Ideals of the Algebra II: Prime Ideal

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Abstract In [4], we have constructed an ideal with respect to a subset of binary operations. In this paper, we construct a prime ideal with respect to a nonempty subset of binary operations in an algebra. Let P and Q be two prime ideals with respect to Φ and Ψ, respectively. Then we have that $P \cup Q$ is a prime ideal if some conditions hold.

CONTENTS

1. Introduction

In [4], we have constructed an ideal with respect to a subset of binary operations. Now, we shall explore other important concepts. We assume that all binary operations are commutative and associative in this paper.

Let $\Delta := \{\beta_0, \beta_1, \ldots, \beta_n\}$ be a nonempty finite set of binary operations, $\sigma \colon \Delta \to \mathbb{Z}$ a map given by $\beta_i \mapsto 2$. Then the ordered pair $\mathfrak{A} := \langle \Delta, \sigma \rangle$ is an algebraic language. Suppose that $\boldsymbol{A}$ is an algebra of the language $\mathfrak{A}$. See notation 2.1 for more details.

Let Φ be a nonempty subset of Δ, and S a nonempty subset of $\boldsymbol{A}$. If an ideal M with respect to Φ is the minimal ideal such that $S \subseteq M$, then the ideal M is said to be generated by the subset S, see definition 3.1 for more details. Let $(S)^{\Phi}$ denote the ideal M. For all subset $S \subseteq M^{\complement}$, if the algebra $\boldsymbol{A}$ is local[4] and M is the maximal ideal, then we have $(S)^{\Phi} = \boldsymbol{A}$, see proposition 3.1 for the details.

In definition 3.2, we construct a prime ideal P with respect to a nonempty subset $\Phi \subseteq \Delta$. Let P and Q be prime ideals with respect to Φ and Ψ, respectively. We have that $\Phi \cup \Psi = \Delta, \Phi \cap \Psi \neq \varnothing$, and $P \cup Q \neq \boldsymbol{A}$ implies that $P \cup Q$ is a prime ideal with respect to $\Phi \cap \Psi$, see proposition 3.3 for more details.

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2. Preliminaries

2.1. **Universal Algebra.** Recall some definitions in universal algebra.

Definition 2.1 ([3, 5])**.** An ordered pair $\langle L, \sigma \rangle$ is said to be a (first-order) **language** provided that

- L is a nonempty set,
- $\sigma : L \to \mathbb{Z}$ is a mapping.

A language $\langle L, \sigma \rangle$ is denoted by $\mathfrak{L}$. If $f \in \mathfrak{L}$ and $\sigma(f) \geq 0$ then f is called an **operation symbol**, and $\sigma(f)$ is called the **arity** of f. If $r \in \mathfrak{L}$ and $\sigma(r) < 0$, then r is called a **relation symbol**, and $-\sigma(r)$ is called the **arity** of r. A language is said to be **algebraic** if it has no relation symbols.

Definition 2.2 ([3])**.** Let X be a nonempty class and n a nonnegative integer. Then an n-ary **partial operation** on X is a mapping from a subclass of X^n to X. If the domain of the mapping is X^n, then it is called an n-ary **operation**. And an n-ary **relation** is a subclass of X^n where $n > 0$. An operation(relation) is said to be **unary**, **binary** or **ternary** if the arity of the operation(relation) is 1, 2 or 3, respectively. And an operation is called **nullary** if the arity is 0.

Definition 2.3 ([3])**.** An ordered pair $\boldsymbol{A} := \langle A, \mathfrak{L} \rangle$ is said to be a **structure** of a language $\mathfrak{L}$ if A is a nonempty class and there exists a mapping which assigns to every n-ary operation symbol $f \in \mathfrak{L}$ an n-ary operation f^A on $\boldsymbol{A}$ and assigns to every n-ary relation symbol $r \in \mathfrak{L}$ an n-ary relation r^A on $\boldsymbol{A}$. If all operation on $\boldsymbol{A}$ are partial operations, then $\boldsymbol{A}$ is called a **partial structure**. A (partial)structure $\boldsymbol{A}$ is said to be a (**partial**)**algebra** if the language $\mathfrak{L}$ is algebraic.

Definition 2.4 (cf. [3, 5])**.** Let X be a nonempty set. Suppose that β is a binary operation on X. Then the 2-ary operation β is **associative** provided that

$$\beta(a, \beta(b, c)) = \beta(\beta(a, b), c) \text{ for every } a, b, c \in X.$$

Definition 2.5 (cf. [3, 5])**.** With the notations of definition 2.4, the 2-ary operation β is **commutative** provided that

$$\beta(a, b) = \beta(b, a) \text{ for every } a, b \in X.$$

2.2. **An Ideal with respect to Φ.** Recall the definition of an ideal in [4].

Convention 2.1. *We assume that all binary operations are associative[definition 2.4] and commutative[definition 2.5] in this paper.*

Notation 2.1. Let $\Delta := \{\beta_1, \beta_2, \ldots, \beta_n\}$ be a set of operation symbols for $n > 0$, and $\sigma : \Delta \to \mathbb{Z}$ a map which assigns to β_i 2 for all $\beta_i \in \Delta$. Then the ordered pair $\mathfrak{A} := \langle \Delta, \sigma \rangle$ is an algebraic language[definition 2.1]. It is clear that all operations of the language $\mathfrak{A}$ are binary operations. Suppose that $\boldsymbol{A}$ is an algebra[definition 2.3] of the language $\mathfrak{A}$.

Definition 2.6 ([4])**.** Let the notations be as in notation 2.1, and $\Phi \subseteq \Delta$ a nonempty subset of 2-ary operations on $\boldsymbol{A}$. A nonempty subalgebra J is said to be an **ideal with respect to Φ** provided that $\beta_i \in \Phi$ implies $\beta_i(a, x) \in J$ for all $a \in J, x \in \boldsymbol{A}$. In this case, we say that the nonempty subset $\Phi \subseteq \Delta$ makes the subalgebra J to be an ideal.

3. Ideals of the Algebra II: Prime Ideal

In [4], we have constructed an ideal with respect to a subset of binary operations. Now, we shall explore other important concepts. The intersection of ideals is an ideal. Hence we have the following definition.

Definition 3.1 (cf. [1,2,4])**.** Let notations be as in notation 2.1, S a nonempty subset of $\boldsymbol{A}$, and Φ a nonempty subset of Δ. Suppose that J is an ideal with respect to Φ in $\boldsymbol{A}$. We say that J is **generated** by S if the ideal J is the minimal ideal with respect to Φ such that $S \subseteq J$. The ideal J is **denoted** by $(S)^{\Phi}$.

Proposition 3.1. *Let the notations be as in notation 2.1, Φ a nonempty subset of Δ. Suppose that $\boldsymbol{A}$ is local[4] with respect to Φ, and M is the maximal ideal in $\boldsymbol{A}$. We have that $(S)^{\Phi} = \boldsymbol{A}$ for all nonempty subset $S \subseteq M^{\complement}$, where $M^{\complement} := \boldsymbol{A} \setminus M$.*

Proof. Immediate from definition 3.1 and [4, Definition 3.3]. □

Definition 3.2 (cf. [1,2,4])**.** Let the notations be as in notation 2.1, $\Phi \subseteq \Delta$ a nonempty subset. An ideal P with respect to Φ in $\boldsymbol{A}$ is said to be a **prime ideal with respect to Φ** provided that $P \neq \boldsymbol{A}$ and $\beta(I,J) \subseteq P$ implies that $I \subseteq P$ or $J \subseteq P$, for all $\beta \in \Phi$ and all ideals I, J with respect to Φ in $\boldsymbol{A}$.

Remark 3.1. With the same notations as in definition 3.2, if an ideal P is prime with respect to Φ, then P is prime with respect to Ψ for all nonempty subset $\Psi \subseteq \Phi$.

Proposition 3.2 (cf. [1])**.** *Let the notations be as in notation 2.1, Φ a nonempty subset of Δ. An ideal $P \neq \boldsymbol{A}$ with respect to Φ is prime if and only if $\beta(i,j) \in P$ implies that $i \in P$ or $j \in P$, for all $\beta \in \Phi$ and all $i,j \in \boldsymbol{A}$.*

Proof. Let $I := (\{i\})^{\Phi}$ and $J := (\{j\})^{\Phi}$ be ideals with respect to Φ generated by $\{i\}$ and $\{j\}$, respectively. We assume that P is a prime ideal with respect to Φ. Since β is commutative and associative, we have that $\beta(i,j) \in P$ implies $\beta(I,J) \subseteq P$, for all $\beta \in \Phi$. Thus we have $I \subseteq P$ or $J \subseteq P$. This suffices to show $i \in P$ or $j \in P$.

Conversely, We assume that $\beta(I,J) \subseteq P$ for all $\beta \in \Phi$. Hence we have that $\beta(x,y) \in P$ for every $x \in I, y \in J$, and $\beta \in \Phi$. Hence we have $x \in P$ or $y \in P$. For a $\beta \in \Phi$, if $J \nsubseteq P$ and $I \nsubseteq P$, then there are $a, b \in I$, and $a', b' \in J$ such that $\beta(a,a'), \beta(b,b') \in P$ implies that $a, b' \in P$ and $a', b \notin P$. Hence we have that $a', b \notin P$, but $\beta(a',b) \in P$. This is a contradiction. Therefore, we have $I \subseteq P$ or $J \subseteq P$. It follows that P is prime. This completes the proof. □

Proposition 3.3. *Let the notations be as in notation 2.1. Suppose that P and Q are prime ideals with respect to Φ and Ψ in $\boldsymbol{A}$, respectively. If $\Phi \cup \Psi = \Delta$, $\Phi \cap \Psi \neq \varnothing$, and $P \cup Q \neq \boldsymbol{A}$, then the subset $P \cup Q$ is a prime ideal with respect to $\Phi \cap \Psi$.*

Proof. We have proved that $P \cup Q$ is an ideal with respect to $\Phi \cap \Psi$ in [4, Proposition 3.4]. Hence it suffices to show that $P \cup Q$ is prime. For every $x,y \in \boldsymbol{A}$, and every $\beta \in \Phi \cap \Psi$, we have that $\beta(x,y) \in P \cup Q$ implies that $\beta(x,y) \in P$ or $\beta(x,y) \in Q$. And the ideals P, Q are prime. Hence we have that $x \in P \cup Q$ or $y \in P \cup Q$ by proposition 3.2. Therefore, the ideal $P \cup Q$ is prime. □

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