Knot in weak field of gravitation with dual Ricci tensor

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We propose that there exists the topological object, a gravitational knot, in weak field of gravitation formulated using dual Ricci tensor in Chern-Simons theory. The Chern-Simons action is interpreted as such a knot.

Keywords: *general theory of relativity, weak field, dual Ricci tensor, Chern-Simons action, knot.*

I. INTRODUCTION

It is commonly believed there exists no topological object in the linear theory, such as Newton's theory of gravitation. It is because a topological theory must be a non-linear theory[1]. How could a gravitational knot exist in Newton's linear theory of gravitation (the weak-field limit of Einstein's non-linear theory of gravitation)?

We consider that identical to the existence of a topological structure in Maxwell's gauge theory in vacuum space[1,2], the curvature especially the metric tensor (the set of the solutions of Einstein field equations) in empty space has a subset field with a topological structure. Empty space here means that there is no matter present and there is no physical fields exist except the weak gravitational field. The weak gravitational field does not disturb the emptiness. But other fields disturb the emptiness[3].

A subset field is locally equal to curvature i.e. curvature can be obtained by patching together subset fields (except in a zero-measure set) but globally different. The difference between the subset fields and the curvature in empty space is global instead of local since the subset fields obey the topological quantum condition but the curvature or the metric tensor does not.

Curvature in Newton's theory of gravitation satisfies a linear field equation, but a subset field satisfies a non-linear field equation. Both, curvature and a subset field, satisfy a linear field equation in the case of the weak field of gravitation. It means that, in the case of the weak field, a non-linear subset field theory reduces to Newton's linear theory of gravitation.

In this article, we propose there exists a knot (a gravitational knot) in Newton's theory of gravitation in empty space. This gravitational knot could exist in Newton's theory of gravitation in empty space because Newton's theory of gravitation in empty space is the weak-field limit[2] of a non-linear subset field theory. To the best of our knowledge[1,4–6], the formulation of such a knot (a weak field gravitational knot) in Newton's theory of gravitation has not been done yet.

II. WEAK-FIELD LIMIT OF GRAVITATION

In the limit of weak gravitational fields, low velocities (of sources), and small pressure, the general theory of relativity reduces to Newton's theory of gravitation[7]. In the case of the weak field, linearization (we assume that we ignore the non-linear terms of connection[8]) of the Ricci curvature tensor yields[7]

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \partial_\nu \Gamma^\alpha_{\mu\alpha} \tag{1}$$

This equation is identical to Abelian field strength in electrodynamics where the curvature (the Ricci tensor), $R_{\mu\nu}$, is identical to the field strength, $F_{\mu\nu}$, and the connection (Christoffel symbol), $\Gamma^\alpha_{\mu\alpha}$, is identical to the gauge potential, A_μ.

In the case of the weak-field limit where the source of gravitation is static[8], we could write Newton's theory of gravitation[7,9,10] as a linear equation written below

$$R_{tt} = \vec{\nabla} \cdot \vec{\nabla}\Phi = \nabla^2\Phi \tag{2}$$

where R_{tt} is the time-time component of the Ricci curvature tensor, ∇^2 (div of grad) is the Euclidean Laplacian operator with respect to space, Φ is the (scalar) potential of gravitation, $\vec{g}$ is the gravitational field, $\vec{g} = \vec{\nabla}\Phi$, and

$$\nabla^2\Phi = 4\pi\rho \tag{3}$$

is Poisson's equation[7], ρ is the mass density. By substituting eq.(3) into eq.(2) we obtain Newton's theory of gravitation expressed as Newtonian field equation[7]

$$R_{tt} = 4\pi\rho \tag{4}$$

We see that eq.(4) as a consequence of the spherical symmetry of eq.(1) i.e. only R_{tt} component is significant and the others are zero. The spherical symmetry is assumed because the form of gravitational objects is assumed to be a sphere at infinite r. The value of $R_{tt} = \nabla^2\Phi$ due to the weak field of gravitation measured or observed at infinite r i.e. far from sources.

III. SUBSET FIELDS PROPERTY AND MAPS $S^3 \to S^2$

Let us consider maps of subset fields (consisting of complex scalar fields) from a finite radius r to an infinite r implies from the stronger field to the weak field.

A scalar field has properties that, by definition, its value for a finite r depends on the magnitude and the direction of the position vector, $\vec{r}$, but for an infinite r it is well-defined[2] (it depends on the magnitude only). In other words, for an infinite r, a scalar field is isotropic. Throughout this article, we will work with the classical scalar field.

The property of such scalar fields can be interpreted as maps $S^3 \to S^2$[1] where S^3 and S^2 are 3-dimensional and 2-dimensional spheres respectively i.e. after identifying via stereographic projection, 3-dimensional physical space, $R^3 \cup \{\infty\}$, with the sphere S^3 and the complete complex plane, $C \cup \{\infty\}$, with the sphere S^2.

These maps $S^3 \to S^2$ can be classified in homotopy classes labeled by the value of the corresponding Hopf indexes, integer numbers, and the topological invariants[1,2]. The other names of the topological invariants are the topological charge, and the winding number (the degree of a continuous mapping)[11]. The topological charge which is independent of the metric tensor could be interpreted as energy[12].

We see there exists (one) dimensional reduction in such maps. We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a scalar field for an infinite r. The property of a scalar field as a function of space seems likely in harmony with the property of space-time itself. Space-time could be locally anisotropic but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

IV. HOPF INVARIANT AND ABELIAN CHERN-SIMONS

Let us discuss the maps above more formally. As we mentioned we have a scalar field as a function of the position vector, $e^a(\vec{r})$, with a property that can be interpreted using the non-trivial Hopf map written below[1,2]

$$e^a(\vec{r}) : S^3 \to S^2 \tag{5}$$

This non-trivial Hopf map is related to the Hopf invariant[13], $\mathcal{H}$, expressed as an integral[13–15]

$$\mathcal{H} = \int_{S^3} \omega \wedge d\omega \tag{6}$$

where ω is a 1-form on S^3[13] and $d\omega$ is a 2-form.

The relation between the Hopf invariant and the Hopf index, h, can be written explicitly as[1]

$$\mathcal{H} = h\ \gamma^2 \tag{7}$$

where γ is the total strength of the field which is the sum of the strengths of all the tubes formed by the integral lines of electric and magnetic fields[1].

Related to gauge theory and magnetohydrodynamics (self-helicity), it can be interpreted naturally that the Hopf invariant has a deep relationship with the Abelian Chern-Simons action (the Abelian Chern-Simons integral)[13].

The Hopf invariant is just the winding number of Gauss mapping[13]. Hopf invariant or the Chern-Simons integral is an important topological invariant to describe the topological characteristics of the knot family[13,16]. In a more precise expression, the Hopf invariant or the Chern-Simons integral is the total sum of all the self-linking and all the linking numbers of the knot family[13,16]. The self-linking and linking numbers by themselves have a topological structure.

V. NON-LINEAR SUBSET FIELD AND LINEARIZED RICCI THEORIES

We assume that a subset field, a scalar field, a component of the curvature, e^a, as a map of the gravitational theory in (3+1) to (2+1)-dimensional space-time written below

$$e^a(\vec{r}, t) : M^{3+1} \to M^{2+1} \tag{8}$$

where M denotes manifold.

The map (8) has a consequence (by considering that the field strength is identical to the curvature) that we could write the Ricci curvature tensor as

$$R^a_{\mu\nu} \approx \frac{\partial_\mu e^{a^*} \partial_\nu e^a - \partial_\nu e^{a^*} \partial_\mu e^a}{(1 + e^{a^*} e^a)^2} \tag{9}$$

where e^a is a subset of Ricci curvature tensor, and e^{a^*} is the complex conjugate of e^a. Eq.(9) is the non-linear equation where the nonlinearity is shown by the $e^{a^*} e^a$ term in the denominator. The superscript index a in e^a represents a set of indices that label the components of the scalar field.

In the case of the weak field, the scalar field is very small, $e^a << 1$, so eq.(9) reduces to a linear equation as written below

$$R^a_{\mu\nu} \approx \partial_\mu e^{a^*} \partial_\nu e^a - \partial_\nu e^{a^*} \partial_\mu e^a \tag{10}$$

This linear equation (10) is equivalent to eq.(1). It means that the linearized Ricci theory (1) could be interpreted as the same as the Ricci theory in the case of the weak field (10).

We see from eq.(5) that a scalar field in a non-trivial Hopf map is written as $e^a(\vec{r})$, i.e. a time-independent scalar field. It differs from a time-dependent scalar field $e^a(\vec{r}, t)$ in eq.(8). This problem could be solved by interpreting some of the quantities that appear in Hopf's theories as Cauchy's initial time values[17].

VI. SCALAR AND TRIAD FIELDS AS POTENTIAL

We consider the scalar field, e^a, as the scalar potential and it could be interpreted similarly to linearized metric perturbation. Linearized metric perturbations take a role as "potentials" in linearized gravitation identical to electric (scalar) and magnetic (vector) potentials in electromagnetism[18]. Linearized metric perturbation can be written as[18]

$$h_{ab} = \rho_{ab}\ e^{i\vec{k}\cdot\vec{r}} \tag{11}$$

where ρ_{ab} is amplitude and $\vec{k}$ is wave vector. In empty space, a weak field, the amplitude is constant. Eq.(11) shows us that the linearized metric perturbation can be understood in terms of the wave.

Analog to eq.(11), we propose that the scalar field and the triad field could be written in terms of the wave, respectively as[19]

$$e^a = \rho^a e^{iq} \tag{12}$$

and

$$e_{\rho a} = f_a\ \partial_\rho q \tag{13}$$

where ρ^a is the amplitude, q is the phase, $f_a = -1/\left\{2\pi[1+(\rho^a)^2]\right\}$, f_a and q are the Clebsch variables[17]. We see from eq.(13) that the triad field could be viewed as vector potential[20]. The subscript index ρ in $e_{\rho a}$ represents space-time coordinates.

We consider that Ricci tensor (10) is identical to the field strength tensor of electromagnetic, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. By using eq.(13), Ricci tensor (10) could be written as[17]

$$R^a_{\mu\nu} \approx \partial_\mu(f_a\ \partial_\nu q) - \partial_\nu(f_a\ \partial_\mu q) \tag{14}$$

This is the Ricci tensor written in term of the Clebsch variables.

VII. DUAL RICCI TENSOR

We define dual Ricci tensor as

$${}^*R^a_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu}{}_{\alpha\beta}\ R^a_{\mu\nu} \tag{15}$$

By substituting eq.(14) into (15, we obtain

$${}^*R^a_{\alpha\beta} = \frac{1}{2}\varepsilon^{\mu\nu}{}_{\alpha\beta}\ \partial_\mu(f_a\ \partial_\nu q) - \partial_\nu(f_a\ \partial_\mu q) \tag{16}$$

VIII. A GRAVITATIONAL KNOT

In the three or (2+1)-dimensional general theory of relativity, the dynamics is topology[21]. Roughly speaking, the (2+1)-dimensional general theory of relativity could be interpreted as a Chern-Simons three form[22] where Chern-Simons theory is topological gauge theory in three dimensions[21]. The Chern-Simons action precisely coincides with the (2+1)-dimensional Einstein-Hilbert action[22,23]. Chern-Simons theory was discovered in the context of anomalies and used as a rather exotic toy model for gauge systems in 2+1 dimensions ever since[24].

The (2+1)-dimensional Abelian Chern-Simons action could be written as[22,23]

$$S_{CS} = \int_M \varepsilon^{\mu\nu\rho}\ e_{\rho a}\ R^a_{\mu\nu}\ d^3r \tag{17}$$

where $\varepsilon^{\mu\nu\rho}$ is the Levi-Civita symbol. By substituting eqs.(13), (14), into eq.(17) we obtain

$$S_{CS} \approx \int_M \varepsilon^{\mu\nu\rho}\ f_a\ \partial_\rho q \left\{\partial_\mu(f_a\ \partial_\nu q) - \partial_\nu(f_a\ \partial_\mu q)\right\}\ d^3r \tag{18}$$

The action, S_{CS}, (18) is related to a topological object i.e. a knot[22], a gravitational knot (a gravitational helicity), an integer number. This integer number is what we mean with the subset fields obeying the topological quantum condition.

IX. A GRAVITATIONAL KNOT WITH DUAL RICCI TENSOR

The (2+1)-dimensional Abelian Chern-Simons action formulated using dual Ricci tensor could be written as

$$S_{CS} = \int_M \varepsilon^{\alpha\beta\rho}\ e_{\rho a}\ {}^*R^a_{\alpha\beta}\ d^3r \tag{19}$$

By substituting eq.(16) into (19), we obtain

$$S_{CS} = \frac{1}{2}\int_M \varepsilon^{\alpha\beta\rho}\ \varepsilon^{\mu\nu}{}_{\alpha\beta} \times f^a\ \partial_\rho q \left\{\partial_\mu(f^a\ \partial_\nu q) - \partial_\nu(f^a\ \partial_\mu q)\right\} d^3r \tag{20}$$

We see that the Chern-Simons action in eq.(20) is the gravitational knot in weak field of gravitation formulated using the dual Ricci tensor.

X. DISCUSSION AND CONCLUSION

The proposal that curvature i.e. Ricci curvature tensor has a subset field, e^a, a scalar field (a scalar potential) has deep and far-reaching consequences. One of the consequences is that we can formulate the Ricci curvature tensor in non-linear form using the scalar field and its conjugate complex field (9).

In the case of empty space or weak field, the non-linear Ricci curvature tensor (9) reduces to the linearized Ricci curvature tensor (10) where Newton's theory of gravitation in the form of a subset field, a scalar field, could be derived from eq.(10). The linearized Ricci curvature tensor (10) is locally equivalent to eq.(1), but globally different. Eq.(1) is no longer valid globally.

We assume that a subset field, a scalar field, or a component of Ricci curvature tensor, as a map of gravitational theory in (3+1) to (2+1)-dimensional space-time. It implies there exists (one) dimensional reduction in such a map. We consider this dimensional reduction as a consequence of the isotropic (well-defined) property of a subset field, a scalar field, for an infinite r i.e. for infinite distance from the source the gravitational field is weak. It implies also that the linearized Ricci curvature tensor and its derived Newton's theory of gravitation can be formulated in (2+1)-dimensional space-time.

The remarkable one, as we mentioned that the (2+1)-dimensional general theory of relativity could be interpreted as a Chern-Simons (topological gauge theory) three form, it has a consequence that we could relate and interpret (2+1)-dimensional linearized Ricci curvature tensor (10) and its derived Newton's theory of gravitation as Chern-Simons three form in (2+1)-dimensional space-time where its action is related to a gravitational knot, an integer number (18). It means that the subset fields obey the topological quantum condition.

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