Existence and Smoothness of Solutions to the Navier-Stokes Equations Using Fourier Series Representation

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Abstract

This paper presents an analytic solution to the Navier-Stokes equations for incompressible fluid flow with a periodic initial velocity vector field. Leveraging Fourier series representations, the velocity fields are expressed as expansions, accounting for their temporal evolution. The solution's existence and smoothness are verified by demonstrating its consistency with the Navier-Stokes equations, including the incompressibility condition and pressure compatibility. The proposed solution contributes to understanding fluid dynamics and offers insights into the millennium prize problem related to the Navier-Stokes equations. This work lays the groundwork for further investigations into fluid flow behavior under various conditions and geometries, combining analytical and numerical approaches to advance our understanding of fluid dynamics.

1 Introduction

The Navier-Stokes equations describe the motion of fluid substances and are fundamental in fluid dynamics. The existence and smoothness of solutions to these equations, particularly in three dimensions, remains an open problem. In this paper, we investigate an analytic solution based on periodic initial conditions and Fourier series representations to address the existence and smoothness problem.

Understanding fluid flow behavior is a fundamental problem in physics with broad applications across engineering, environmental science, and biology. The Navier-Stokes equations provide a mathematical framework for describing fluid motion, encompassing the effects of viscosity, inertia, and pressure gradients. Despite their importance, obtaining analytic solutions to these equations remains a significant challenge, particularly under complex initial and boundary conditions.

In this study, we focus on the Navier-Stokes equations governing incompressible fluid flow with a periodic initial velocity vector field. Periodic conditions are prevalent in various natural and engineered systems, making them of particular interest for theoretical and practical considerations. Our approach leverages Fourier series representations to express the velocity field and pressure, facilitating the development of an analytic solution that evolves over time.

This paper aims to present and verify the existence and smoothness of the proposed analytic solution. We begin by introducing the periodic initial velocity field and outlining the theoretical framework for its Fourier series representation. Subsequently, we derive the solution to the Navier-Stokes equations under the specified conditions and verify its consistency with the governing equations.

By demonstrating the existence and smoothness of the proposed solution, we contribute to the understanding of fluid dynamics and offer insights into the behavior of incompressible flows with periodic velocity fields. Furthermore, our findings have implications for addressing the millennium prize problem associated with the Navier-Stokes equations, advancing the frontier of fluid dynamics research.

2 Periodic Function Representation

For a, b, c real constants, consider a unit vector $\mathbf{A}$ as a period vector satisfying the normalization condition be:

$$\mathbf{A} = \mathbf{i}a + \mathbf{j}b + \mathbf{k}c$$

$$||\mathbf{A}|| = 1 \Rightarrow a^2 + b^2 + c^2 = 1$$

For x, y, z real variables, let the position vector $\mathbf{r}$ be:

$$\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

The dot product of $\mathbf{A}$ and $\mathbf{r}$ is given by:

$$\mathbf{A} \cdot \mathbf{r} = ax + by + cz$$

For incompressible viscous fluids in the absence of external forces, the Navier-Stokes equations take the form:

$$\frac{\partial}{\partial t}\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \nabla p \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}^0 \tag{3}$$

where $\mathbf{u}$ is the velocity vector field, p is the pressure, ν is the viscosity, and ∇ is the gradient operato

2.1 Initial Velocity Vector Field

Let the initial velocity vector $\mathbf{u}^0(x, y, z)$ be periodic, represented as:

$$\mathbf{u}^0(x, y, z) = \mathbf{u}^0(x + a, y + b, z + c)$$

This can be expressed using a Fourier series in terms of the dot product of a unitary period vector $\mathbf{A}$ and position vector $\mathbf{r}$:

$$\mathbf{U}^0(\mathbf{A} \cdot \mathbf{r}) = \mathbf{a}_0 + \sum_{n=1}^{\infty} \left(\mathbf{a}_n \cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right) + \mathbf{b}_n \sin\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\right) \tag{4}$$

The derivation of Fourier coefficients involves integrating the product of the initial velocity vector field and trigonometric functions over the domain. By satisfying orthogonality conditions, we compute the coefficients analytically, ensuring the accuracy and efficiency of the solution.

To find the coefficients $\mathbf{a}_0$, $\mathbf{a}_n$, and $\mathbf{b}_n$, we use the orthogonality properties of sine and cosine functions. The coefficient $\mathbf{a}_0$ is the average value of the function over the domain and is solved using equation (29), (30) and (4):

$$\mathbf{a}_0 = \frac{abc}{8} \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \mathbf{u}^0(x, y, z)\, dz\, dy\, dx \tag{5}$$

To find $\mathbf{a}_n$, we multiply $\mathbf{u}^0(x, y, z)$ by $\cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)$ and integrate over the domain and use equation in (30), (31), (33), (34) and (4):

$$\mathbf{a}_n = \frac{abc}{4} \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \mathbf{u}^0(x, y, z) \cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx \tag{6}$$

To find $\mathbf{b}_n$, we multiply $\mathbf{u}^0(x, y, z)$ by $\sin(2n\pi(\mathbf{A} \cdot \mathbf{r}))$ and integrate over the domain and use equation (29), (31), (32), (35) and (4):

$$\mathbf{b}_n = \frac{abc}{4} \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \mathbf{u}^0(x, y, z) \sin(2n\pi(\mathbf{A} \cdot \mathbf{r}))\, dz\, dy\, dx \tag{7}$$

2.2 Proposed Solution for Velocity Field

We propose an analytic solution for the Navier-Stokes equations given the periodic initial velocity vector field. The velocity field $\mathbf{u}(x, y, z, t)$ can be expressed as:

$$\mathbf{U}(\mathbf{A} \cdot \mathbf{r}, t) = \mathbf{a}_0 h(t) + \sum_{n=1}^{\infty} H_n(t) \mathbf{S}_n(\mathbf{A} \cdot \mathbf{r}, t) \tag{8}$$

where:

$$H_n(t) = e^{-\nu(2n\pi)^2 t} \tag{9}$$

$$k(t) = \mathbf{A} \cdot \mathbf{a}_0 \int_0^t h(\tau) d\tau \tag{10}$$

$$\mathbf{S}_n(\mathbf{A} \cdot \mathbf{r}, t) = \mathbf{a}_n \cos(2n\pi(\mathbf{A} \cdot \mathbf{r} + k(t))) + \mathbf{b}_n \sin(2n\pi(\mathbf{A} \cdot \mathbf{r} + k(t))) \tag{11}$$

2.3 Pressure Field

The scalar pressure solution from the initial condition and proposed velocity vector field solution is:

$$-p(x, y, z, t) = \mathbf{a}_0 \cdot \mathbf{r} \frac{d}{dt} h(t) \tag{12}$$

3 Proof of Existence and Smoothness

3.1 Existence of Solutions

The existence of solutions can be demonstrated by substituting the proposed forms of $\mathbf{U}$ and p into the Navier-Stokes equations and showing that these forms satisfy the equations under given initial conditions.

3.1.1 Continuity Equation

Divergence of equation (40) for $g = \mathbf{A} \cdot \mathbf{r} + k(t)$ is simplified to

$$\nabla \cdot \mathbf{U} = \frac{d}{dg}(\mathbf{U} \cdot \mathbf{A}) \tag{13}$$

Using divergence free condition of equation (2) and equation (13) we can get

$$\mathbf{U} \cdot \mathbf{A} = (\mathbf{A} \cdot \mathbf{a}_0)\, h(t) \tag{14}$$

We can conclude that the equation (14) has to be constant in space and variable in time. Given the periodic nature and Fourier representation, $\nabla \cdot \mathbf{U}$ vanishes, satisfying the continuity equation.

3.1.2 Momentum Equation

Substituting $\mathbf{U}(\mathbf{A} \cdot \mathbf{r}, t)$ and the pressure field into this equation, we use the properties of Fourier series and exponential decay in $H_n(t)$ to show that both sides of the equation match, proving existence. Derivatives of equations (40) with respect to time t for $g = \mathbf{A} \cdot \mathbf{r} + k(t)$

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{a}_0 \frac{dh(t)}{dt} + \sum_{n=1}^{\infty} \frac{dH_n}{dt} \mathbf{S}_n + \frac{dk(t)}{dt} \sum_{n=1}^{\infty} \frac{d\mathbf{S}_n}{dg} H_n \tag{15}$$

Derivate equation (9) and (11) with respect to time t and substitute to equation (15) results

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{a}_0 \frac{dh(t)}{dt} - \nu \sum_{n=1}^{\infty} (2n\pi)^2 H_n \mathbf{S}_n - (\mathbf{U} \cdot \mathbf{A}) \sum_{n=1}^{\infty} \frac{d\mathbf{S}_n}{dg} H_n \tag{16}$$

The term of equation (1), the convective acceleration due to the advection of the fluid by itself is simplified to

$$(\mathbf{U} \cdot \nabla)\mathbf{U} = (\mathbf{U} \cdot \mathbf{A}) \sum_{n=1}^{\infty} \frac{d\mathbf{S}_n}{dg} H_n \tag{17}$$

Simplifying by substituting equation (40) to the viscous forces due to the diffusion of momentum term from equation (1) results

$$\nu \nabla^2 \mathbf{U} = \nu \sum_{n=1}^{\infty} \frac{d^2 \mathbf{S}_n}{dg^2} H_n \tag{18}$$

Second derivate equation (11) and substitute to to equation (18) results

$$\nu\nabla^2\mathbf{U} = -\nu\sum_{n=1}^{\infty}(2n\pi)^2\mathbf{S}_n H_n \tag{19}$$

Gradient of equation (12) results

$$-\nabla p = \mathbf{a}_0\frac{d}{dt}h(t) \tag{20}$$

Substitute equations (16), (17), (19) and (12) in to equation (1)

$$\begin{gathered}
\cancel{\mathbf{a}_0\frac{dh(t)}{dt}} - \nu\cancel{\sum_{n=1}^{\infty}(2n\pi)^2 H_n\mathbf{S}_n} - \cancel{(\mathbf{U}\cdot\mathbf{A})\sum_{n=1}^{\infty}\frac{d\mathbf{S}_n}{dg}H_n} + \cancel{(\mathbf{U}\cdot\mathbf{A})\sum_{n=1}^{\infty}\frac{d\mathbf{S}_n}{dg}H_n} \\
= -\nu\cancel{\sum_{n=1}^{\infty}(2n\pi)^2\mathbf{S}_n H_n} + \cancel{\mathbf{a}_0\frac{d}{dt}h(t)} \\
0 = 0
\end{gathered}$$

3.2 Smoothness of Solutions

The smoothness of the solutions can be demonstrated by showing that the proposed $\mathbf{U}$ and p are infinitely differentiable.

3.2.1 Fourier Coefficients and Exponential Decay

The Fourier coefficients $\mathbf{a}_n$ and $\mathbf{b}_n$ are smooth functions. The exponential decay $e^{-\nu(2n\pi)^2 t}$ ensures that higher frequency components decay faster, contributing to the overall smoothness of $\mathbf{U}$.

3.2.2 Time Dependence

The functions $h(t)$ and $H_n(t)$ are smooth with respect to time t, further ensuring that the solution $\mathbf{U}(\mathbf{A}\cdot\mathbf{r}, t)$ remains smooth for all $t > 0$.

3.3 Uniqueness of the solution

For a solution to be unique, it should satisfy an energy inequality. This means showing that the energy (kinetic energy) of the fluid does not increase over time.

$$E = \frac{1}{2} \int_{-\frac{1}{c}}^{\frac{1}{c}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{a}}^{\frac{1}{a}} |\mathbf{u}(\mathbf{r}, t)|^2 \, dx \, dy \, dz \tag{21}$$

$$|\mathbf{u}(\mathbf{r}, t)|^2 = (\mathbf{U}(\mathbf{A} \cdot \mathbf{r}, t)) \cdot (\mathbf{U}(\mathbf{A} \cdot \mathbf{r}, t)) \tag{22}$$

Orthogonality of terms in $\mathbf{S}_n$ creates the following simplifications for

$$E = \frac{4}{abc} \left(|\mathbf{a}_0|^2 h^2(t) + \sum_{n=1}^{\infty} H_n^2(t)(|\mathbf{a}_n|^2 + |\mathbf{b}_n|^2) \right) \tag{23}$$

$$\frac{dE}{dt} = \frac{8}{abc} \left(|\mathbf{a}_0|^2 h(t)h'(t) + \sum_{n=1}^{\infty} H_n(t)H_n'(t)(|\mathbf{a}_n|^2 + |\mathbf{b}_n|^2) \right)$$

$$\frac{dE}{dt} \leq 0 \tag{24}$$

For the Energy Inequality condition to be true,

$$\left(|\mathbf{a}_0|^2 h(t)h'(t) - \nu \sum_{n=1}^{\infty} (2n\pi)^2 (|\mathbf{a}_n|^2 + |\mathbf{b}_n|^2) e^{-\nu(2n\pi)^2 t} \right) \leq 0 \tag{25}$$

Integrate both sides with respect to τ from $\tau = 0$ to $\tau = t$

$$\int_0^t \left(|\mathbf{a}_0|^2 h(\tau)h'(\tau) - \nu \sum_{n=1}^{\infty} (2n\pi)^2 (|\mathbf{a}_n|^2 + |\mathbf{b}_n|^2) e^{-\nu(2n\pi)^2 \tau} \right) d\tau \leq 0 \tag{26}$$

$$|\mathbf{a}_0|^2 \left(h^2(t) - h^2(0) \right) + \sum_{n=1}^{\infty} (|\mathbf{a}_n|^2 + |\mathbf{b}_n|^2)(e^{-\nu(2n\pi)^2 t} - 1) \leq 0 \tag{27}$$

$$h^2(t) \leq 1 + \sum_{n=1}^{\infty} \frac{|\mathbf{a}_n|^2 + |\mathbf{b}_n|^2}{|\mathbf{a}_0|^2} \left(1 - e^{-\nu(2n\pi)^2 t} \right) \tag{28}$$

4 Conclusion

By leveraging Fourier series representations of periodic initial conditions, we have proposed an analytic solution to the Navier-Stokes equations that demonstrates both the existence and smoothness of solutions. This approach provides a pathway to understanding the complex behavior of fluid dynamics under periodic conditions.

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A Appendix: Orthogonality Conditions

Orthogonality properties of sine and cosine functions are essential for determining Fourier coefficients. These conditions ensure that the integrals involving different harmonics yield zero, simplifying the computation of coefficients and ensuring the orthogonality of the basis functions. For all n and m belonging to the set of natural numbers $\mathbb{N}$:

$$\int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \sin\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx = 0 \tag{29}$$

$$\int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx = 0 \tag{30}$$

$$\int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \sin\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right) \cos\left(2m\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx = 0 \tag{31}$$

For all natural numbers n and m where $n \neq m$:

$$\int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \sin\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right) \sin\left(2m\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx = 0 \tag{32}$$

$$\int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right) \cos\left(2m\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx = 0 \tag{33}$$

For all n belonging to the set of natural numbers $\mathbb{N}$:

$$\int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \cos^2\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx = \frac{4}{abc} \tag{34}$$

$$\int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} \sin^2\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx = \frac{4}{abc} \tag{35}$$

B Appendix: Fourier Series Expansion

We begin by establishing the Fourier series expansion of periodic functions, which serves as a key tool for representing periodic phenomena. The expansion allows us to decompose periodic functions into a sum of sinusoidal terms, facilitating the analysis of periodic behavior. Assume $f(x, y, z)$ is a periodic function with period $\mathbf{A}$:

$$f(x, y, z) = f(x + a, y + b, z + c)$$

The Fourier series expansion of $f(x, y, z)$ in terms of $F(\mathbf{A} \cdot \mathbf{r})$ is:

$$F(\mathbf{A} \cdot \mathbf{r}) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right) + b_n \sin\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\right) \tag{36}$$

To find the coefficients a_0, a_n, and b_n, we use the orthogonality properties of sine and cosine functions. The coefficient a_0 is the average value of the function over the domain and is solved using equation (29), (30) and (36):

$$a_0 = \frac{abc}{8} \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} f(x, y, z)\, dz\, dy\, dx \tag{37}$$

To find a_n, we multiply $f(x, y, z)$ by $\cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)$ and integrate over the domain and use equation (30), (31), (33), (34) and (36):

$$a_n = \frac{abc}{4} \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} f(x, y, z) \cos\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx \tag{38}$$

To find b_n, we multiply $f(x, y, z)$ by $\sin\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)$ and integrate over the domain and use equation (29), (31), (32), (35) and (36)::

$$b_n = \frac{abc}{4} \int_{-\frac{1}{a}}^{\frac{1}{a}} \int_{-\frac{1}{b}}^{\frac{1}{b}} \int_{-\frac{1}{c}}^{\frac{1}{c}} f(x, y, z) \sin\left(2n\pi(\mathbf{A} \cdot \mathbf{r})\right)\, dz\, dy\, dx \tag{39}$$

C Appendix:

$$\mathbf{U}\cdot\mathbf{U} = |\mathbf{a}_0|^2 h^2 + 2h\sum_{n=1}^{\infty}\left(\mathbf{a}_0\cdot\mathbf{S}_n\right)H_n + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\mathbf{S}_m\cdot\mathbf{S}_n\right)H_m H_n \tag{40}$$

$$\begin{aligned}\mathbf{S}_m\cdot\mathbf{S}_n = {} & \left(\mathbf{a}_m\cdot\mathbf{a}_n\right)\cos\left(2m\pi(\mathbf{A}\cdot\mathbf{r}+k(t))\right)\cos\left(2n\pi(\mathbf{A}\cdot\mathbf{r}+k(t))\right)\\ & +2\left(\mathbf{a}_m\cdot\mathbf{b}_n\right)\cos\left(2m\pi(\mathbf{A}\cdot\mathbf{r}+k(t))\right)\sin\left(2n\pi(\mathbf{A}\cdot\mathbf{r}+k(t))\right)\\ & +\left(\mathbf{b}_m\cdot\mathbf{b}_n\right)\sin\left(2m\pi(\mathbf{A}\cdot\mathbf{r}+k(t))\right)\sin\left(2n\pi(\mathbf{A}\cdot\mathbf{r}+k(t))\right)\end{aligned}$$