Algebraic Orientation, Invariance and Variance Within the Family $\mathscr{H}$ of Hypercomplex Algebras

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Abstract

It was brought to my attention that in previous papers I put in the public domain (references [1] through [8]), I did a rather piecemeal presentation of the critical concepts of algebraic orientation, algebraic invariance and algebraic variance, as well as the generalization of Cayley-Dickson algebras I referred to as Cayley-Dickson without emphasizing the more general nature of the presentation, which was done for reader familiarity purposes. I will attempt to remedy this within this document starting at the beginning: the basic definition of an algebra, carried through to a full definition and discussion of the general family of hypercomplex algebras I call $\mathscr{H}$ which subsumes all division algebras and traditional Cayley-Dickson doubled forms. Taking into account all structural and algebraic orientation options exposes beautiful structure revealed through group theoretical aspects of $\mathscr{H}$ construction. The concepts presented are essential to Octonion Algebra mathematical physics.

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The general concept of an algebra

An alternative to the n-tuple representation of an n-dimensional object as an ordered set of elements comma separated such as $(A_0, A_1, \ldots A_{n-1})$ is the representation referred to as an *algebraic element* defined by the sum

$$\mathbf{A} = A_0\, e_0 + A_1\, e_1 + \ldots + A_{n-1}\, e_{n-1}$$

Each $A_i\, e_i$ are called *components* of the algebraic element, defining the dimensional partitioning within the summation. The A_i are called *coefficients*, and are members of some mathematical field. Once the dimension n and particular field are declared, the descriptive *an n dimensional algebra over the field…* is made. Typical field types are real numbers and complex numbers. The e_i are called *basis elements*.

With this representation, *an algebra* is defined by three rules on algebraic elements

1. Algebraic element addition/subtraction: $\mathbf{A} \pm \mathbf{B} = (A_0 \pm B_0)\, e_0 + (A_1 \pm B_1)\, e_1 + \ldots + (A_{n-1} \pm B_{n-1})\, e_{n-1}$
2. Multiplication by a field member: $k\,\mathbf{A} = kA_0\, e_0 + kA_1\, e_1 + \ldots + kA_{n-1}\, e_{n-1}$
3. Algebraic element multiplication: $\mathbf{A} * \mathbf{B} = (A_i\, B_j)\, (e_i * e_j)$ sum i,j: 0 to n-1

The first two rules are common to any n dimensional algebra. They serve no purpose defining possible differences between two n dimensional algebras. Rules 1 and 2 are sufficient for the important subset *linear algebra*, but the real fun begins with rule 3, which is a statement of the distributive rule of multiplication over addition. It is within rule 3 that the salient characteristics of an algebra are defined. The coefficients multiply like real numbers by the definition of any mathematical field, so are separated off from the products $(e_i * e_j)$, which we have yet to define. The only limit on the products $(e_i * e_j)$ is they must not create new basis elements, that is, the set of basis elements is required to be *closed* for the operation *. The most general form this requirement can take is given by the following, where the n^3 values s_{ijk} are called *structure constants*

$$(e_i * e_j) = s_{ijk}\, e_k$$

An n dimensional algebra is fully defined by rules 1, 2 and 3 once all structure constants are assigned. The difference, if any, between two n dimensional algebras then comes down to a comparison of the set of structure constants defining each. Do understand that the basis element enumerations are not defining characteristics in and of themselves. We could permute their indexes along with a compatible permutation of the structure

constants without actually changing the algebraic structure in in a meaningful way, although the two structure constant sets will look different. We say then these two structure constant representations are non-identical but *equivalent representations*. Two n dimensional algebras are *isomorphic* if and only if their structure constant representations are equivalent representations.

The general form for basis element product results is a linear combination of some subset of, or all basis elements. Define an *intrinsic basis set* as one where the product of any two intrinsic basis elements is within sign only one element of the basis element set. We can then define intrinsic basis element products as

$(e_i * e_j) = s_{ijk}\, e_k$ where s_{ijk} defines both a single k value for given i,j and a choice s_{ijk} = one of ±1.

Intrinsic basis sets lend themselves to what is called a *basis element multiplication table*, where the rows and columns are labelled by the basis elements, and each table entry is either $+e_i$ or $-e_i$. This representation is equivalent to the structure constant form, and provides in some ways a more illuminating view of the algebra..

Bifurcating the intrinsic basis product structure definition, partial definition for the algebra $\mathscr{H}_n$

From the definition just provided for an intrinsic basis set, we see there are two "things" going on which we can take on separately. One is a "numbers thing" where we must define which k for given i, j produces a non-zero s_{ijk}. The other is a "sign thing", where we choose whether this $s_{ijk} = +1$ or -1. The algebra is not defined until both are determined for all s_{ijk}.

Before continuing, restrict the definition of $\mathscr{H}$ algebras to an algebra where one basis element, specifically e_0, is defined as $e_0 = +1$. This basis element is referred to as the *scalar basis element*, and all others if any are *non-scalar basis elements*. This sets up an algebra where one dimension is given up as a representation of the field the algebra is over. This is also a declaration that our algebras of concern $\mathscr{H}_n$ are a type of *hypercomplex algebra*, understanding $\mathscr{H}_1$ is the algebra of the field defined, e.g., the algebra of real numbers. This single definition algebra requires no additional discussion.

For our "numbers thing", our goal is to partially determine the result of $(e_i * e_j)$ by having a rule, or function if you will, which specifies a single index k given the two indexes i and j. This can be expressed as k = F(i, j) where F(i, j) is some function of the given indexes. We can easily determine this function where one or both indexes are 0, for some non-zero index a:

F(0, 0) = 0
F(0, a) = a
F(a, 0) = a

At this point we further restrict our $\mathscr{H}_n$ algebra definition such that for integer $0 \le b < n$, we have F(b, b) = 0.

We now require the dimension of algebra $\mathscr{H}_n$ to be $n = 2^m$ for integer $m \ge 0$. Then our numeric basis element indexes will range from $0 \rightarrow 2^m - 1$, and can be represented by binary numbers of dimension m. This binary nature of $\mathscr{H}_n$ might get one thinking about Boolean algebra, and the recognition the F() rules above can be satisfied if F() is given by the bit-wise binary exclusive-or (xor) of the two index numbers. Xor is and will be here represented by the standard computer software operator ^.

As it turns out, we can without issue, simply and quite optimally define for all $\mathscr{H}_n$ F(i, j) = i^j for any two indexes $0 \le i, j < n$. The reason this is an optimal representation is the n integers ranging $0 \rightarrow 2^m - 1$ form a group X_n of order n under the group operation ^. Moreover, we can assign a one-to-one correspondence between every subgroup of X_n and every subalgebra of $\mathscr{H}_n$. I showed this in detail within reference [2].

Covering the rules for xor, we find a^a = 0, 0^a = a^0 = a, a^b = b^a, and a^(b^c) = (a^b)^c, the operator is fully commutative and associative. Every permutation of the order of application for a given set of variables will give the same result.

If the two indexes are $i \neq j \neq 0$, then i^j will always equal a third different non-zero result. Using i^j = k as above, taking the xor by k on both sides gives us i^j^k = 0, since k^k = 0 for any k. This is extremely important, since it tells us given any two of the three the other is single rule determinate. Bringing in now basis element e_0, the basis element set $\{e_0, e_i, e_j, e_k)$ is closed for the xor of its basis element indexes. Its indexes are then members of the xor group X_4, and the basis set $\{e_0, e_i, e_j, e_k)$ where $i \neq j \neq k \neq 0$ and i^j^k = 0 becomes the general specification for the basis element set of any $\mathscr{H}_4$ (sub)algebra. This is actually quite nice, because as we shall see $\mathscr{H}_4$ (sub)algebras are where the algebraic orientation choices for any orientable $\mathscr{H}_n$ live, and all such $\mathscr{H}_4$ subalgebras of a higher dimension $\mathscr{H}_n$ must be identified before its algebraic orientation can be classified.

Algebraic orientation: the "sign choice thing"

To complete the definition for $\mathscr{H}_n$, we must make the one of the ± sign assignments after determining the "numbers thing" for $(e_i * e_j) = \pm e_{i\wedge j}$. We will find some sign choices are fixed for the family of algebras $\mathscr{H}_n$ while others allow choices. Define the full set of sign assignments where there is a choice as the *algebraic orientation* of the algebra $\mathscr{H}_n$.

We find for any xor group X_n for $n \geq 2$, the integers 0 and any single member $m \neq 0$ form an order two (sub)group, and thus after algebraic orientation form a two-dimensional (sub)algebra $\mathscr{H}_2$. Our sole sign choice here is the sign assignment on $(e_m * e_m) = \pm e_{m\wedge m} = \pm e_0$. With the ultimate goal of subsuming the division algebras, the obvious choice is to singularly define for any $0 < m < n$, $(e_m * e_m) = -e_0$. This declares that every $\mathscr{H}_2$ (sub)algebra is the algebra of complex numbers.

$\mathscr{H}_n$ starts having multiple non-scalar basis elements for $n \geq 4$. Within these algebras we call all products $(e_a * e_b)$ where $a \neq b \neq 0$ *oriented products*, where we have the binary ± sign choice in $(e_a * e_b) = \pm e_{a\wedge b}$. All fixed definition products where one or both of a and b are 0, or a = b, which define the $\mathscr{H}_1$ and $\mathscr{H}_2$ subalgebras of any $\mathscr{H}_n$, are called *not oriented products*.

As stated above, any X_4 (sub)group can be uniquely determined by any two of its non-zero members. We desire the same uniqueness for our algebraic orientation choice by devising a rule where for any single oriented basis product, the free binary ± choice for $(e_a * e_b) = \pm e_{a\wedge b}$ determines the algebraic orientation for all oriented basis element product rules within the $\mathscr{H}_4$ related to the X_4 that indexes a and b specify. Turned around the other way, the single algebraic orientation assigned to a particular $\mathscr{H}_4$ can then be associated with each of its oriented basis element products.

W. R. Hamilton of course determined the algebra satisfying these requirements when he devised Quaternion Algebra, and we will take this as the definition of any $\mathscr{H}_4$. We must then declare for any $\mathscr{H}_n$ for $n \geq 4$ that every oriented product anti-commutes, a statement that $(e_a * e_b) = -(e_b * e_a)$. The specification of algebraic orientation is complete by adding to the requirement indexes $a \neq b \neq c \neq 0$ and a^b^c = 0, the following cyclic shift equivalences exist where the same +1 or –1 choice is used for each ±, thus the choice of one determines the others.

$(e_a * e_b) = \pm e_c$
$(e_b * e_c) = \pm e_a$
$(e_c * e_a) = \pm e_b$

From the algebra defining structure constants perspective, this is a statement that $s_{abc} = s_{bca} = s_{cab}$. The full set of

order permutations on the set {a, b, c} is described by the symmetric group S_3. Its three even permutations are the cyclic shifts just presented, and its odd permutations are their reversed orders, flipping the even permutation sign choice. So, we have for the six members of the group S_3 operating on the index set {a, b, c} defining the six oriented products defined by a particular $\mathscr{H}_4$

$s_{abc} = s_{bca} = s_{cab} = -s_{cba} = -s_{bac} = -s_{cba}$

With this set of equalities for any $\mathscr{H}_4$ subalgebra, the algebraic orientation of any $\mathscr{H}_4$ (sub)algebra $\{e_0, e_a, e_b, e_c\}$ is given by the free binary choice of $s_{abc} = +1$ or $s_{abc} = -1$. This all tells us the two basis indexes of any oriented product uniquely specifies the $\mathscr{H}_4$ (sub)algebra the product belongs to, we can assign an orientation description unique to this particular $\mathscr{H}_4$, and it can be commonly associated with all six oriented products this $\mathscr{H}_4$ defines.

When $n \geq 8$, every oriented product resides in a single $\mathscr{H}_4$ subalgebra of this $\mathscr{H}_n$, and the set of all $\mathscr{H}_4$ subalgebras fully cover all possible basis element product descriptions in a way that the algebraic orientation of one does not confront the algebraic orientation of another. This leads us to a very important observation.

Rule: The algebraic orientation options for any $\mathscr{H}_n$ where $n \geq 4$ are fully given by the free choice of algebraic orientation for each of its $\mathscr{H}_4$ subalgebras.

This tells us if $\mathscr{H}_n$ has q $\mathscr{H}_4$ subalgebras, $\mathscr{H}_n$ specifies a family of 2^q different algebraic orientations of similarly structured n dimensional algebras.

The full definition of the family of hypercomplex algebras $\mathscr{H}_n$

$\mathscr{H}_n$ is defined on algebraic elements of dimension $n = 2^m$ where m = 0, 1, 2, 3, … where basis element e_0 is the real number +1. All basis element products within the sequentially indexed basis element set e_x for $0 \leq x < n$ are specified by the following rules

$(e_0 * e_a) = (e_a * e_0)$ for any index $0 < a < n$
$(e_0 * e_0) = +e_0$
$(e_b * e_b) = -e_0$ for any index $0 < b < n$
$(e_a * e_b) = -(e_b * e_a)$ for $a \neq b$ and $0 < a, b < n$
$(e_a * e_b) = s_{ab(a\hat{}b)}\, e_{a\hat{}b}$ for $a \neq b$ and $0 < a, b < n$, structure constant $s_{ab(a\hat{}b)}$ = algebraic orientation +1 or –1
$s_{abc} = s_{bca} = s_{cab} = -s_{cba} = -s_{acb} = -s_{bac}$ for $a \neq b \neq c \neq 0$ and $a\hat{}b\hat{}c = 0$
The algebraic orientation of any $\mathscr{H}_n$ is specified by a free algebraic orientation choice $s_{abc} = +1$ or –1 for each of its $\mathscr{H}_4$ subalgebras defined by basis elements indexed $a \neq b \neq c \neq 0$ and $a\hat{}b\hat{}c = 0$, if any.

Algebraic element conjugation is the standard definition where the coefficients attached to all non-scalar basis elements are negated. Define the conjugate of $\mathbf{A} = \underline{\mathbf{A}}$. For any $\mathbf{A}$, the product $\mathbf{A} * \underline{\mathbf{A}}$ = the sum of squares of the coefficients scaling each basis element. The norm of $\mathbf{A} = |\mathbf{A}|$ is given by $(\mathbf{A} * \underline{\mathbf{A}})^{1/2}$.

We find $\mathscr{H}_n$ to be the generalization of Cayley-Dickson algebras. The Cayley-Dickson dimension doubling process uses an obtuse basis element multiplication rule to generate only a single algebraic orientation from a particular doubled dimension algebra. On the other hand, $\mathscr{H}_n$ has simple basis element product rules and a simple methodology to assign all possible algebraic orientation choices.

The algebra family $\mathscr{H}_8$

Group theory tells us X_8 has seven X_4 subgroups, meaning $\mathscr{H}_8$ has seven $\mathscr{H}_4$ subalgebras, and the algebraic orientation of each is a free choice. This gives us $2^7 = 128$ different algebraic orientations for $\mathscr{H}_8$. We rightfully expect to find all possible Octonion Algebra algebraic orientations within this set. A relatively simple computer program doing a proof by exhaustion on the limited set of options tells us only 16 of 128 algebraic orientations demonstrate the required Octonion norm composition rule $|\mathbf{A} * \mathbf{B}| = |\mathbf{A}|\,|\mathbf{B}|$ for any two algebraic elements.

In reference [3] I demonstrated each of the other 112 $\mathscr{H}_8$ non-Octonion algebraic orientations I called Broctonion Algebras are seen to be one $\mathscr{H}_4$ (Quaternion) subalgebra algebraic orientation off of a proper Octonion algebraic orientation. Unlike Octonions, they are not division algebras, producing 24 primitive zero divisors. With 16 proper Octonion orientations, each with seven Quaternion subalgebras, this gives the total of 16(1+7) = 128 covering all $\mathscr{H}_8$ algebraic orientations.

Because the set of division algebras are of superior importance, we will focus in on the algebraic orientation characteristics of this subset of $\mathscr{H}_8$ for the remainder.

Octonion Algebra algebraic orientations

Each non-scalar basis element of $\mathscr{H}_8$ shows up in three of seven $\mathscr{H}_4$ subalgebras, specifying all combinations of unlike non-scalar basis products with each of the other six not e_0, two up for each subalgebra. We defined the $\mathscr{H}_4$ structure constant algebraic orientation choices such that cyclic shifts of its index set indicate identical orientations. We are free then to take the algebraic orientation defining structure constants for the three $\mathscr{H}_4$ subalgebras any given non-scalar basis element belongs to, and cyclically shift their three indexes if necessary to place the common index in the central position, without confronting their assigned algebraic orientations.

Doing this, we can observe the set of 16 proper Octonion algebraic orientations indicate two different observable structures. Using any choice of member non-scalar common basis element index, eight of 16 Octonion algebraic orientations will indicate the three non-scalar basis element indexes of a fourth $\mathscr{H}_4$ subalgebra only on the right side of the three triplet structure constants, and the other eight will indicate the same but only on the left side.

To clarify this, jumping ahead a bit to the specification of Octonion algebraic orientation **R0**, the first three are set for common index 4 being in the central position. We have

s_{642}
s_{541}
s_{743}

The right side indexes of these three are {2, 1, 3} which are the indexes of a fourth Quaternion triplet. The left side indexes are {6, 5, 7} which are not triplets of a Quaternion subalgebra. Choosing instead a common index of 1 we have

s_{541}
s_{123}
s_{761}

Cyclically shifting to place 1 centrally we have

s_{415}

s_{312}
s_{617}

Once again, the right side indexes are {5, 2, 7}, the triplet of another Quaternion subalgebra and the left side {4, 3, 6} are not.

This is a consistent structural difference between halves of the full set of 16 proper Octonion orientations despite the fact all are proper Octonion. The right and left sided structures cannot possibly be different representations of equivalent basis element product tables, Right Octonion algebraic orientations *are not isomorphic* with Left Octonion algebraic orientations. We should then specify which chiral type we are dealing with when discussing an Octonion algebraic orientation.

We can do this by assigning a number, one of 0 to 7 attached to the declaration R for right and L for left. Which number to attach to which algebraic orientation can be optimally chosen by first identifying fundamental Quaternion algebraic orientation negations that do not move us from a proper Octonion into a Broctonion Algebra orientation. Two fundamental morphs achieve this. We can negate the four Quaternion subalgebra orientations excluding any single basis element, or negate the three Quaternion subalgebra orientations that include any single basis element, and any number of combinations thereof. The four-morph is a map within Right or within Left proper Octonion algebraic orientations, and the three-morph is a map between Right and Left proper Octonion algebraic orientations.

We can arbitrarily choose any one of the Right proper Octonion algebraic orientations and declare it **R0**. The optimal choice for enumerating the remaining seven is to pick for **Ra** the four-morph on **R0** that excludes the basis element indexed a. The composition of the three-morph and four-morph using the same included/excluded basis index is the negation of all seven Quaternion subalgebra algebraic orientations. We can use this involution to map between **Lb** and **Rb**, allowing us to define all algebraic orientations for enumerated Left Octonions from and consistent with our chosen optimal enumerations for Right Octonions.

My algebraic orientation enumerations for $\mathscr{H}_8$ proper Octonions are defined by the following structure constants where each structure constant equals +1

R0:	s_{642}	s_{541}	s_{743}	s_{123}	s_{572}	s_{761}	s_{653}
R1:	s_{246}	s_{541}	s_{347}	s_{123}	s_{275}	s_{761}	s_{356}
R2:	s_{642}	s_{145}	s_{347}	s_{123}	s_{572}	s_{167}	s_{356}
R3:	s_{246}	s_{145}	s_{743}	s_{123}	s_{275}	s_{167}	s_{653}
R4:	s_{642}	s_{541}	s_{743}	s_{321}	s_{275}	s_{167}	s_{356}
R5:	s_{246}	s_{541}	s_{347}	s_{321}	s_{572}	s_{167}	s_{653}
R6:	s_{642}	s_{145}	s_{347}	s_{321}	s_{275}	s_{761}	s_{653}
R7:	s_{246}	s_{145}	s_{743}	s_{321}	s_{572}	s_{761}	s_{356}
L0:	s_{246}	s_{145}	s_{347}	s_{321}	s_{275}	s_{167}	s_{356}
L1:	s_{642}	s_{145}	s_{743}	s_{321}	s_{572}	s_{167}	s_{653}
L2:	s_{246}	s_{541}	s_{743}	s_{321}	s_{275}	s_{761}	s_{653}
L3:	s_{642}	s_{541}	s_{347}	s_{321}	s_{572}	s_{761}	s_{356}
L4:	s_{246}	s_{145}	s_{347}	s_{123}	s_{572}	s_{761}	s_{653}
L5:	s_{642}	s_{145}	s_{743}	s_{123}	s_{275}	s_{761}	s_{356}
L6:	s_{246}	s_{541}	s_{743}	s_{123}	s_{572}	s_{167}	s_{356}
L7:	s_{642}	s_{541}	s_{347}	s_{123}	s_{275}	s_{167}	s_{653}

The order 8 Hadamard Group and the enumeration of proper Octonion algebraic orientations

All non-oriented basis element products have no connection to any proper Octonion algebraic orientation specification, so are consistently defined across all Right and Left orientations. The oriented products take on the algebraic orientation associated with the $\mathscr{H}_4$ algebra they belong to. We can separately chart both against Right and Left enumerations from the above definitions by using table entry +1 representing a not oriented products or if the $\mathscr{H}_4$ for a particular Right or Left Octonion enumeration is oriented the same as its same X_4 $\mathscr{H}_4$ subalgebra in some reference Octonion Right or Left enumeration. We would then use –1 if the algebraic orientation is the negation of that reference orientation. We will use our **R0** and **L0** algebraic orientations for our references. Doing so produces the following tables

𝕆	**R0**	**R1**	**R2**	**R3**	**R4**	**R5**	**R6**	**R7**
[0]:Not oriented	+1	+1	+1	+1	+1	+1	+1	+1
Triplet[1]: s_{642}	+1	–1	+1	–1	+1	–1	+1	–1
Triplet[2]: s_{541}	+1	+1	–1	–1	+1	+1	–1	–1
Triplet[3]: s_{743}	+1	–1	–1	+1	+1	–1	–1	+1
Triplet[4]: s_{123}	+1	+1	+1	+1	–1	–1	–1	–1
Triplet[5]: s_{572}	+1	–1	+1	–1	–1	+1	–1	+1
Triplet[6]: s_{761}	+1	+1	–1	–1	–1	–1	+1	+1
Triplet[7]: s_{653}	+1	–1	–1	+1	–1	+1	+1	–1

𝕆	**L0**	**L1**	**L2**	**L3**	**L4**	**L5**	**L6**	**L7**
[0]:Not oriented	+1	+1	+1	+1	+1	+1	+1	+1
Triplet[1]: s_{246}	+1	–1	+1	–1	+1	–1	+1	–1
Triplet[2]: s_{145}	+1	+1	–1	–1	+1	+1	–1	–1
Triplet[3]: s_{347}	+1	–1	–1	+1	+1	–1	–1	+1
Triplet[4]: s_{321}	+1	+1	+1	+1	–1	–1	–1	–1
Triplet[5]: s_{275}	+1	–1	+1	–1	–1	+1	–1	+1
Triplet[6]: s_{167}	+1	+1	–1	–1	–1	–1	+1	+1
Triplet[7]: s_{356}	+1	–1	–1	+1	–1	+1	+1	–1

The two tables have identical arrays of +1 and –1 entries, which can be seen to form an 8x8 Hadamard Matrix. This matrix ends up in what is referred to as systematic format because we used the particular enumeration rule assigning m for **Rm** and **Lm**, and used the particular enumeration order of the row labelling structure constants.

Hadamard matrices demonstrate a composition of two rows(columns) where the entries in common column(row) are multiplied to form the composition result same column(row). This composition will always result in one of the matrix rows(columns). Clearly the “Not oriented” row and the **R0**/**L0** columns define the identity member for these compositions. The composition is clearly commutative and associative. Staying with only row compositions for our two tables, we have an identity member row, each row composed with itself

yields this identity row so each member row has its inverse in the set of rows, and the row set is closed for the composition of rows. The rows then form a group under the stated composition as the group operation, the order 8 Hadamard Group.

Taking the [x: 0 to 7] indexes in the table row labels as our row enumerations and using the operator • to specify the composition of rows, we find row[a] • row[b] = row[a^b]. Once again, the xor operation comes into play. This only is the case because the Quaternion subalgebra triplet algebraic orientations were enumerated as they were, optimally since it leads to this beautiful structure. Hadamard matrices appear in many places within division algebras, I covered this in some detail in reference [4].

These enumerations also have another remarkable feature. Take index m ≠ 0 to select Triplet[m]. If we take the set {a, b, c} to be the triplet indexes specified by Triplet[m], we will find that index m is a member of Triplet[a], Triplet[b] and Triplet[c].

Now Octonion Algebra is not fully associative for *, so it cannot directly take on the characteristics of a group since group theory requires associativity of the group operation on its members. But as we have just demonstrated, splitting * in two, to the "numbers thing" and "sign thing", we find both are individually represented by groups for the largest dimension division algebra, and hence all division algebras.

Multiple * products and the composition of algebraic orientation indicators

When we form the product **A** * **B** of two Octonion algebraic elements, the result will be a sum of product terms that are either oriented or not oriented, scaling the same resultant basis element. Our fundamental definition of an algebra says the different types of coefficients simply add without consideration as to whether they are oriented or not as long as they scale the same basis element. My position on this is if you plan on doing mathematical physics with Octonion Algebra, you must track and maintain the orientation characteristics of every in-progress product term coefficient.

We can do this by appending an *algebraic orientation indicator* to every coefficient, and agree to extend our fundamental algebra rules to state two coefficients scaling the same basis element only add or can be reduced (e.g., a trig reduction) if they have identical algebraic orientation indicators. At the end of the process, we can form sets of product terms with the same algebraic orientation indicator as an important partition of the result.

Each product term in **A** * **B** is of the form $A_r\, e_r * B_s\, e_s$ and the product of the two basis elements tells us if this is an oriented or not oriented product. If oriented, indexes r and s give us the $\mathscr{H}_4$ subalgebra in play, and the orientation of that $\mathscr{H}_4$ subalgebra in our particular choice of using either **Rn** or **Ln** Octonion Algebra used gives us the sign of the basis element product result $e_{r\hat{}s}$.

It would make sense then to make the algebraic orientation indicator a function of the [] indexes in the two tables above. The fact that we pick the same one of eight indexes independent of Right or Left considerations is a nod to the fact they are not isomorphic representations of Octonion Algebra, but all Right Octonions are isomorphic yet different, and all Left Octonions are isomorphic yet different. The index rule resides within isomorphic sets.

When we form multiple * products, each final product term has its separate product history, including the number of oriented products of two basis elements along the way, or more specifically whether this number is odd or even. To track this odd/even parity along with algebraic orientation, we will take the [] table index indicated by the indexes of the two basis elements multiplied, and multiply it by two, or from the binary number perspective shift the bits 1 to the left, freeing up the least significant bit for our parity designation.

So, we have 16 different algebraic orientation indicators, and without too much confusion with three index structure constants we can label them s_0 through s_{15}, that is index 0 through fifteen. This so far begs the question; how do we multiply algebraic orientation indicators? Without too much head scratching with the understanding the row composition result is indexed by the bit-wise xor of the row indexes and parity combinations also are a single bit xor we have the simple rule for algebraic orientation indicator products

$s_a\ s_b = s_{a\hat{}b}$

By virtue of the fact the compositions form a group, we can do as many repeated products as we please.

We can now write the general product term result including initial algebraic orientation indicators for oriented or not oriented constituent basis element products as follows. First, we must appreciate we always work within a single Octonion algebraic orientation, and it will determine the sign of the basis element product result based on its rules and the basis element order, so it must be included in the result indicating only the resultant basis element product sign. We have the following product term rule for oriented or not oriented $e_u * e_v$ for each step through a product term product history.

$A_u\ s_a\ e_u * B_v\ s_b\ e_v$
=
$s_{uv(u\hat{}v)}\ A_u\ B_v\ s_{a\hat{}b}\ e_{u\hat{}b}$ for not oriented product ($e_u * e_v$)
$s_{uv(u\hat{}v)}\ A_u\ B_v\ s_{a\hat{}b\hat{}1}\ e_{u\hat{}b}$ for oriented product ($e_u * e_v$)

There is no loss in generality defaulting our calculations to **R0** when we maintain the algebraic orientation indicators. They effectively sieve the results into sets whose members either all change sign or all do not change sign if we were to use a different Octonion Algebra orientation. The indicators are agnostic to whether we are using a Right Octonion or Left Octonion orientation. We must however have results that properly track what would happen if we changed between Right and Left. This is where the parity partition comes into play. The map between **Rm** and **Lm** negates all oriented products. This negation will only change the sign of product terms whose product history has an odd number of oriented products. Sieving into separate odd/even parity bins takes care of this.

Whether or not the product terms within a given algebraic orientation indicator set change signs is deterministic for any change in applied Octonion orientation. If required, the results using any other Octonion orientation can simply be determined from the **R0** results.

Algebraic Invariance and Algebraic Variance

If a product term in a final result ends up with an algebraic orientation indicator s_0 or s_1, it means the result is independent of $\mathscr{H}_4$ subalgebra orientations. If we changed Octonion orientations within the Right Octonion set or within the Left Octonion set, these product terms would not change sign. If the final indicator is s_1, only changes between Right and Left orientations would induce a sign change.

Every final product term with algebraic orientation indicator s_0 remains unchanged for any change in Octonion orientation applied. These product terms are thus *algebraic invariants*. Each of the other 15 algebraic orientation indicators flag product terms that are *algebraic variants*.

From the mathematical physics perspective, we were given a heads up on why this delineation is significant by the cross-product rule explaining the deflection of a charged particle moving through a magnetic field. The particle velocity is in a fixed observable direction as is the force causing the deflection. Both are observables that must be independent of the definition of the cross-product, which is an oriented product. This forces the

magnetic field to be an oriented vector such that the cross-product of the velocity and field vectors is an orientation invariant.

More than simple right- or left-handed cross-product rules, Octonion Algebra has 16 different algebraic orientations, and there is absolutely nothing that can lead us to believe only one orientation is physical. We must then construct Octonion mathematical physics in a way that all observables end up as algebraic invariants. If the explanation of something observable ends up as an Octonion algebraic variant, one should question the legitimacy of the approach.

Another consideration is there may be structures that seem to be required yet do not lend themselves to direct experimental observation, like fractional charge quarks. If describable within an Octonion setting, one might expect them to appear in some algebraic variant subset along with the explanations for observables in the algebraic invariant subset.

Additionally, any final result can be made into an algebraic invariant by requiring the sum of all members in each algebraic variant subset to be 0. Then, when the Octonion orientation is changed up, it can only result in a sign change to 0, i.e., no change. I have called these *homogeneous equations of algebraic constraint*. Without the typical experimental observation suggesting an ansatz mathematical theory path, due to their algebraic variant character, this would provide an additional paradigm for theoretical development.

Conclusions

There are some that claim all Octonion variations are isomorphic, so there is no need to consider the differences. Besides being a patently false statement, it is excessively naïve to think so. Right and Left Octonion algebraic orientations are not isomorphic, they have the defined structural difference presented above. One might claim this structure is unimportant since the only thing that matters is all forms are Octonion, all forms are division algebras, and that is all that matters. A person taking this path does so at their own peril. They may never reach the conclusions they seek, or worse may develop a methodology that falls apart when Octonion definition variations are brought in.

Even the differences within isomorphic Right Octonion algebraic orientations and isomorphic Left Octonion algebraic orientations are critical, because consideration of the differences within provides the entrée for the extremely important theoretical tool of group theory as twice shown above. These group theoretical aspects will most likely have ramifications beyond the scope of this document.

If you wish to use Octonion Algebra, you should have a complete understanding of it. What is presented here is meat on the bone.

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