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# BEYOND GÖDEL

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## ABSTRACT

In 1931, Gödel wrote a landmark paper exhibiting a formal system that contains statements that cannot be proven. However, the deficiency of formal systems is much greater. In this paper starting with the same set, we develop two logically valid arguments simultaneously, proving a statement to be both true and false. This result is possible because we validly draw a false conclusion from true premises. This result is profound. It is no wonder that the most lucid arguments still sometimes result in hung juries, and earnest people can disagree on the most fundamental issues. Truth is a much deeper concept than logical validity. Inconsistency in mathematical systems has a long heritage. From the earliest times division by zero led to many contradictions, which could only be rectified by banning division by zero. Gottlob Frege developed the first formal system of arithmetic at the beginning of the twentieth century. However, it contains a contradictory self-referential set. Gödel proved that any formal system containing arithmetic (although not necessarily inconsistent) is incomplete.

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## 1 Introduction

Inconsistency in mathematical systems has a long heritage. From the earliest times division by zero led to many contradictions [1], which could only be rectified by banning division by zero.

$$\text{(start with)} \quad x = 1 \tag{1}$$

$$\text{(multiply by } x) \quad x^2 = x \tag{2}$$

$$\text{(subtract 1)} \quad x^2 - 1 = x - 1 \tag{3}$$

$$\text{(divide by } x - 1 \text{ or } 0) \quad x + 1 = 1 \tag{4}$$

$$\text{(subtract 1 and end with)} \quad x = 0 \tag{5}$$

Gottlob Frege developed the earliest formal system of arithmetic at the beginning of the twentieth century. However, it contained a contradictory self-referential set [2]: the set of all sets that do not contain themselves. Does it contain itself? If it does, then it does not. If it does not, then it does. Gödel proved that any formal system containing arithmetic (though not necessarily inconsistent) is incomplete [3].

We make a logically valid argument that sets of rationals  $(0, a)$  with  $a < 100$  have largest elements. Many people will find it repugnant that a false statement has been validly deduced from previous true statements. However, perhaps people will become more tolerant when they realize that in discussing anything there may be a logically valid argument using the same premises arriving at an opposite conclusion. This also shows that truth is a much deeper concept than logical validity.

## 2 Inconsistency

For rational numbers  $a$  in  $(0, 100)$  let the collection of  $Ra$  sets be  $\{y \text{ is a rational number} \mid 0 \leq y < a\}$

### Argument#1: No $Ra$ set contains a largest element.

- 1) Suppose there is a largest element  $b$  in some individual  $Ra$ .
- 2)  $b < (a + b)/2 < a$ .
- 3) Let  $c = (a + b)/2$ .
- 4) Then  $c$  is in  $Ra$  and  $b < c$ .
- 5) Therefore, no  $Ra$  set contains a largest element.

### Argument#2: Each $Ra$ set contains a largest element.

- 1) Below each  $Ra$  set for all rationals  $0 < x < a$  is a collection of  $Rx$  subsets  $\{z \text{ is a rational number} \mid 0 \leq z < x\}$ .
- 2) Each chosen  $Ra$  set and its collection of  $Rx$  subsets comprise a nested descending set hierarchy and for every  $x < a$  there is an  $d = (a + x)/2$  in  $Ra$  that is not in  $Rx$ .  $\lim_{x \rightarrow a} (a + x)/2 = a$  so  $d$  can get arbitrarily close to  $a$ .
- 3) Since the chosen  $Ra$  set contains the  $x$  indices missing from the  $Rx$  subsets below it in the set hierarchy, there is at least one  $Ra$  set element not contained in the  $Rx$  subsets.
- 4) Let  $e$  and  $f$  be two elements of the chosen  $Ra$  set with  $e > f$ .
- 5) Substituting  $e$  for  $x$ ,  $f$  is an element of  $Re$ : a proper subset of the chosen  $Ra$  set.
- 6) By Steps 4) and 5), there is at most one  $Ra$  element missing from all its  $Rx$  subsets.
- 7) By Steps 3) and 6), the chosen  $Ra$  set contains a largest element  $b$  not in any  $Rx$  subset below it in the nested descending set hierarchy.
- 8) There is no  $c = (a + b)/2$  as in Argument#1. It would be a second element not in an  $Rx$  subset below the chosen  $Ra$  set, which we know from step 6) is not possible.

## 3 Conclusion

Argument#1 is correct and its conclusion is true. In argument#2 the first two statements of are true, but the second half of the third statement is false. No infinite set of rationals in an open interval can have a largest element. Most people dismiss argument#2. They are unable to accept that a false statement can be a valid logical deduction drawn from true statements directly proceeding it. However, based on the hierarchical structure of the chosen  $Ra$  set and its  $Rx$  subsets, it is a valid logical deduction that the chosen  $Ra$  set contains an element not found in any of the  $Rx$  subsets. Moreover, if there is not a largest element in each  $Ra$  set, the entire hierarchy collapses.

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