N-complex number, N-dimensional polar coordinate and 4D Klein bottle with 4-complex number

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Abstract: While a 3D complex number would be useful, it does not exist. Recently, I have constructed the N-complex number, which has demonstrated high efficiency in computations involving high-dimensional geometry. The N-complex number provides arithmetic operations and polar coordinates for N-dimensional spaces, akin to the classic complex number. In this paper, we will explain how these systems work and present studies on 4D Klein bottles and hyperspheres to illustrate the advantages of these systems

Summary

1 Introduction

The classic complex number system is a remarkable mathematical tool because it allows for the addition and rotation of vectors in two-dimensional space, following the same rules as real numbers for addition and multiplication. However, in three-dimensional space, it is impossible to manipulate vectors with similarly intuitive arithmetic operations because such a system does not currently exist. The development of a three-dimensional complex number system, analogous to the two-dimensional one, would represent a significant advancement in mathematics.

In 2022, I constructed a system of complex numbers for spaces with any number of dimensions, which I call the "N-complex number system." Edgar Malinovsky used this system to create many beautiful 3D objects (see «Rendering of 3D Mandelbrot, Lambda and other sets using 3D complex number system»[4]). Figure 1 shows the 3D Mandelbrot set he created. Computing 3D fractal objects is very time-consuming; he would not have succeeded in this work without the 3-complex number system. His work demonstrates that the 3-complex number system significantly accelerates computations in 3D space.

I have worked on 4D Klein bottles by extending a 3D Klein bottle (see Figure 2) into 4D space. I rotated the 4D Klein bottles in 4D space and showcased the rotation in my video animation "Observing a 4D Klein Bottle in 4-Dimension" [5]. This work would have been impossible without the 4-complex number system. In addition to N-complex numbers, the new system provides a polar coordinate system for N-dimensional spaces, which was previously missing in mathematics.

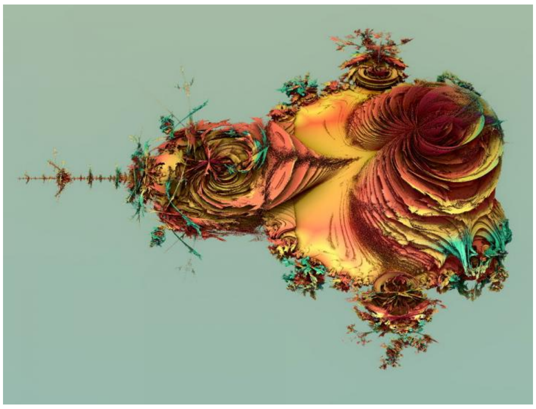


Figure 1 Mandelbrot set in 3D

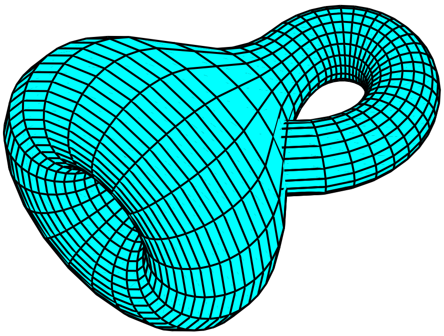


Figure 2 Klein bottle

Below, I will briefly explain the principles of the N-complex number system, which I described in «Extending complex number to spaces with 3, 4 or any number of dimensions»[1]. Please refer to this article for more details. Then, I will present my work on 4D Klein bottles and the polar coordinate system for N-dimensional spaces. To make the presentation clearer and simpler, I will refer to a complex number in N-dimensional space as N-complex, or as 2-complex, 3-complex, and 4-complex when N equals 2, 3, or 4, respectively. I will abbreviate N-dimensional as ND, and use 2D, 3D, and 4D when N equals 2, 3, or 4, respectively. In the following discussion, all spaces are Euclidean

2 N-complex Number System

2.1 Three-complex Number System

The N-complex number system is constructed recursively, meaning that the ND system is built from the (N-1)D system. First, we start with the 1-complex number system (the real number system) and construct the 2D system upon it. Then, we construct the 3D system upon the 2D system, and so on.

2.1.1 One-complex Number System

The 1-complex number system is based on the x-axis. A 1-complex number is simply a real number.

2.1.2 Two-complex Number System

To construct the 2-complex number system, we first create a 2D space by adding the y-axis to the existing 1D space. The y-axis is perpendicular to the x-axis. A point in the 2D space is defined by the 2-complex number $\boldsymbol{Z}_2$:

$$\boldsymbol{Z}_2 = x + iy \quad (1)$$

where x and y are real numbers, and i is the unit imaginary number.

Moreover, $\boldsymbol{Z}_2$ can be represented as a vector of length r:

$$r = |\boldsymbol{Z}_2| \quad (2)$$

With its direction determined by the angle θ between $\boldsymbol{Z}_2$ and the positive x-axis, the components of $\boldsymbol{Z}_2$ can be expressed as:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \quad (3)$$

and $\boldsymbol{Z}_2$ expressed as r multiplied by the Euler's formula $\boldsymbol{e}^{i\theta}$:

$$\begin{aligned} \boldsymbol{Z}_2 &= r(\cos(\theta) + i\sin(\theta)) \\ &= r \cdot e^{i\theta} \end{aligned} \quad (4)$$

$$e^{i\theta} = \cos\theta + i\sin\theta \quad (5)$$

Geometrically speaking, $\boldsymbol{Z}_2$ is a vector in the 2D space and makes an angle θ with the positive x-axis. Let us take the real number r as a vector along the positive x-axis and denote it by $\boldsymbol{Z}_1$:

$$\boldsymbol{Z}_1 = r \quad (6)$$

When we rotate $\boldsymbol{Z}_1$ by the angle θ towards the y-axis, we obtain $\boldsymbol{Z}_2$. So, the multiplication by Euler's formula $\boldsymbol{e}^{i\theta}$ of a vector of the 1D space ($\boldsymbol{Z}_1$) is equivalent to rotating it by the angle θ towards the y-axis (see (4)) :

$$\boldsymbol{Z}_2 = \boldsymbol{Z}_1 \cdot e^{i\theta} \quad (7)$$

The unit complex number along $\boldsymbol{Z}_2$ is:

$$\begin{aligned} \boldsymbol{U}_2 &= \frac{\boldsymbol{Z}_2}{r} = e^{i\theta} \\ &= \cos\theta + i\sin\theta \end{aligned} \quad (8)$$

2.1.3 Three-complex Number System

To construct the 3-complex number system, we begin by introducing the z-axis to the existing 2D space. The z-axis is orthogonal to both the x and y axes. We define a 3-complex number $\boldsymbol{Z}_3$ as:

$$\boldsymbol{Z}_3 = x + iy + jz \quad (9)$$

where x, y and z are Cartesian components, i and j are the unit imaginary numbers along the y and z axes respectively, with i^2=-1 and j^2=-1.

Currently, no 3-complex number exists; therefore, $\boldsymbol{Z}_3$ and x, y and z have to be constructed. Because the unit imaginary number j is perpendicular to the existing 2D space, it is perpendicular to any vector in this space. Let us take a 2-complex number (vector) $\boldsymbol{Z}_2$, and rotate it by the angle ϕ towards the z-axis, or alternately j (see Figure 3). The rotation is done in the plane ($\boldsymbol{Z}_2$, j) as shown in Figure 4. From this rotation we derive the expression of $\boldsymbol{Z}_3$:

$$\begin{aligned} \boldsymbol{Z}_3 &= \boldsymbol{Z}_2\cos(\phi) + j\,|\boldsymbol{Z}_2|\sin(\phi) \\ &= |\boldsymbol{Z}_2|\left(\frac{\boldsymbol{Z}_2}{|\boldsymbol{Z}_2|}\cos(\phi) + j\sin(\phi)\right) \end{aligned} \quad (10)$$

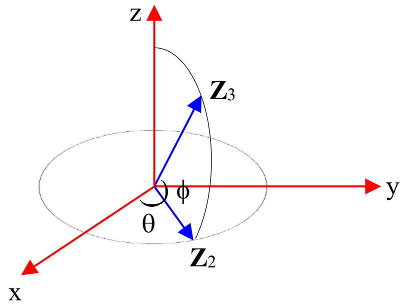


Figure 3

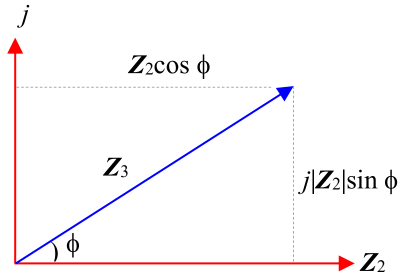


Figure 4

The rotation does not alter the length of the rotated vector:

$$|\boldsymbol{Z}_3| = |\boldsymbol{Z}_2| = r \quad (11)$$

Then, the unit vector $\boldsymbol{U}_3$ equals:

$$\begin{aligned}\boldsymbol{U}_3 &= \frac{\boldsymbol{Z}_3}{|\boldsymbol{Z}_3|} = \frac{\boldsymbol{Z}_3}{|\boldsymbol{Z}_2|} \\ &= \frac{\boldsymbol{Z}_2}{|\boldsymbol{Z}_2|}\cos(\phi) + j\sin(\phi)\end{aligned} \quad (12)$$

Because $\frac{\boldsymbol{Z}_2}{|\boldsymbol{Z}_2|}$ is the unit vector $\boldsymbol{U}_2$, the unit vector $\boldsymbol{U}_3$ equals:

$$\boldsymbol{U}_3 = \boldsymbol{U}_2\cos(\phi) + j\sin(\phi) \quad (13)$$

For extracting a general pattern from the construction of $\boldsymbol{Z}_3$, let us take the plane ($\boldsymbol{Z}_2$, j) as a complex plane with $\boldsymbol{U}_2$ and j being its basis vectors. In this complex plane $\boldsymbol{U}_3$ equals (see (13)):

$$\boldsymbol{U}_3 = \cos(\phi) + j\sin(\phi) \quad (14)$$

We notice that $\boldsymbol{U}_3$ is an Euler's formula with ϕ being the angle of rotation (the argument) and j the unit imaginary number with j^2=-1. So, in the plane ($\boldsymbol{Z}_2$, j), $\boldsymbol{U}_3$ can be written as :

$$\boldsymbol{U}_3 = e^{j\phi} \quad (15)$$

Therefore, Euler's formula can define the rotation of a unit vector in any plane, provided that the basis vectors of the plane are the unit vector to be rotated ($\boldsymbol{U}_2$) and a unit imaginary number j perpendicular to $\boldsymbol{U}_2$. This means that using Euler's formula, we can rotate any unit vector $\boldsymbol{U}_2$ from the 2D space towards j, which creates the 3-complex number $\boldsymbol{U}_3$. So, $\boldsymbol{U}_3$ is constructed in a similar way as previously for the 2-complex number $\boldsymbol{Z}_2$, which results from a multiplication of a 1-complex number $\boldsymbol{Z}_1$ by Euler's formula.

Then, we define the rotation of a unit vector $\boldsymbol{U}_2$ from the 2D space into 3D space as the multiplication of $\boldsymbol{U}_2$ by Euler's formula with argument ϕ and j.

Definition 1: 3-complex multiplication

Let $\boldsymbol{U}_2$ be a unit 2-complex number and j the unit imaginary number parallel to the z-axis. We rotate $\boldsymbol{U}_2$ by the angle ϕ towards the z-axis (the unit imaginary number j). This rotation is defined by the Euler's formula $\boldsymbol{e}^{j\phi}$:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \quad (16)$$

The **3-complex multiplication** of $\boldsymbol{U}_2$ by $\boldsymbol{e}^{j\phi}$ is defined as below:

$$\begin{aligned}\boldsymbol{U}_2 \bullet e^{j\phi} &= \boldsymbol{U}_2 \bullet (\cos(\phi) + j\sin(\phi)\,) \\ &= \boldsymbol{U}_2 \cos(\phi) + j\sin(\phi)\end{aligned} \qquad (17)$$

We use the symbol • to designate the operator of this multiplication.

End of the definition

The unit 3-complex number $\boldsymbol{U}_3$ is the result of the rotation of $\boldsymbol{U}_2$ and equals $\boldsymbol{U}_2 \bullet e^{j\phi}$:

$$\begin{aligned}\boldsymbol{U}_3 &= \boldsymbol{U}_2 \bullet e^{j\phi} \\ &= \boldsymbol{U}_2 \cos(\phi) + j\sin(\phi)\end{aligned} \qquad (18)$$

Because $\boldsymbol{Z}_2 = r\,\boldsymbol{U}_2$ and $\boldsymbol{Z}_3 = r\,\boldsymbol{U}_3$ (see (11)), we multiply both sides of equation (18) by r :

$$r\boldsymbol{U}_3 = r\boldsymbol{U}_2 \bullet e^{j\phi} \qquad (19)$$

and obtain the 3-complex number $\boldsymbol{Z}_3$ expressed as the product of 3-complex multiplication of $\boldsymbol{Z}_2$ and $\boldsymbol{e}^{j\phi}$:

$$\boldsymbol{Z}_3 = \boldsymbol{Z}_2 \bullet e^{j\phi} \qquad (20)$$

We develop $\boldsymbol{Z}_3$ with (19) and (17) and get the expression for $\boldsymbol{Z}_3$:

$$\begin{aligned}\boldsymbol{Z}_3 &= r\boldsymbol{U}_3 \\ &= r(\boldsymbol{U}_2 \cos(\phi) + j\sin(\phi)) \\ &= r\boldsymbol{U}_2 \cos(\phi) + j\,r\sin(\phi)\end{aligned} \qquad (21)$$

Because $\boldsymbol{Z}_2 = r\boldsymbol{U}_2$ and $|\boldsymbol{Z}_2| = r$, equation (21) gives:

$$\boldsymbol{Z}_3 = \boldsymbol{Z}_2 \cos(\phi) + j|\boldsymbol{Z}_2|\sin(\phi) \qquad (22)$$

The expression in (22) is identical to that in (10). Therefore, the 3-complex multiplication is correctly defined and gives the correct result, and the 3-complex multiplication is equivalent to the rotation of the 2-complex number $\boldsymbol{Z}_2$ by the angle ϕ towards j.

Let us express $\boldsymbol{Z}_3$ with the modulus r and the two arguments θ and ϕ by replacing $\boldsymbol{Z}_2$ with its expression in (7) :

$$\begin{aligned}\boldsymbol{Z}_3 &= \boldsymbol{Z}_2 \bullet e^{j\phi} \\ &= re^{i\theta} \bullet e^{j\phi}\end{aligned} \qquad (23)$$

For the sake of simplicity, when there is no possible confusion, we can write the 3-complex multiplication without the operator symbol • :

$$\begin{aligned}\boldsymbol{Z}_3 &= \boldsymbol{Z}_2 e^{j\phi} \\ &= re^{i\theta}e^{j\phi}\end{aligned} \qquad (24)$$

2.1.3.1 Conversion of Z_3

When we need $\boldsymbol{Z}_3$ in Cartesian form while we have $\boldsymbol{Z}_3$ in exponential form, $\boldsymbol{Z}_3$ should be converted. The formula for conversation is derived from the expression of $\boldsymbol{U}_3$ given by equation (18) in which $\boldsymbol{U}_2$ is replaced with its expression given in equation (8):

$$\begin{aligned}\boldsymbol{U}_3 &= \boldsymbol{U}_2 \cos(\phi) + j\sin(\phi) \\ &= (\cos(\theta) + i\sin(\theta))\cos(\phi) + j\sin(\phi) \\ &= \cos(\theta)\cos(\phi) + i\sin(\theta)\cos(\phi) + j\sin(\phi)\end{aligned} \qquad (25)$$

For expressing $\boldsymbol{Z}_3$ in Cartesian form we multiply $\boldsymbol{U}_3$ by r:

$$\begin{aligned} \boldsymbol{Z}_3 &= r\,\boldsymbol{U}_3 \\ &= r\cos(\theta)\cos(\phi) + i\,r\sin(\theta)\cos(\phi) + j\,r\sin(\phi) \end{aligned} \qquad (26)$$

The Cartesian form of $\boldsymbol{Z}_3$ is then:

$$\boldsymbol{Z}_3 = x + iy + kz \qquad (27)$$

$$\begin{aligned} x &= r\cos(\theta)\cos(\phi) \\ y &= r\sin(\theta)\cos(\phi) \\ z &= r\sin(\phi) \end{aligned} \qquad (28)$$

Equation (26) is the formula that converts $\boldsymbol{Z}_3$ into Cartesian form and vice versa.

2.1.3.2 Rule for Multiplication

Multiplication of 3-complex numbers must be done in exponential form. For multiplying the two 3-complex numbers z_1 and z_2:

$$\begin{aligned} \boldsymbol{z}_1 &= r_1 e^{i\theta_1} e^{j\phi_1} \\ \boldsymbol{z}_2 &= r_2 e^{i\theta_2} e^{j\phi_2} \end{aligned} \qquad (29)$$

We write:

$$\begin{aligned} \boldsymbol{z}_1 \bullet \boldsymbol{z}_2 &= r_1 e^{i\theta_1} e^{j\phi_1} \bullet r_2 e^{i\theta_2} e^{j\phi_2} \\ &= r_1 r_2 e^{i\theta_1} e^{i\theta_2} e^{j\phi_1} e^{j\phi_2} \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} e^{j(\phi_1+\phi_2)} \end{aligned} \qquad (30)$$

We notice that the exponentials for the unit imaginary number i ($e^{i1\theta 1}$ and $e^{i2\theta 2}$) gather together, and those for j ($e^{j1\phi 1}$ and $e^{j2\phi 2}$) gather together. This is because the arguments for i correspond to the rotation towards i and those for j correspond to the rotation towards j. These rotations are for different dimensions which are independent, so the imaginary units i and j do not cross multiply, that is, $i \bullet j$ is not defined for 3-complex number system.

Due to this property, if we reverse the order of z_1 and z_2 we will get the same result. Thus, 3-complex multiplication is commutative and associative, but not distributive.

2.1.3.3 Rule for Addition

Addition of 3-complex numbers must be done in Cartesian form. For adding the two 3-complex numbers z_1 and z_2:

$$\begin{aligned} \boldsymbol{z}_1 &= x_1 + iy_1 + jz_1 \\ \boldsymbol{z}_2 &= x_2 + iy_2 + jz_2 \end{aligned} \qquad (31)$$

The Cartesian components for each dimension add together:

$$\boldsymbol{z}_1 + \boldsymbol{z}_2 = (x_1 + x_2) + i(y_1 + y_2) + j(z_1 + z_2) \qquad (32)$$

2.2 N-Complex Number System

2.2.1 Generalization

We summarize the steps of the construction of the 3-complex number systems as below:

1. The 1-complex number system is the real number system on the x-axis.
2. The 2D space is created by adding the y-axis perpendicularly to the 1D space.
3. A 2-complex number $\boldsymbol{Z}_2$ is constructed by rotating a 1-complex number $\boldsymbol{Z}_1$ by the angle θ towards the y-axis.
4. The rotation of $\boldsymbol{Z}_1$ is equivalent to the multiplication of $\boldsymbol{Z}_1$ by $e^{i\theta}$.
5. The 3D space is created by adding the z-axis perpendicularly to the 2D space.
6. A 3-complex number $\boldsymbol{Z}_3$ is constructed by rotating a 2-complex number $\boldsymbol{Z}_2$ by the angle ϕ towards the z-axis.
7. The rotation of $\boldsymbol{Z}_2$ is equivalent to the 3-complex multiplication of $\boldsymbol{Z}_2$ by $e^{j\phi}$.

The steps 5, 6 and 7 construct the 3-complex number system from the 2-complex number system. By extension, an N-complex number is constructed from an existing (N-1)-complex number with the following recursive procedure:

1. The ND space is created by adding the n^{th} axis perpendicularly to the (N-1)D space.
2. An N-complex number is constructed by rotating an (N-1)-complex number $\boldsymbol{Z}_{n-1}$ by the angle θ_{n-1} towards the n^{th} axis.
3. The rotation of $\boldsymbol{Z}_{n-1}$ is equivalent to the multiplication of $\boldsymbol{Z}_{n-1}$ by $\boldsymbol{e}^{i_{n-1}\theta_{n-1}}$ resulting in the N-complex number $\boldsymbol{Z}_n$.

The third step involves the multiplication for N-complex number, which we define below by extending the 3-complex multiplication (see Definition 1).

Definition 2: N-complex multiplication
Let $\boldsymbol{U}_{n-1}$ be a unit (N-1)-complex number and i_{n-1} the unit imaginary number parallel to the n^{th} axis. The n^{th} axis is associated with i_{n-1} because the first axis is the real number axis. We rotate $\boldsymbol{U}_{n-1}$ by the angle θ_{n-1} towards the n^{th} axis. This rotation is defined by the Euler's formula below:

$$e^{i_{n-1}\theta_{n-1}} = \cos(\theta_{n-1}) + i_{n-1}\sin(\theta_{n-1}) \quad (33)$$

The **N-complex multiplication** of $\boldsymbol{U}_{n-1}$ by $\boldsymbol{e}^{i_{n-1}\theta_{n-1}}$ is defined as:

$$\boldsymbol{U}_{n-1} \bullet e^{i_{n-1}\theta_{n-1}} = \boldsymbol{U}_{n-1}\cos(\theta_{n-1}) + i_{n-1}\sin(\theta_{n-1}) \quad (34)$$

End of the Definition

The result of (34) is the unit N-complex number $\boldsymbol{U}_n$ in the ND space:

$$\begin{aligned} \boldsymbol{U}_n &= \boldsymbol{U}_{n-1} \bullet e^{i_{n-1}\theta_{n-1}} \\ &= \boldsymbol{U}_{n-1}\cos(\theta_{n-1}) + i_{n-1}\sin(\theta_{n-1}) \end{aligned} \quad (35)$$

For constructing the N-complex number $\boldsymbol{Z}_n$, we multiply both sides of equation (35) by r (modulus of $\boldsymbol{Z}_n$):

$$r\boldsymbol{U}_n = r\boldsymbol{U}_{n-1} \bullet e^{i_{n-1}\theta_{n-1}} \quad (36)$$

Given that $\boldsymbol{Z}_n = r\,\boldsymbol{U}_n$ and $\boldsymbol{Z}_{n-1} = r\,\boldsymbol{U}_{n-1}$, the expression of $\boldsymbol{Z}_n$ is (see (35) and (36)):

$$\begin{aligned} \boldsymbol{Z}_n &= \boldsymbol{Z}_{n-1} \bullet e^{i_{n-1}\theta_{n-1}} \\ &= r\boldsymbol{U}_{n-1}\cos(\theta_{n-1}) + i_{n-1}r\ \sin(\theta_{n-1}) \\ &= \boldsymbol{Z}_{n-1}\cos(\theta_{n-1}) + i_{n-1}|\boldsymbol{Z}_{n-1}|\sin(\theta_{n-1}) \end{aligned} \quad (37)$$

The N-complex number system is constructed upon the (N-1)-complex number system, which is constructed upon the (N-2)-complex number system and so on until the 1-complex number system. Let us construct $\boldsymbol{Z}_2$ from the 1-complex number $\boldsymbol{Z}_1$ which is the real number r:

$$\boldsymbol{Z}_1 = r \quad (38)$$

We multiply $\boldsymbol{Z}_1$ by $\boldsymbol{e}^{i_1\theta_1}$ and obtain $\boldsymbol{Z}_2$:

$$\boldsymbol{Z}_2 = \boldsymbol{Z}_1 \bullet e^{i_1\theta_1} \quad (39)$$

We multiply $\boldsymbol{Z}_2$ by $\boldsymbol{e}^{i_2\theta_2}$ and obtain $\boldsymbol{Z}_3$:

$$\begin{aligned} \boldsymbol{Z}_3 &= \boldsymbol{Z}_2 \bullet e^{i_2\theta_2} \\ &= \boldsymbol{Z}_1 \bullet e^{i_1\theta_1} \bullet e^{i_2\theta_2} \\ &= re^{i_1\theta_1} \bullet e^{i_2\theta_2} \end{aligned} \quad (40)$$

We repeat the multiplications until $\boldsymbol{Z}_n$ and obtain its expression:

$$\boldsymbol{Z}_n = re^{i_1\theta_1} \bullet \cdots \bullet e^{i_{n-1}\theta_{n-1}} \quad (41)$$

which can be written without the symbol • :

$$\begin{aligned}\boldsymbol{Z}_n &= re^{i_1\theta_1} \dots e^{i_{n-1}\theta_{n-1}} \\ &= re^{(i_1\theta_1+\cdots+i_{n-1}\theta_{n-1})}\end{aligned} \quad (42)$$

The unit imaginary numbers from the second to Nth dimension are $i_1, \dots, i_{n-1}$, and are defined with the relations: $i_1^2=-1, \dots, i_{n-1}^2=-1$.

2.2.2 Conversion of Z_n

The formula for converting the N-complex number $\boldsymbol{Z}_n$ from exponential form into Cartesian form is derived from equation (42), and is developed in the same way as equations (25) and (26):

$$\boldsymbol{Z}_n = r\left(\left(\left((\cos\theta_1 + i_1\sin\theta_1)\cos\theta_2 + i_2\sin\theta_2\right)\cdots\right)\cos\theta_{n-1} + i_{n-1}\sin\theta_{n-1}\right) \quad (43)$$

The Cartesian expression of $\boldsymbol{Z}_n$ is:

$$\boldsymbol{Z}_n = x_0 + i_1x_1 + \cdots + i_{n-1}x_{n-1} \quad (44)$$

By comparing the equations (43) and (44) we derive $x_0, x_1, \dots, x_{n-1}$ from r and $\theta_1, \dots, \theta_{n-1}$ and vice versa.

2.2.3 Arithmetic Operations

Multiplication of N-complex numbers must be done in exponential form. Two N-complex numbers $\boldsymbol{X}$ and $\boldsymbol{Y}$ are expressed below:

$$\begin{aligned}\boldsymbol{X} &= r_xe^{i_1\alpha_1} \dots e^{i_{n-1}\alpha_{n-1}} \\ \boldsymbol{Y} &= r_ye^{i_1\beta_1} \dots e^{i_{n-1}\beta_{n-1}}\end{aligned} \quad (45)$$

They multiply as follow:

$$\begin{aligned}\boldsymbol{X} \bullet \boldsymbol{Y} &= r_xe^{i_1\alpha_1} \dots e^{i_{n-1}\alpha_{n-1}} \bullet r_ye^{i_1\beta_1} \dots e^{i_{n-1}\beta_{n-1}} \\ &= r_x\, r_y\, e^{i_1(\alpha_1+\beta_1)} \dots e^{i_{n-1}(\alpha_{n-1}+\beta_{n-1})} \\ &= r_x\, r_y\, e^{(i_1(\alpha_1+\beta_1)+\cdots+i_{n-1}(\alpha_{n-1}+\beta_{n-1}))}\end{aligned} \quad (46)$$

Addition of N-complex numbers must be done in Cartesian form. Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be:

$$\begin{aligned}\boldsymbol{X} &= x_0 + i_1x_1 + \cdots + i_{n-1}x_{n-1} \\ \boldsymbol{Y} &= y_0 + i_1y_1 + \cdots + i_{n-1}y_{n-1}\end{aligned} \quad (47)$$

They add as follow:

$$\boldsymbol{X} + \boldsymbol{Y} = (x_0 + y_0) + i(x_1 + y_1) + \cdots + i_n(x_{n-1} + y_{n-1}) \quad (48)$$

For more details about the N-complex number system, please see «Extending complex number to spaces with 3, 4 or any number of dimensions»[1].

3 Möbius Strip and Klein Bottle with Three-Complex Number

The N-complex number system is very convenient for describing geometric forms in higher-dimensional spaces. For demonstrating the capability of this new tool, let us first create a Möbius strip and a Klein bottle.

3.1 Möbius strip

Our Möbius strip is generated by a vector moving on a circle, with the circle serving as the directrix and the moving vector as the generatrix. In Figure 5, the generatrix (depicted by the blue arrows) rotates half a turn as it completes a full rotation on the directrix (represented by the black circle).

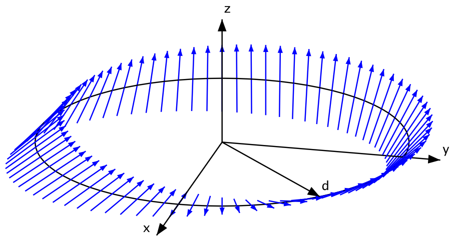


Figure 5, Möbius strip with generatrix vectors

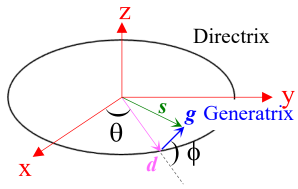


Figure 6, 3-complex numbers for Möbius strip

The directrix is defined by a vector of length 1 that rotates in the plane (x, y). Therefore, the directrix circle is defined by the unit complex number $\boldsymbol{d}$ (see Figure 6):

$$\boldsymbol{d} = e^{i\theta} \quad (49)$$

where θ is the angle between $\boldsymbol{d}$ and the positive x-axis.

The generatrix is the vector $\boldsymbol{g}$ that moves along the directrix (see Figure 6). Meanwhile, $\boldsymbol{g}$ rotates in the plane ($\boldsymbol{d}$, z) or alternately the plane ($\boldsymbol{d}$, j). Therefore, $\boldsymbol{g}$ is defined by the 3-complex number below (see (20)):

$$\begin{aligned} \boldsymbol{g} &= p\, \boldsymbol{d} \bullet e^{j\phi} \\ &= p\, e^{i\theta} e^{j\phi} \end{aligned} \quad (50)$$

where ϕ is the angle between $\boldsymbol{g}$ and $\boldsymbol{d}$ (see Figure 6) and p the distance between a point and the directrix (circle) along $\boldsymbol{g}$.

The rate of rotation of ϕ equals half that of θ, so $\phi = \theta/2$ and $\boldsymbol{g}$ equals:

$$\boldsymbol{g} = p\, e^{i\theta} e^{j\theta/2} \quad (51)$$

The sum of $\boldsymbol{d}$ and $\boldsymbol{g}$ defines a point on the Möbius strip (see Figure 6), which we denote by $\boldsymbol{s}$:

$$\begin{aligned} \boldsymbol{s} &= \boldsymbol{d} + \boldsymbol{g} \\ &= e^{i\theta} + p\, e^{i\theta} e^{j\theta/2} \end{aligned} \quad (52)$$

where $0 < \theta < 2\pi$ and $-0.2 < p < 0.2$.

Because $\boldsymbol{g}$ is a 3-complex number, $\boldsymbol{s}$ is also a 3-complex number. As $\boldsymbol{g}$ and $\boldsymbol{d}$ are well defined by θ, ϕ and p, all points of the Möbius strip are well defined by $\boldsymbol{s}$. The radius of the Möbius strip is 1, and the width of the strip is 0.4.

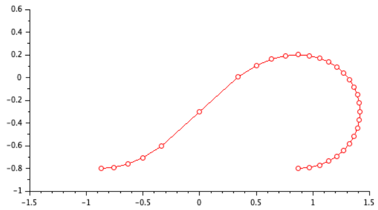
Compared to the usual definition of a Möbius strip[1], the 3-complex function (52) is much simpler and more intuitive, thereby demonstrating the advantage of the 3-complex number system.

3.2 Three-dimensional Klein Bottle

3.2.1 Geometric Construction of a Klein Bottle

A 3D Klein bottle comprises two parts: the neck and the body (see Figure 2). The neck is a curved tube connected to the bottle-like body. For constructing the neck, we first define its center line, which is the curve shown in Figure 7. Then we draw circles centered on the center line, creating the mesh of the neck as depicted in Figure 8. The markers (small circles) on the curve represent the centers of the circles, which are perpendicular to the local tangent of the center line. These circles serve as the generatrix of the neck.

[1] Möbius strip, https://en.wikipedia.org/wiki/M%C3%B6bius_strip#Sweeping_a_line_segment



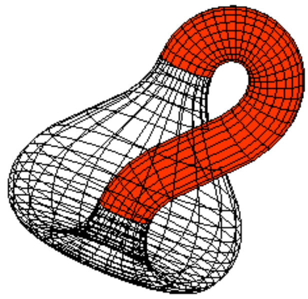
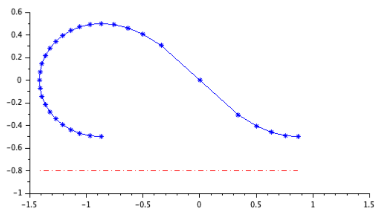


Figure 7 Directrix curve of the neck   Figure 8 Neck of the Klein bottle

The surface of the body is created by rotating a generatrix around a directrix. As shown in Figure 9, the generatrix is represented by the blue curve, and the directrix is depicted as the red straight line. When the blue curve is rotated, it creates a surface around the red straight line, forming the surface of the body, as illustrated in Figure 10 (the blue surface). The markers (small stars) draw circles of different radii, which together constitute the mesh of the surface of the body.



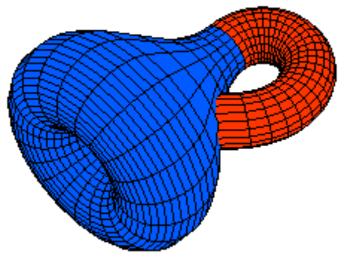


Figure 9 Directrix and generatrix of the body   Figure 10 Body of the Klein bottle

3.2.2 Directrix and Generatrix Curves

The curve in Figure 11 is a Lemniscate of Bernoulli. The directrix (Center line) of the neck is represented by the red part of this curve, while the generatrix (Rotating curve) of the body is its blue part. This curve is defined by the function below:

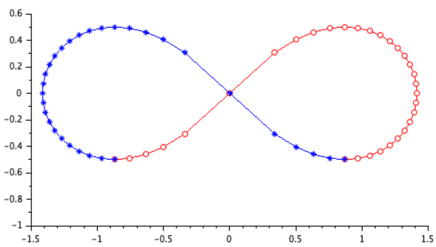
$$\rho = \pm b\sqrt{\cos(2\beta)} \qquad (53)$$

A point on the Lemniscate of Bernoulli is defined by the complex number ***B***:

$$\boldsymbol{B} = \rho e^{i\beta} \qquad (54)$$

We replace ρ with its expression (see (53)) in (54) and obtain the expression for ***B***:

$$\begin{aligned} \boldsymbol{B} &= \pm b\sqrt{\cos(2\beta)}\,(\cos(\beta) + i\sin(\beta)) \\ &= \pm b\sqrt{\cos(2\beta)}\cos(\beta) \pm i\, b\sqrt{\cos(2\beta)}\sin(\beta) \qquad (55) \\ &= x_B + iy_B \end{aligned}$$



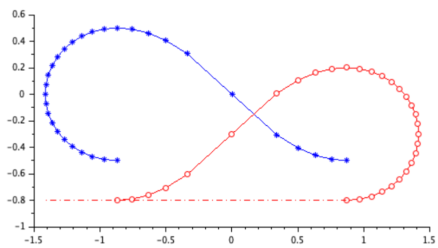


Figure 11 Lemniscate of Bernoulli   Figure 12 Red lines : directrix of the neck and body

3.2.3 Three-complex Surfaces

3.2.3.1 The Neck

The surfaces of the neck and the body are specified by 3-complex functions that we will define below. The radius of the neck is denoted by r_N, which is a constant. To ensure that the neck is seamlessly connected to the body, we shift the red curve downward along the *y*-axis by a distance of r_N. This connects the directrix of the neck (red curve) to that of the body (red straight line, see Figure 12).

The lowered red curve is defined by the complex function below (see (54)):

$$\boldsymbol{d} = \pm b\sqrt{\cos(2\beta)}\, e^{i\beta} - i\, r_N \qquad (56)$$

The generatrix of the neck is described as a circle perpendicular to the local tangent of the red curve, thus also perpendicular to the (*x, y*) plane. The plane of the circle is determined by its intersection with the (*x, y*) plane. Let ***U*** represent the unit complex number along the intersection line. ***U*** is defined by Euler's formula below:

$$\boldsymbol{U} = e^{i\theta_N} \qquad (57)$$

where θ_N signifies the angle the intersection line makes with the positive *x*-axis.

The unit imaginary number along the *z*-axis is *j* and is perpendicular to ***U***. Thus, the plane of the circle is defined by ***U*** and *j* and is denoted as (***U**, j*) plane. Let ***g*** be the 3-complex number that represents the circle. The length of ***g*** equals the radius of the neck r_N, and the angle ***g*** makes with ***U*** is ϕ. Therefore, ***g*** is defined in the (***U***, *j*) plane as the product of the 3-complex multiplication of the the 3-complex number $r_N\boldsymbol{U}$ by Euler's formula $\boldsymbol{e}^{j\phi}$:

$$e^{j\phi} = \cos(\phi) + j\sin(\phi) \qquad (58)$$

$$\begin{aligned} \boldsymbol{g} &= r_N\boldsymbol{U} \bullet e^{j\phi} \\ &= r_N\boldsymbol{U}\cos(\phi) + j r_N\sin(\phi) \end{aligned} \qquad (59)$$

In (59) we replace ***U*** with its expression (see (57)) and write ***g*** without the symbol • :

$$\boldsymbol{g} = r_N e^{i\theta_N} e^{j\phi} \qquad (60)$$

The angle θ_N depends on the parameter β of each marker, and the angle ϕ varies from 0 to 2π to draw a full circle.

3.2.3.2 The Body

The body is formed by rotating the blue curve around the red straight line, which is positioned below the Lemniscate of Bernoulli at a distance r_N (see Figure 12). The minimum *y*-value of the Lemniscate of Bernoulli is denoted by y_{min}. Thus, the *y*-coordinate of the red straight line is determined as follows:

$$y = y_{min} - r_N \qquad (61)$$

The *x* coordinate of each marker on the blue curve is the x_B defined by equation (55):

$$x = \pm b\sqrt{\cos(2\beta)}\cos(\beta) \qquad (62)$$

As the blue curve rotates, each marker draws a circle. Because the circle is perpendicular to the directrix, the *x* coordinate of its center equals that of the marker (see (62)). Moreover, since the center lies on the directrix, its *y*-coordinate equals that of the directrix (see (61)). Let the complex number ***d*** define the center of the circle:

$$\begin{aligned} \boldsymbol{d} &= x + iy \\ &= \pm b\sqrt{\cos(2\beta)}\cos(\beta) + i(y_{min} - r_N) \end{aligned} \qquad (63)$$

Let r_B denote the radius of the circle. Then, r_B equals the *y* coordinate of the marker (see (55)) minus that of the directrix (see (61)) :

$$r_B = b\sqrt{\cos(2\beta)}\sin(\beta) - (y_{min} - r_N) \quad (64)$$

The plane of the circle is perpendicular to the x-axis, so it intersects with the (x, y) plane on a line that is perpendicular to the x-axis. Let $\boldsymbol{U}$ be the unit complex number along the intersection line. $\boldsymbol{U}$ makes an angle of $\pi/2$ with the positive x-axis:

$$\boldsymbol{U} = e^{i\pi/2} \quad (65)$$

Therefore, the circle is defined by the 3-complex number $\boldsymbol{g}$, which equals the product of the 3-complex multiplication of $r_B\boldsymbol{U}$ and $\boldsymbol{e}^{j\phi}$:

$$\begin{aligned} \boldsymbol{g} &= r_B \boldsymbol{U} \bullet e^{j\phi} \\ &= r_B e^{i\pi/2} e^{j\phi} \\ &= \left(b\sqrt{\cos(2\beta)}\sin\beta - (y_{min} - r_N)\right) e^{i\pi/2} e^{j\phi} \end{aligned} \quad (66)$$

where ϕ is the angle between $\boldsymbol{g}$ and $\boldsymbol{U}$ (with $0 < \phi < 2\pi$), and r_B is the radius of the circle.

3.2.3.3 The Klein Bottle

The surfaces of both the neck and the body of the Klein bottle are formed using a directrix and a generatrix, denoted by the complex functions $\boldsymbol{d}$ and $\boldsymbol{g}$. The general expression of the 3-complex function $\boldsymbol{g}$ is:

$$\boldsymbol{g} = re^{i\theta}e^{j\phi} \quad (67)$$

Therefore, a point on the surface of the Klein bottle is defined by the sum of $\boldsymbol{d}$ and $\boldsymbol{g}$:

$$\begin{aligned} \boldsymbol{s} &= \boldsymbol{d} + \boldsymbol{g} \\ &= \boldsymbol{d} + re^{i\theta}e^{j\phi} \end{aligned} \quad (68)$$

where θ is defined by equation (57) for the neck and equals $\pi/2$ for the body, while r is defined by equation (64) for the body and equals r_N for the neck.

The functions $\boldsymbol{d}$ is defined by equations (56) for the neck and (63) for the body. Then the surface of the Klein bottle is defined by the 3-complex functions $\boldsymbol{s}_{\text{neck}}$, $\boldsymbol{d}_{\text{neck}}$ and $\boldsymbol{g}_{\text{neck}}$ for the neck, and $\boldsymbol{s}_{\text{body}}$, $\boldsymbol{d}_{\text{body}}$ and $\boldsymbol{g}_{\text{body}}$ for the body, as shown below:

$$\begin{aligned} \boldsymbol{s}_{\text{Neck}} &= \boldsymbol{d}_{\text{Neck}} + \boldsymbol{g}_{\text{Neck}} \\ \boldsymbol{d}_{\text{Neck}} &= \pm b\sqrt{\cos(2\beta)}\, e^{i\beta} - i\, r_N \\ \boldsymbol{g}_{\text{Neck}} &= r_N e^{i\theta_N} e^{j\phi} \end{aligned} \qquad \begin{aligned} &\beta\text{: the red markers} \\ &\theta_N\text{: angle of } \boldsymbol{U} \\ &\phi = 0 \rightarrow 2\pi \end{aligned} \qquad (69)$$

$$\begin{aligned} \boldsymbol{s}_{\text{Body}} &= \boldsymbol{d}_{\text{Body}} + \boldsymbol{g}_{\text{Body}} \\ \boldsymbol{d}_{\text{Body}} &= \pm b\sqrt{\cos(2\beta)}\cos\beta + i(y_{min} - r_N) \\ \boldsymbol{g}_{\text{Body}} &= \left(b\sqrt{\cos(2\beta)}\sin\beta - (y_{min} - r_N)\right) e^{i\pi/2} e^{j\phi} \end{aligned} \qquad \begin{aligned} &\beta\text{: the blue markers} \\ &\phi = 0 \rightarrow 2\pi \end{aligned} \qquad (70)$$

The definition of the surface of the Klein bottle with $\boldsymbol{s}$, $\boldsymbol{d}$ and $\boldsymbol{g}$ is much simpler and more intuitive than the usual formula for a Klein bottle[2].

[2] Formula For A Klein Bottle, https://en.wikipedia.org/wiki/Klein_bottle#Bottle_shape

4 Four-dimensional Klein Bottle

4.1 Construction

The previous Klein bottle is a 3D surface where the neck intersects the body. A true Klein bottle possesses four dimensions and does not intersect with itself. However, no concrete representation of 4D Klein bottle exists because the fourth dimension of such a structure has never been properly defined. Now, with the advent of the 4-complex number system, we have the capability to define a function for the fourth dimension of a 4D Klein bottle by transforming the 3-complex function $\boldsymbol{s}$ of the 3D Klein bottle (see (68)) into a 4-complex function.

To define the fourth dimension of a 4D Klein bottle, we utilize a 4-complex function obtained by applying Euler's formula $\boldsymbol{e}^{k\psi}$ (as described in equation (71). This formula defines the rotation towards the fourth dimension:

$$e^{k\psi} = \cos(\psi) + k\sin(\psi) \qquad (71)$$

where k is the unit imaginary number of the fourth dimension, and ψ signifies the angle of rotation towards the fourth dimension.

To transform the 3D Klein bottle, we rotate the 3-complex function $\boldsymbol{g}$ (see (69) and (70)) by multiplying it by $\boldsymbol{e}^{k\psi}$ resulting in the 4D generatrix function $\boldsymbol{g}_4$:

$$\begin{aligned} \boldsymbol{g}_4 &= \boldsymbol{g} \bullet e^{k\psi} \\ &= r_4\left(e^{i\theta}e^{j\phi}\right) \bullet e^{k\psi} \\ &= r_4\left(e^{i\theta}e^{j\phi}\cos(\psi) + k\sin(\psi)\right) \\ &= r_4\ \cos(\psi)\, e^{i\theta}e^{j\phi} + k\, r_4 \sin(\psi) \end{aligned} \qquad (72)$$

where r_4 is the modulus of $\boldsymbol{g}_4$ still to be defined.

The generatrix function $\boldsymbol{g}_4$ can be further divided into a 3-complex function $\boldsymbol{g}_3$ and a function for the fourth dimension $\boldsymbol{g}_w$:

$$\begin{aligned} \boldsymbol{g}_4 &= \boldsymbol{g}_3 + \boldsymbol{g}_w \\ \boldsymbol{g}_3 &= r_4\cos(\psi)\ e^{i\theta}e^{j\phi} \\ \boldsymbol{g}_w &= k\, r_4 \sin(\psi) \end{aligned} \qquad (73)$$

The 4D Klein bottle is defined by $\boldsymbol{s}_4$, which is the sum of $\boldsymbol{d}$ and $\boldsymbol{g}_4$:

$$\begin{aligned} \boldsymbol{s}_4 &= \boldsymbol{d} + \boldsymbol{g}_4 \\ &= \boldsymbol{d} + \boldsymbol{g}_3 + \boldsymbol{g}_w \\ &= \boldsymbol{d} + r_4\cos(\psi)\ e^{i\theta}e^{j\phi} + k\, r_4\sin(\psi) \end{aligned} \qquad (74)$$

From equation (74), we derive the function $\boldsymbol{s}_p$, which defines the projection of the 4D Klein bottle onto the subspace (x, y, z) of (x, y, z, w):

$$\begin{aligned} \boldsymbol{s}_p &= \boldsymbol{d} + \boldsymbol{g}_3 \\ &= \boldsymbol{d} + r_4\cos\psi\ e^{i\theta}e^{j\phi} \end{aligned} \qquad (75)$$

For the 4D Klein bottle to be coherent with the 3D Klein bottle, the function $\boldsymbol{s}_p$ should equal $\boldsymbol{s}$ (see (68)):

$$\boldsymbol{s}_p = \boldsymbol{s} \Rightarrow\ \boldsymbol{d} + r_4\cos(\psi)\ e^{i\theta}e^{j\phi} = \boldsymbol{d} + re^{i\theta}e^{j\phi} \qquad (76)$$

Thus, the 4D Klein bottle must respect this condition:

$$r_4\cos(\psi) = r \qquad (77)$$

where r_4 is the modulus of the 4-complex function $\boldsymbol{g}_4$ and r is that of the 3-complex function $\boldsymbol{g}$ of the 3D Klein bottle.

To fulfill this condition, we initially opt for the simplest function for r_4, which is a constant:

$$r_4 = r_0 \quad (78)$$

Then, we determine the angle ψ using the relation provided in equation (77):

$$\cos(\psi) = \frac{r}{r_0} \quad (79)$$

The function for the fourth dimension is established as described in equation (73):

$$\begin{aligned} \boldsymbol{g}_w &= k\, r \sin(\psi) \\ &= k\, r_0 \sqrt{1 - \cos^2(\psi)} \end{aligned} \quad (80)$$

With this function in place, the 4D Klein bottle is defined accordingly, as shown in equations (73) and (80):

$$\boldsymbol{s_4} = \boldsymbol{d} + r_0 \cos(\psi)\, e^{i\theta} e^{j\phi} + k\, r_0 \sqrt{1 - \cos^2(\psi)} \quad (81)$$

When ψ varies in accordance with r to fulfill the condition (79), the sum of the first two terms in equation (81) mirrors that in equation (68) ensuring that the projection of the 4D Klein bottle onto the 3D space (*x, y, z*) remains identical to that of the 3D Klein bottle.

4.2 Visualization in Four-dimension

4.2.1 Slices in three-dimension

Since we cannot directly perceive 4D objects, even though we can calculate all the points of the 4D Klein bottle defined by equation (81) in the 4D space (*x, y, z, w*), we cannot produce a visual representation encompassing all four dimensions. However, we can project its geometry onto the 3D subspaces (*x, y, z*), (*x, y, w*), (*x, w, z*), and (*w, y, z*). Notably, the last three of these four subspaces fully incorporate the fourth dimension. Therefore, we can observe the fourth dimension of the 4D Klein bottle in these 3D projections.

A common way to represent the fourth dimension of the 4D Klein bottle is by showing slices of it with increasing values of the fourth dimension. Each slice is a 3D object that we can see. The slices are defined by equation (81) with various values of ψ and are shown one at a time. As the value of the fourth dimension increases, the slices move in 3D space. The best way to show the motion of the slices is through video animation, which we have created and posted on YouTube, see 0:27 of "Observing a 4D Klein Bottle in 4-Dimension" [5].

Figure 13, Figure 14 and Figure 15 show three slices corresponding to the values of ψ:

$$0° < \psi < 7°,\ 32° < \psi < 52° \text{ and } \psi = 77°$$

respectively. The cyan bands represents the slices, and the mesh of the 3D Klein bottle serves as a reference for the position of each slice.

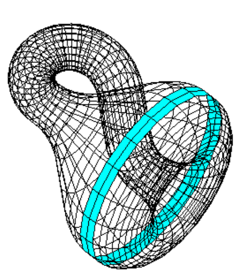


Figure 13 **0° < ψ < 7°**

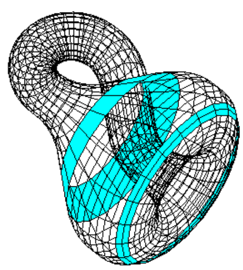


Figure 14 **32° < ψ < 52°**

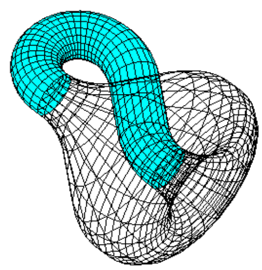


Figure 15 **ψ = 77°**

These three figures effectively illustrate why the neck does not intersect the body in 4D space: the slice of the neck does not simultaneously appear with any slice of the body, meaning that they are not at the same value of the fourth dimension (w). Consequently, the neck and the body cannot intersect each other in the 4D space.

The fourth dimension of each slice is best shown with the projections of the 4D Klein bottle onto the subspace (x, y, w), where the z dimension is replaced by the w dimension. Figure 16, Figure 17 and Figure 18 display these projections, which reveal that the 4D Klein bottle forms a closed strip without self-intersection. As w increases, the slices ascend along the strip. The ascension of the slices along the fourth dimension is well demonstrated in the video animation we have created and posted on YouTube. see [0:43](#) of ["Observing a 4D Klein Bottle in 4-Dimension"](#) [5].

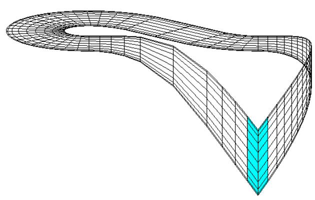


Figure 16 **$0^\circ < \psi < 7^\circ$**

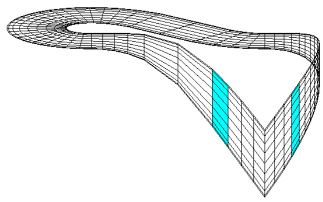


Figure 17 **$32^\circ < \psi < 52^\circ$**

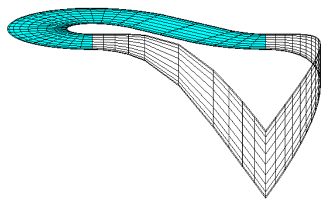


Figure 18 **$\psi = 77^\circ$**

4.2.2 Klein Bottle Becomes Möbius Strip

Upon close examination of the projection of the 4D Klein bottle onto the subspace (x, y, w), we notice that the two faces of the strip are inverted after a turn, indicating that this strip is a Möbius strip. However, it possesses an angular turn.

Equation (77) represents the condition that the 4D Klein bottle should adhere to in order to remain coherent with the original 3D Klein bottle. Because the dimensions x, y and z are independent from w, the function $\boldsymbol{g}_w$ (the fourth dimension of the 4D Klein bottle, see (73)) can have any value without altering the projection onto the subspace (x, y, z). Consequently, one 3D Klein bottle can be extended into an infinite number of 4D Klein bottles because $\boldsymbol{g}_w$ can take any function.

To illustrate the multiplicity of 4D Klein bottles associated with a single 3D Klein bottle, we will define the following function for the fourth dimension of the 4D Klein bottle:

$$\boldsymbol{g}_w = g_0 \cos(\alpha), \text{ with } g_0 \text{ being constant}, \ \alpha = 0 \to 2\pi \qquad (82)$$

This $\boldsymbol{g}_w$ is different from that given in (80), and the resulting 4D Klein bottle is referred to as the second 4D Klein bottle. Its projection onto the 3D subspace (x, y, w) is a smooth Möbius strip (see Figure 19). Figure 20 displays its projection onto the subspace (x, w, z), where the dimension y is replaced by the fourth dimension w. This projection reveals that the neck maintains its tubular form in this subspace.

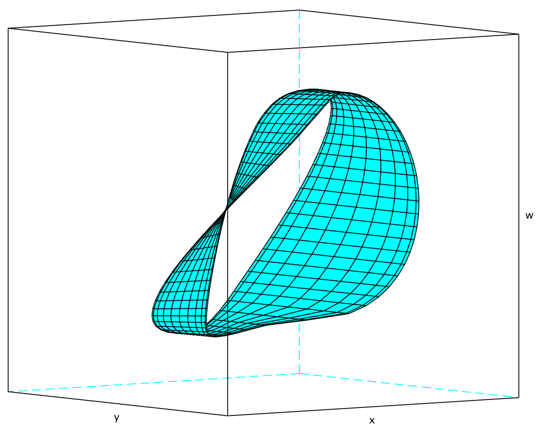


Figure 19 Projection (x, y, w)

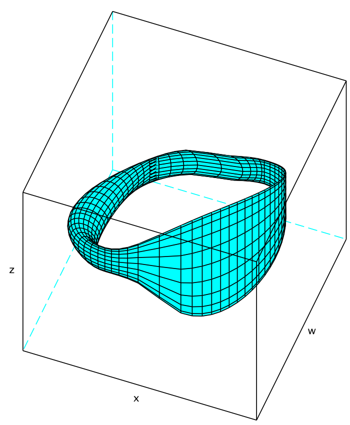


Figure 20 Projection (x, w, z)

4.2.3 Rotation of a Four-dimensional Klein Bottle

The Möbius strip in Figure 16 appears far from resembling a bottle, despite being a view of a Klein bottle. This phenomenon arises because the projection of a 4D Klein bottle onto different subspaces takes various forms. One might feel confused: how can a view of a bottle-like Klein bottle appear as a strip?

To clarify this concept, let us consider the circle shown in Figure 21. This circle represents the projection of a 3D object onto the plane (x, y). When the circle is rotated around the y-axis, it manifests as a spiral (see Figure 22); a rotation of 90° reveals a sine curve (see Figure 23). In reality, this object is a helix: the circle represents its projection onto the plane (x, y), and the sine curve its projection onto the plane (y, z). We have created a video animation showcasing the rotation of the helix; see 1:21 of "Observing a 4D Klein Bottle in 4-Dimension" [5]. This example illustrates that the projections of the same object onto different subspaces can take entirely different forms, providing a concrete understanding of the diverse views of a 4D Klein bottle.

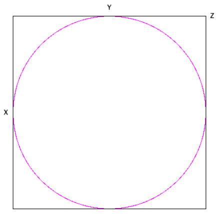


Figure 21

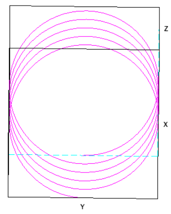


Figure 22

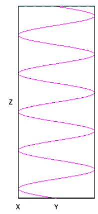


Figure 23

Since the bottle-like Klein bottle and the Möbius strip are different views of the same 4D Klein bottle, we can create a continuous transformation of views from one to the other. Let us consider the coordinate system: S_1 with coordinates (x, y, z, w), which is the coordinate system of the 4D Klein bottle, and S_2 with coordinates (X, Y, Z, W). Initially, the axes of both systems are aligned as follows:

$$X = x,\ Y = y,\ Z = z,\ W = w \quad (83)$$

We observe the subspace (X, Y, Z) of S_2. At the start, we see the subspace (x, y, z) of S_1, the projection we see is the familiar bottle-like Klein bottle. The 4D Klein bottle is immobile in the coordinate system S_1, which we rotate in the (Z, W) plane. Upon reaching an angle of -90°, the axes align as follows:

$$X = x,\ Y = y,\ Z = w,\ W = -z \quad (84)$$

At this point, the projection we see in the subspace (X, Y, Z) of S_2 is that onto the subspace (x, y, w) of S_1, which forms the Möbius strip.

By computing 20 images of the intermediate steps of the rotation, we have created a video animation that demonstrates the transformation of the projection resulting from the rotation of the 4D Klein bottle in the (Z, W) plane. You can view this animation at 1:05 of "Observing a 4D Klein Bottle in 4-Dimension" [5]. In 3D space, this transformation appears as a morphing of the bottle-like Klein bottle into the Möbius strip.

Additionally, below are three images depicting key steps of the rotation: the bottle-like Klein bottle before the rotation (Figure 24), an intermediate step (Figure 25), and the Möbius strip at the end (Figure 26). The rotated 4D Klein bottle is the second Klein bottle, whose fourth dimension is defined in equation (82). These three figures, along with the video animation, provide a more intuitive understanding of the 4D Klein bottle in four dimensions.

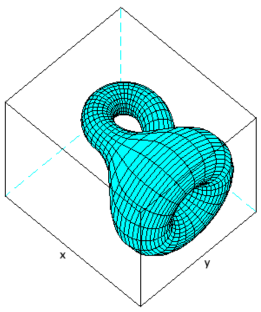


Figure 24

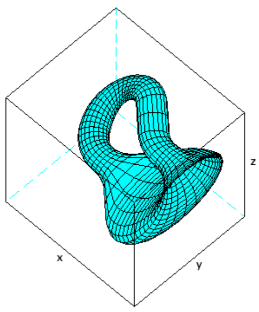


Figure 25

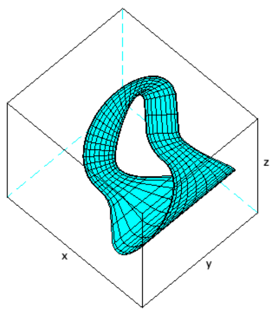


Figure 26

5 General Polar Coordinate System

5.1 Polar Coordinate System for N-Dimensional Space

An N-complex number specifies a geometric point by its radial distance from the origin and its angles of rotation in the ND space (see (41) and (42)). Therefore, the N-complex number system serves as a polar coordinate system for N-dimensional space, extending from the classic polar coordinate system in 2D plane and the spherical coordinate system in 3D space without an upper limit. This provides a significant advancement to mathematical systems that mathematicians have sought for a long time. I propose to name this system the “General polar coordinate system”.

For this new system to be valid, it must be able to specify all points in N-dimensional space. The formulas by which the N-complex number specifies a point in space are given below (see (43)):

Arbitrary point in the general polar coordinate system	$\boldsymbol{Z}_n = re^{i_1\theta_1} \cdots e^{i_{n-1}\theta_{n-1}}$ $= r\left(\left(\left((\cos\theta_1 + i_1 \sin\theta_1)\cos\theta_2 + i_2 \sin\theta_2\right)\cdots\right)\cos\theta_{n-1} + i_{n-1}\sin\theta_{n-1}\right)$	(85)
Arbitrary point in the Cartesian coordinate system	$\boldsymbol{Z}_n = x_0 + i_1 x_1 + \cdots + i_{n-1} x_{n-1}$	(86)

Equation (85) defines an N-complex number $\boldsymbol{Z}_n$ (see (43)), while equation (86) expresses it in Cartesian form. Both equations specify the same point in an ND space. In my paper «Extending complex number to spaces with 3, 4 or any number of dimensions»[1] I have proven that the expressions in equations (85) and (86) are equivalent. This equivalence arises from our ability to derive the modulus r and the arguments $\theta_1, \ldots, \theta_{n-1}$ from the Cartesian components $x_0, x_1, \ldots, x_{n-1}$, and vice versa. Thus, the ND polar coordinate system is a valid coordinate system.

To demonstrate the advantages of the new ND polar coordinate system, let us investigate hyperspheres.

5.2 Some Properties of Hyperspheres

5.2.1 Interior and Exterior of Hyperspheres

In 3D space, a sphere (2-sphere) is a closed surface because a point cannot move from inside to outside without crossing its surface. Therefore, we say that the sphere separates its interior from the exterior. However, when placed in 4D space, the 2-sphere becomes an open surface because a point can transition from inside to outside without crossing its surface. Given that hyperspheres exist in higher-dimensional space, it is uncertain whether they separate their interior from the exterior.

Let us consider the case of a 3-sphere in 4D space. A point in 4D space is defined by the 4D-complex number:

$$\boldsymbol{Z}_4 = re^{i\theta}e^{j\phi}e^{k\psi} \quad (87)$$

A unit 3-sphere is defined by keeping $r = 1$ in (87):

$$\boldsymbol{U}_4 = e^{i\theta}e^{j\phi}e^{k\psi} \quad (88)$$

When the point $\boldsymbol{Z}_4$ is inside the unit 3-sphere its modulus r is smaller than 1, ($r < 1$). To transition from inside the unit 3-sphere to the outside, r must continuously increase until it becomes greater than 1, ($r > 1$). In this process r must pass through the point where $r = 1$, at which $\boldsymbol{Z}_4$ crosses the surface of the unit 3-sphere.

Because the arguments θ, ϕ and ψ are arbitrary, the direction of $\boldsymbol{Z}_4$ is arbitrary. Therefore, regardless of its direction, $\boldsymbol{Z}_4$ cannot avoid encountering the surface of the the unit 3-sphere, meaning that no point can transition from inside the unit 3-sphere to the outside without crossing it. Thus, the unit 3-sphere effectively separates its interior from the exterior in 4D space, or the hypersphere is completely sealed.

We extend this property to N-sphere (hyperspheres) using equations (85) and (86), and conclude that N-spheres separate their interior from the exterior in (N+1)D space.

5.2.2 Thicknesses of Hyperspheres

A 2-sphere is a 2D surface and thus inherently possesses zero thickness. Hyperspheres, being higher-dimensional spaces, constitute volumes rather than mere surfaces. Intuitively, volumes should possess a non-zero thickness. However, as previously discussed, an unit N-sphere is defined by all points where $r = 1$. Consequently, its thickness equals that of a point, which is indeed zero. This observation leads to the counterintuitive conclusion that the thickness of the unit N-sphere, and by extension the ND space it occupies, is zero.

This conclusion may seem perplexing at first glance. How can a surface with zero thickness be a space with more than two dimensions? To address this question, let us consider the three free arguments θ, ϕ and ψ of the unit 3-sphere (see (88)). If the unit 3-sphere were limited to two dimensions, one of these arguments, let us say ψ, would need to remain constant. Consequently, the points on the unit 3-sphere corresponding to the free ψ could not be reached by $\boldsymbol{U}_4$. Therefore, the unit 3-sphere is incompletely defined with only two dimensions, and to fully encompass all its points, the unit 3-sphere must possess three dimensions.

5.2.3 Holes in 2-sphere and Hyperspheres

Spheres are typically closed surfaces, meaning they do not have holes. However, when a sphere is placed in 4D space, points can move from the inside to the outside without crossing it. This suggests the existence of “holes” through which points pass. First, let us define a unit 2-sphere in 4D space using (88) with ψ kept at zero:

$$\boldsymbol{U}_{2s} = e^{i\theta} e^{j\phi} e^{k0} \quad (89)$$

Now, consider the trajectory of a point initially inside the unit 2-sphere:

$$\boldsymbol{Z}_{2s} = r_{2s} e^{i\theta} e^{j\phi} e^{k0} \quad (90)$$

where $r_{2s} < 1$.

To exit the unit 2-sphere, we add to $\boldsymbol{Z}_{2s}$ a new 4-complex number with nonzero ψ and obtain a new point $\boldsymbol{Z'}_{2s}$:

$$\begin{aligned} \boldsymbol{Z'}_{2s} &= \boldsymbol{Z}_{2s} + r' e^{i\theta} e^{j\phi} e^{k\psi} \\ &= r_{2s} e^{i\theta} e^{j\phi} e^{k0} + r' e^{i\theta} e^{j\phi} e^{k\psi} \\ &= r'_{2s} e^{i\theta'} e^{j\phi'} e^{k\psi'} \end{aligned} \quad (91)$$

The value of r' increases until the modulus of $\boldsymbol{Z'}_{2s}$ (r'_{2s}) exceeds 1. We then rotate $\boldsymbol{Z'}_{2s}$ back to ψ=0, returning it to the space that the unit 2-sphere occupies. The final $\boldsymbol{Z'}_{2s}$ is outside the unit 2-sphere because the modulus of $\boldsymbol{Z'}_{2s}$ is bigger than 1, ($r'_{2s} > 1$). However, the trajectory of $\boldsymbol{Z'}_{2s}$ doesn't intersect the unit 2-sphere because ψ is nowhere zero on the trajectory except at the start and the end. This suggests that the trajectory of $\boldsymbol{Z'}_{2s}$ passes through a “hole” in the unit 2-sphere.

Points passing through non-existing “holes” appears intriguing. To gain an intuitive sense of this concept, let us consider a circle, which is a 1-sphere in a plane. The interior of a circle forms a disc, but when viewed from a 3D space (see Figure 21), the disc is a hole. Similarly, the interior of a 2-sphere forms a ball, but from a region in the 4D space where ψ is not equal to zero, the ball is a hole. Thus, 2-spheres do have holes when placed in 4D space, even though they are invisible to us 3D beings.

By generalizing this idea to N-spheres, we conclude that all N-spheres have holes when placed in (N+1)D spaces. Extending this concept to a 3D Klein bottle, we can think that the neck passes into the body of a 3D Klein bottle through a hole that exists only in 4D space.

5.2.4 Remark about the Proof of the Poincaré Conjecture

All these explorations about hyperspheres make me think of the Poincaré conjecture, the proof of which was quite laborious. Now that we have N-complex numbers and N-polar coordinate systems, could a simpler proof be within reach? Perhaps the development of more straightforward and concise proofs would pique the interest of mathematicians seeking to tackle this challenging problem.

5.3 Unification of 3-Dimensional Polar Coordinate Systems

We live in a 3D space, and our work involves the frequent use of 3D polar coordinate systems. The spherical coordinate system is the most commonly used, but there are multiple other spherical coordinate systems in use as well. For instance, to specify the orientation of a rigid body, such as an aircraft, we use two different systems of angles: Euler angles and Tait–Bryan angles. Similarly, to specify the orbit of a celestial body around the Sun, we use the system known as Keplerian elements.

Each of these systems works well for its specific purpose. However, despite the fact that the direction of a point in space is mathematically the same across all fields, each system follows different conventions for the angles. Therefore, specific methods and algorithms are required for each field. The multiplicity of spherical coordinate systems complicates the work within each field and hinder communication and exchange of ideas among scientists from different disciplines.

To address this problem, I propose our 3D polar coordinate system to be the standard polar coordinate system for 3D space. Can we simply retain the classic spherical coordinate system? No, because the system of angles of the classic spherical coordinate system follows a convention different from that used in the 3-complex number system. Therefore, the classic spherical coordinates of a point cannot be manipulated with 3-complex arithmetic operations.

Another advantage of our 3D polar coordinate system is that it maintains coherence among all N-complex number systems, as the following list shows:

- The 1-complex number system (real number system) works with the 1D polar coordinate system (number line).
- The 2-complex number system (classic complex number system) works with the 2D polar coordinate system.
- The 3-complex number system works with the 3D polar coordinate system.
- ...
- The N-complex number system works with the ND polar coordinate system.

If we retain the classic spherical coordinate system, this coherence would be broken.

Our 3D polar coordinate system would unify spherical coordinate systems across different fields. Furthermore, it would introduce arithmetic operations to 3D space akin to the classic polar coordinate system, marking a breakthrough in 3D geometry. Adopting the 3D polar coordinate system would simplify mathematical formulations, enhance 3D geometry with arithmetic operations, and accelerate numerical computations, as demonstrated by the work of Edgar Malinovsky[4]. Therefore, implementing the 3-complex number system in general-purpose computer software would immediately enhance numerical computations worldwide.

In my articles, « Computing orientation with complex multiplication but without trigonometric function »[2] and «Determination of the relative roll, pitch and yaw between arbitrary objects using 3D complex number»[3], I have explained methods for computing the orientation of rigid bodies. These techniques could prove useful to scientists working with computing the orientation of rigid bodies.

6 Discussion

In this article, we have provided a brief overview of the new N-complex number system, which extends the classic complex number system to ND space (see «Extending complex number to spaces with 3, 4 or any number of dimensions»[1]). The introduction of the N-complex number system has facilitated the creation of a Möbius strip in 3D space and two different 4D Klein bottles. Furthermore, we have demonstrated the computation of their projections onto subspaces of the 4D space (x, y, z, w). Additionally, we have computed the rotation of the 4D Klein bottles in the subspaces (x, y, w) and (x, w, z), and produced a video animation titled "Observing a 4D Klein Bottle in 4-Dimension" [5]. This animation illustrates the result of the rotation in 4D space, revealing a morphing of the familiar 3D Klein bottle into a Möbius strip.

The N-complex number system is equipped with arithmetic operations that function in ND geometry similarly to the classic complex number system in 2D plane. Additionally, it provides the long-missing polar coordinate system for ND space. Through this framework, we have deduced three properties of hyperspheres:

1. The surface of hyperspheres serves as a boundary, separating their interior from the exterior or completely sealing them off.
2. Despite having more than two dimensions, hyperspheres possess zero thickness.
3. An N-sphere, when placed in an (N+1)D space, exhibits at least one hole. A point initially located inside the N-sphere can traverse to the outside through this hole without crossing the N-sphere itself.

We propose the adoption of the 3D polar coordinate system as the standard polar coordinate system in 3D space. Standardization offers the advantage of reducing the undesirable complexity associated with the multiplicity of spherical coordinate systems. Furthermore, it enables arithmetic operations for 3D geometry and has the potential to accelerate numerical computations in 3D space.

The 3-complex number system and 3D polar coordinate system could be particularly beneficial for CAD (Computer-Aided Design) software used in mechanical and architectural design, control software for polar robotic arms, and electronic game development. These applications extensively utilize spherical coordinate systems to generate 3D images from numerical positions of points. Conversely, the new systems can be helpful for image recognition software that computes 3D geometric positions of points from 2D images—a complex computation in spherical coordinate systems. Image recognition plays an essential role in automatic driving, AI, and various other fields.

In conclusion, the N-complex number system and ND polar coordinate systems have significant potential for mathematical and engineering applications, promising to streamline computations and enhance problem-solving capabilities across various disciplines.

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Letter to readers

Dear readers,

The purpose of this article is to provide examples of the use of my N-complex number system, which I constructed in 2022. Indeed, without concrete examples of its application, no one can fully appreciate the potential of a new theory.

A concrete representation of a 4D Klein bottle has been desired by many but has never been presented. So, I decided to dive into the Klein bottle. Working with the Klein bottle was my first opportunity to practice with this system. To my surprise, the ease with which it allowed me to create 4D Klein bottles was remarkable. The 4D Klein bottles were generated smoothly without the slightest hitch. My video animations of the rotating 4D Klein bottle in 4D space, as well as the 3D slices ascending in the 4D space, were also computed effortlessly.

Edgar Malinovsky was the first to use my N-complex number system. He created many beautiful 3D fractal objects, which are showcased in «Rendering of 3D Mandelbrot, Lambda and other sets using 3D complex number system» .

When I created the N-complex number system, my aim was to develop a system that enables arithmetic operations in 3D space. Initially, I was not aware that the system I had created also functions as a polar coordinate system for ND space; it just happened to work out that way.

Since I was already exploring 4D space with the Klein bottle, I decided to use it to investigate hyperspheres. This led me to discover three surprising properties. I believe the N-complex number system has the potential to yield many more discoveries in hypergeometry. For example, it might offer a simpler proof for the Poincaré conjecture.

As I am not a professional mathematician, I do not know of other interesting conjectures in hypergeometry. However, the new hypergeometric tools—my N-complex number system and the polar coordinate system for ND space—will surely be helpful in solving some of them.

In fact, because I'm not a professional mathematician, I'm not able to write my articles in rigorous academic form. For example, the three properties of hyperspheres could be considered new theorems, but I lack the appropriate training to formulate them in theorem form. Consequently, all my previous articles were rejected by the editors of mathematical journals, which really hurt me. So, I have decided not to submit to mathematical journals but instead to post them online.

Since my articles are not published through academic channels, you cannot cite them in the reference list if you develop upon my work. Nevertheless, you can still mention my work in a footnote if it had been useful to you.

Kuan Peng, 4 June 2024