,(2\,n+1)\,a\,e^{2}\right)\ ,\ \ a > 0 \tag{34}$$

$$\Gamma\left(0,\,(2\,n+1)\,a\,e^{-2},\,(2\,n+1)\,a\,e^{2}\right) = \Gamma\left(0,\,(2\,n+1)\,a\,e^{-2}\right) - \Gamma\left(0,\,(2\,n+1)\,a\,e^{2}\right) \tag{35}$$

where $\Gamma(a, b)$ is the incomplete Gamma function.

$$\pi = \int_{-\infty}^{\infty} \left(\frac{e^{-2-e^{x-2}}}{1+e^{-2\,e^{x-2}}} - \frac{e^{2-e^{x+2}}}{1+e^{-2\,e^{x+2}}} \right) e^x\,x\,dx \tag{36}$$

$$\pi = \int_0^{\infty} \left(\frac{e^{-2-e^{-2}\,x}}{1+e^{-2\,e^{-2}\,x}} - \frac{e^{2-e^{2}\,x}}{1+e^{-2\,e^{2}\,x}} \right) \ln(x)\,dx \tag{37}$$

$$\pi = \frac{1}{2} \int_0^{\infty} \left(\frac{e^{-2}}{\cosh\left(e^{-2}\,x\right)} - \frac{e^{2}}{\cosh\left(e^{2}\,x\right)} \right) \ln(x)\,dx \tag{38}$$

$$\pi = \int_0^{\infty} \left(\frac{e^{-2-e^{x-2}+x}}{1+e^{-2\,e^{x-2}}} - \frac{e^{2-e^{x+2}+x}}{1+e^{-2\,e^{x+2}}} - \frac{e^{-2-e^{-x-2}-x}}{1+e^{-2\,e^{-x-2}}} + \frac{e^{2-e^{-x+2}-x}}{1+e^{-2\,e^{-x+2}}} \right) x\,dx \tag{39}$$

$$\pi = \frac{1}{2} \int_0^{\infty} \left(\frac{e^{-2+x}}{\cosh\left(e^{-2+x}\right)} - \frac{e^{2+x}}{\cosh\left(e^{2+x}\right)} - \frac{e^{-2-x}}{\cosh\left(e^{-2-x}\right)} + \frac{e^{2-x}}{\cosh\left(e^{2-x}\right)} \right) x\,dx \tag{40}$$

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots \tag{41}$$

IV. References

1. M. Abramowitz, and I.A. Stegun, eds. Handbook of mathematical functions with formulas, graphs, and mathematical tables, US Government printing office, 1968.
2. G.E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press, Cambridge, 2006.
3. A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, Higher Transcendental Functions, Volumes 1-3, McGraw-Hill, 1953.
4. I.S. Gradshteyn, and I.M. Ryzhik, Table of Integrals, Series, and Products, 6th. ed., Academic Press, Boston, 2000.