On the diophantine equation $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=n$

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Abstract

In this paper, we proved that there are infinitely many integers n such that $a+b+c=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=n$ has infinitely many rational solutions.

1. Introduction

The following appeared in the problems section of the March 2015 issue of the American Mathematical Monthly[1]. "Show that there are infinitely many rational triples (a,b,c) such that $a+b+c=abc=6$". The source of this equation is problem $D.16$ in Guy's book[2]. In 1996, Schinzel[4] has proved the problem $D.16$. We worked on the equation $a+b+c=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=n$, inspired by this problem. It appears that nobody has studied our problems yet. Our goal is to prove that there are infinitely many integers n such that $a+b+c=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=n$ has infinitely many rational solutions. The first attempt is to prove that there are infinitely many integers n such that $a+b+c=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=n$ has rational solutions. The first result obtained is related to the Pell equation $x^2-5y^2=4$. Next, we shall achieve our goal by extending the result of first attempt. Finally, the two kinds of computer search for $n<100$ are done using method-1 and method-2.

2. Preliminaries

Theorem 2.1. *There are infinitely many integers n such that $a+b+c=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=n$ has rational solutions.*

Proof.

$$\begin{cases} a+b+c=n & (1)\\ \dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=n & (2)\end{cases}$$

Let $a=\frac{n}{r},\ b=-\frac{n^2}{p},\ c=\frac{n^2}{q}$, and $q=p-n$. Then, from equation (1) we get

$$r=\frac{p(-p+n)}{pn+n^2-p^2}.$$

From equation (2) we get

$$p=\frac{n\pm\sqrt{5n^2+4}}{2}.$$

To get the rational solution for p, $5n^2+4$ must be a perfect square number. An equation $5n^2+4=u^2$ is known as Pell's equation, and it has infinitely many integer solutions. Hence, we can obtain an infinitely many integers n such that $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=n$. We know $\frac{1+\sqrt{5}}{2}$ is a fundamental unit of $t^2-5n^2=4$, then all integer solutions are given as follows.

$$\frac{t+\sqrt{5}n}{2}=\Big(\frac{1+\sqrt{5}}{2}\Big)^k$$

where k is even.

□

Example 1

Table 1: $t^2 - 5n^2 = 4$

k	t	n	a	b	c
4	7	3	3/10	-9/5	9/2
6	18	8	8/65	-64/13	64/5
8	47	21	21/442	-441/34	441/13
10	123	55	55/3026	-3025/89	3025/34
12	322	144	144/20737	-20736/233	20736/89
14	843	377	377/142130	-142129/610	142129/233
16	2207	987	987/974170	-974169/1597	974169/610
18	5778	2584	2584/6677057	-6677056/4181	6677056/1597

Theorem 2.2 (Nagell/Lutz Theorem)). *Suppose E is an elliptic curve over Q whose Weierstrass form has integer coefficients, and let be the discriminant of E. If $P = (x, y)$ is a rational point of finite order, then x and y are integers. Furthermore, either $y = 0$ or y^2 divides D.*

3. Main Results

Theorem 3.1. *There are infinitely many integers n such that $a+b+c = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = n$ has infinitely many rational solutions.*

Proof. Let $a = \frac{n}{r}$, $b = -\frac{mn}{p}$, $c = \frac{mn}{q}$, and $q = p - n$. Then, from equation (1) we get

$$r = \frac{p(p-n)}{-mn + p^2 - pn}.$$

From equation (2) we get

$$(n - m + mn^2)p^2 + (-n^2 + mn - mn^3)p - m^2n^3 - mn^2 = 0.$$

To get the rational solution for p, discriminant must be rational square, we have

$$V^2 = (-4n + 4n^3)m^3 + (6n^2 - 3 + n^4)m^2 + (2n + 2n^3)m + n^2.$$

Let $X = (-4n + 4n^3)mandY = (-4n + 4n^3)v$, we have the elliptic curve

$$E : Y^2 = X^3 + (6n^2 - 3 + n^4)X^2 + (4n^2(n^4 + 2n^2 - 3) + 4n^6 - 8n^4 + 4n^2)X + 4n^2(4n^6 - 8n^4 + 4n^2).$$

The discriminant D is given $16(n^4 - 10n^2 + 9)n^2$. From Theorem 1.1, we know the point $(X, Y) = ((-4n + 4n^3)n, 4n^3v(n^2 - 1))$ where $4 + 5n^2 = v^2$. By Nagell-Lutz theorem, Y coordinate is nonzero and its square does not divide D, then the point (X, Y) has infinite order. Thus, an elliptic curve E has infinitely many rational points.

□

Example 2

The case for $n = 8$: An elliptic curve is given

$$E : X^3 + 4477X^2 + 2096640X + 260112384.$$

E has rank 1 and generator is $P(X,Y)=(16128,2322432)$. We can obtain infinitely many rational points using group law. Since the rational points become very huge, only the case for $2P$ and $3P$ are shown. We have $2P$ and $3P$ using using group law.

$$\begin{aligned}
2P(X,Y) &= \Big(\frac{1776313}{576},\frac{3876444467}{13824}\Big),\\
(p,q,r) &= \Big(\frac{698855}{74898},\frac{99671}{74898},\frac{17567680}{253759}\Big),\\
(a,b,c) &= \Big(\frac{253759}{2195960},-\frac{259369}{197760},\frac{1882159}{204672}\Big),\\
3P(X,Y) &= \Big(\frac{942479720365824}{1152069489025},-\frac{918649153177999889080 32}{1236568025697538625}\Big),\\
(p,q,r) &= \Big(\frac{17327086176467}{2064829196885},\frac{808452601387}{2064829196885},\frac{34244478655559}{409062378631}\Big),\\
(a,b,c) &= \Big(\frac{3272499029048}{34244478655559},-\frac{3855363198784}{9965839809835},\frac{32822956698304}{3958724982485}\Big).
\end{aligned}$$

4. Find the numerical solutions

4.1. Method-1

Substitute $c=n-a-b$ into $a+b+c=(\frac{1}{a}+\frac{1}{b}+\frac{1}{c})$.

$$(-nb+1)a^2+(b+bn^2-b^2n-n)a+b^2-nb=0. \tag{3}$$

To get the rational solution for a, the discriminant must be a rational square.
Thus, there must exist $v\in Q$ such that

$$Q: v^2=n^2b^4+(-2n^3+2n)b^3+(-3+n^4)b^2+(-2n^3+2n)b+n^2.$$

The quartic equation is birationally equivalent to an elliptic curve.

$$E: Y^2+(-2n^2+2)YX+(-4n^4+4n^2)Y=X^3+(-4+2n^2)X^2-4n^4X+16n^4-8n^6. \tag{4}$$

$$b=\frac{2nX-8n+4n^3}{Y},$$

$$v=\frac{nX^3-12nX^2+6n^3X^2+32nX-32Xn^3-8n^3Y+8nY+12n^5X-16n^5+8n^7}{Y^2}.$$

We searched the rationa ponits of equation (4) for $1<n<100$ with height $(X)<1000000$.
For $n=1,2,3,4,5,6,7,9,11,12,13,15,16,17,19,20,26,27,28,30,31,33,36,37,38,40,41,42,44,46,49,$ $50,51,52,53,54,57,60,62,65,66,67,71,72,74,76,77,78,80,82,84,86,87,88,89,91,94,95,96,98,99,$
E has rank 0 and torsion point of order 6 $(X,Y)=(2n^2,0)$ gives no non trivial solution.
Hence, we need the points of infinite order for the non-trivial solution.

4.2. Method-2

Search for n which failed to appear in method 1.

Let $a = \frac{n}{r}$, $b = -\frac{mn}{p}$, $c = \frac{mn}{q}$, and $r = pq$. Then, from equation (1) we get

$$q = \frac{1 + mp}{m + p}.$$

From equation (2) we get

$$(m^2 - 1)p^2 + (-mn^2 + m)p - m^2n^2 + 1 = 0. \tag{5}$$

For the quadratic in p equation (5) to have rational solutions, the discriminant must be a rational square. Thus there must exist $v \in Q$ such that

$$Q : v^2 = 4m^4n^2 + (-6n^2 - 3 + n^4)m^2 + 4. \tag{6}$$

We searched the rational ponits of equation (6) for $n = 18, 24, 45, 63, 64, 79$ with height $(m) < 100000$.

4.3. Search results

Search range: $n < 100, (a, b, c) < 1000000$

Table 2: Small solutions

n	a	b	c
8	64/5	-64/13	8/65
10	175/16	-35/34	25/272
14	320/21	-64/49	10/147
18	9879760/482517	-354320/140049	103114/1891773
21	441/13	-441/34	21/442
23	3978/161	-765/437	130/3059
24	11613784/477081	-1166848/3062277	326144/8669991
25	4807/175	-627/250	69/1750
28	9071/296	-193/72	47/1332
29	3553/116	-627/377	51/1508
34	11362/289	-7904/1479	736/25143
35	82615/1064	-110825/2597	11275/394744
43	559/12	-559/155	43/1860
45	219712122/4833865	-3265137/6889675	581994/27404975
47	4277/85	-611/183	329/15555
55	3025/34	-3025/89	55/3026
56	5312/91	-1992/833	192/10829
58	905840/14297	-121136/22533	190970/11108769
59	33099/553	-3009/3458	649/39026
63	27784341/278075	-7153731/193697	2491911/157032925
64	92568783/1364992	-1769673/461824	598879/38475712
69	27347/396	-667/9516	943/78507
72	53761/624	-15983/1128	407/29328
79	10440087/57013	-2040017/19591	116841/9230923
85	18800/221	-235/3009	400/39117
90	180081/2000	-5049/101840	2889/318250
92	93081/800	-53889/2212	4807/442400

The other large solutions are

$$(n,a,b,c) = (61, 1839878299779/30138554138, -180752198601/3001548136699, 14520329819/1126625024822),$$

$$\begin{aligned}(n,a,b,c) = &(70, 115148911166312615068925101503496507 2/16448733703744484903901651805459165,\\ &3547674799466427178523807068637387 2/5733154784860145737311544952102626225,\\ &-2523654174409881112161644274867222/2312107612889860767701679739880759 75),\end{aligned}$$

$$\begin{aligned}(n,a,b,c) = &(73, -90518708890610/86867723277193, 124437696960115/9201689556492827,\\ &1559559265989062/21067009890419),\end{aligned}$$

$$\begin{aligned}(n,a,b,c) = &(75, 123859705839353/1587069791475, -63406997310943/20746400481000,\\ &1179240742511/88813786257000),\end{aligned}$$

$$\begin{aligned}(n,a,b,c) = &(83, 11717358486721200474/37080311991617453, -5276919784626712217 5/226466060739568507,\\ &19184368806141136 50/15923242330588061503 1),\end{aligned}$$

$$\begin{aligned}(n,a,b,c) = &(93, 7052305354829229/71933121545305, -47532343770639/9411446212348,\\ &6219625284873/579593202983140).\end{aligned}$$

5. Concluding Remarks

Although we looked for the numerical solutions in the serarch range, there was no positive solution. The only trivial positive solution is $1+1+1 = 1/1+1/1+1/1 = 3$. We state a conjecture as follows.
Conjecture:
There is no integer n such that $a+b+c = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} = n$ has distinct positive rational solutions.

References

[1] American Mathematical Monthly,Volume 122,2015,Problems and Solutions, https://www.tandfonline.com/doi/abs/10.4169/amer.math.monthly.122.03.284

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[4] A. Schinzel, Triples of positive integers with the same sum and the same product, Serdica Math. J., 22(1996) 587-588. MR1483607