Gaps and Overlappings

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Abstract

In the first part, we investigate the tiling of the plane by convex polygons, and we introduce many constants. At the end, we calculate one. We provide an example, where we cover the plane with convex 8-gons.
In a second part, we take other curves and convex polygons.

Keywords and phrases: polygon; plane; tiling

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1 Introduction

It is well-known that we can tile the plane $\mathbb{R}^2$ with n-gons, where n is a natural number larger than $\mathsf{2}$, see [1], p. 11. If we restrict our efforts to convex polygons, in most cases it is impossible to cover the plane completely without overlappings. It is known that we can tile the plane with squares and regular $\mathsf{6}$-gons. We also can tile the plane with convex $\mathsf{5}$-gons, see [2]. For natural numbers larger than $\mathsf{6}$ we believe that it is impossible to tile the plane completely with convex n-gons. Either we have to leave gaps or some polygons overlap to cover $\mathbb{R}^2$ completely.

For additional information, see [3].

We believe that it is useful to repeat the definition of a *simple polygon*.
A simple polygon with n vertices consists of n different points of the plane (x_1, y_1), (x_2, y_2), $\ldots$ $(x_{\mathsf{n}-1}, y_{\mathsf{n}-1})$, $(x_{\mathsf{n}}, y_{\mathsf{n}})$, called *vertices*, and the straight lines between (x_i, y_i) and (x_{i+1}, y_{i+1}) for $1 \leq i \leq \mathsf{n}-1$, called *edges*. Also the straight line between $(x_{\mathsf{n}}, y_{\mathsf{n}})$ and (x_1, y_1) belongs to the polygon. We demand that it is homeomorphic to a circle, and that there are no three consecutive collinear points $(x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2})$ for $1 \leq i \leq \mathsf{n}-2$. Also the three points $(x_{\mathsf{n}}, y_{\mathsf{n}}), (x_1, y_1), (x_2, y_2)$ and $(x_{\mathsf{n}-1}, y_{\mathsf{n}-1}), (x_n, y_n), (x_1, y_1)$ are not collinear.
We call this just described simple polygon an n-gon.

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Let r be any real numbers larger or equal to 1, and k a natural number larger than 2.

Note that we work exclusively with convex curves. With 'Polygon' we always mean a convex simple polygon. With 'k-gon' we always mean a convex k-gon.

We define a set of simple polygons.

Definition 1.1.
We define $\mathsf{k}-\mathsf{Polygons}$ as the class of k-gons.

Remark 1.2. In the next definition, we define some constants. They are actual percentages, but we prefer numbers from 0 to 1.

The area of $\mathbb{R}^2$ is regarded as 1.

We try to tile $\mathbb{R}^2$ with simple polygons. For $\mathsf{k} > 6$, either we fix tiles without any overlapping, or we cover $\mathbb{R}^2$ completely, where there may be overlappings.

Definition 1.3.
Let $\mathsf{k}\ \mathsf{gap}(\mathsf{r})$ be the supremum of the covered part of $\mathbb{R}^2$. The polygons do not overlap. We use elements from the class $\mathsf{k}-\mathsf{Polygons}$, where the quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.
Let $\mathsf{k}\ \mathsf{overlap}(\mathsf{r})$ be the infimum of the part of $\mathbb{R}^2$ which is covered by polygons from the class $\mathsf{k}-\mathsf{Polygons}$ at least twice, where $\mathbb{R}^2$ is covered completely, and the quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.
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Conjecture 1.4. $\mathsf{k}\ \mathsf{gap}(\mathsf{r}) = 1$ and $\mathsf{k}\ \mathsf{overlap}(\mathsf{r}) = 0$ holds for all k for suitable numbers r.

For polygons with 8 vertices, see Proposition 2.3.

Let n be a natural number larger than 2.

Definition 1.5. A *cyclic polygon* is defined as a polygon such that all vertices are on a circle.
We define an *elliptical polygon* as a polygon such that all vertices are on an ellipse.
We define a *convex Cassini polygon* as a polygon such that all vertices are on a convex Cassini curve.
We call a *regular* n*-gon* a regular polygon which has precisely n vertices.
We call a *cyclic* n*-gon* a cyclic polygon which has precisely n vertices.
We call an *elliptical* n*-gon* an elliptical polygon wich has precisely n vertices.
We call a *convex Cassini* n*-gon* a convex Cassini polygon wich has precisely n vertices.

We define a set of shapes.

Definition 1.6. $\mathsf{shapes} := \{\textit{circle}, \textit{ellipse}, \textit{ convex Cassini curve }\}$

Definition 1.7. Let Curves be the set of curves of a shape from the set shapes.
Let $\mathsf{gap}_{\mathsf{XXX}}(\mathsf{r})$ be the supremum of the covered part of $\mathbb{R}^2$, where we use curves from the set Curves of shape XXX, where XXX is an element of shapes. The curves do not overlap. The quotient of the arc lengths of two curves is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.
Let $\mathsf{overlap}_{\mathsf{XXX}}(\mathsf{r})$ be the infimum of the part of $\mathbb{R}^2$ which is covered at least twice, where we use curves from the set Curves of shape XXX, where XXX is an element of shapes, and $\mathbb{R}^2$ is covered completely. The quotient of the arc lengths of two curves is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.

Definition 1.8.
We define $\mathsf{k\ reg}(\mathsf{r})$ as the supremum of the covered part of the plane, where we use regular k-gons. The polygons do not overlap. The quotient of two edges of the used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.
We define $\mathsf{k\ overlap\ reg}(\mathsf{r})$ as the infimum of the part of the plane which is covered at least twice, where we use regular k-gons. The quotient of two edges of the used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$, and $\mathbb{R}^2$ is covered completely.
We define $\mathsf{k\ cyclic}(\mathsf{r})$ as the supremum of the covered part of the plane, where we use cyclic k-gons. The polygons do not overlap. The quotient of two edges of one or two used polygons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.
We define $\mathsf{k\ overlap\ cyclic}(\mathsf{r})$ as the infimum of the part of the plane which is covered at least twice. We use cyclic k-gons. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$, and $\mathbb{R}^2$ is covered completely.
We define $\mathsf{k\ elliptical}(\mathsf{r})$ as the supremum of the covered part of the plane, where we use elliptical k-gons. The polygons do not overlap. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.
We define $\mathsf{k\ overlap\ elliptical}(\mathsf{r})$ as the infimum of the part of the plane which is covered at least twice. We use elliptical k-gons. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$, and $\mathbb{R}^2$ is covered completely.
We define $\mathsf{k\ Cassini}(\mathsf{r})$ as the supremum of the covered part of the plane, where we use convex Casssini k-gons. The polygons do not overlap. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$.
We define $\mathsf{k\ overlap\ Cassini}(\mathsf{r})$ as the infimum of the part of the plane which is covered at least twice. We use convex Cassini k-gons. The quotient of two edges of one or two used k-gons is in the interval $\left[\frac{1}{\mathsf{r}}, \mathsf{r}\right]$, and $\mathbb{R}^2$ is covered completely.

Remark 1.9. Note that $\mathsf{r} = 1$ means that all edges of the polygons or all curves, respectively, have the same length.

Remark 1.10. The used polygons or curves, respectively, can not be arbitrarily small since r is a positive number.

We suggest the name 'The first Thuerey constant' for $\mathsf{5\ reg}(1)$, and for $\mathsf{5\ overlap\ reg}(1)$ we suggest 'The second Thuerey constant'.

2 Propositions

Proposition 2.1. The following equations hold for all r.

$$\mathsf{3\ reg}(\mathsf{r}) = \mathsf{4\ reg}(\mathsf{r}) = \mathsf{6\ reg}(\mathsf{r}) = \mathsf{3\ gap}(\mathsf{r}) = \mathsf{4\ gap}(\mathsf{r}) = \mathsf{6\ gap}(\mathsf{r}) = 1$$

as well as

$$\mathsf{3\ overlap\ reg}(\mathsf{r}) = \mathsf{4\ overlap\ reg}(\mathsf{r}) = \mathsf{6\ overlap\ reg}(\mathsf{r}) = \mathsf{3\ overlap}(\mathsf{r}) = \mathsf{4\ overlap}(\mathsf{r}) = \mathsf{6\ overlap}(\mathsf{r}) = 0$$

Proof. Well-known. □

Proposition 2.2. The following equations hold for all r.

$$5\ \mathsf{gap}(\mathsf{r}) = 1 \qquad \text{and} \qquad 5\ \mathsf{overlap}(\mathsf{r}) = 0$$

Hint. See the 'Cairo Tiling' in [2].

Proposition 2.3. It holds

$$8\ \mathsf{overlap}(\mathsf{r}) = 0$$

for all r equal or larger than 2.

Proof. At first we tile the plane with squares of sidelength 1. Into every square we inscribe a regular octogon of sidelength $\sqrt{2} - 1$. These octogons cover a part of $\mathbb{R}^2$. We call them 'old' octogons.
Please see Figure 1. There the right square has vertices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$. Two vertices of the right octogon are $\mathbf{E}$ and $\mathbf{F}$. $\mathbf{G}$ is a vertex of the left octogon. We add another point $\mathbf{W}$ on one diagonal of the right square. We connect $\mathbf{E}$ and $\mathbf{W}$ and also $\mathbf{F}$ and $\mathbf{W}$. We add a point called $\mathbf{X}$ on one diagonal of the left square. We connect $\mathbf{E}$ and $\mathbf{X}$ and also $\mathbf{G}$ and $\mathbf{X}$. We define two more points $\mathbf{Y}$ and $\mathbf{Z}$ on the diagonals of other squares, and in this way, we generate a 'new' 8-gon. Seven of its vertices are $\mathbf{W}, \mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{E}, \mathbf{F}$, and $\mathbf{G}$. The tuples $\mathbf{X}$, $\mathbf{A}$, $\mathbf{Z}$ and $\mathbf{Y}$, $\mathbf{A}$, $\mathbf{W}$, $\mathbf{C}$ are collinear. By this way, we generate infinite many 'new' 8-gons between the 'old' 8-gons. The 'new' 8-gons cover the area which is not yet covered. With the 'new' 8-gons together with the 'old' 8-gons $\mathbb{R}^2$ is covered completely, where parts of $\mathbb{R}^2$ are covered twice. The number of 'new' 8-gons is countable. We can choose $\mathbf{W}$ such that the area of the triangle with vertices $\mathbf{E}, \mathbf{F}$, and $\mathbf{W}$ is arbitrarily small. Hence, we can choose $\mathbf{W}, \mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ such that the area of $\mathbb{R}^2$ which is covered twice is less than $\frac{1}{2}$. In the next 'new' 8-gon we can choose four vertices of the 8-gon such that the area which is covered twice is less than $\frac{1}{4}$, et cetera.
Therefore, the part of $\mathbb{R}^2$ which is covered twice can be made arbitrarily small. □

Figure 1:

We see two squares

which are partially

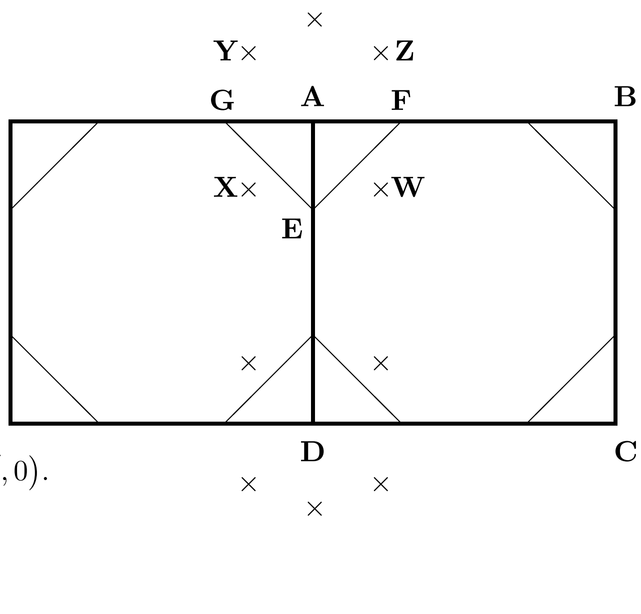
covered by regular 8-gons.

We set $\mathbf{A} = (0, 0)$ and $\mathbf{B} = (1, 0)$.

It holds

$\mathbf{E} = \left(0, \frac{1}{2} \cdot \sqrt{2} - 1\right)$ and $\mathbf{F} = \left(1 - \frac{1}{2} \cdot \sqrt{2}, 0\right)$.

$\mathbf{W}, \mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ are not fixed.



The concept can be generalized into higher dimensions.

References

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