On the diophantine equation $X^6 + Y^6 = W^n + Z^n,\ n = 2, 3, 4$

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Abstract

In this paper, we proved that there are infinitely many integer solutions of $X^6 + Y^6 = W^n + Z^n,\ n = 2, 3, 4$.

1. Introduction

The diophantine equation $x^n + y^n = z^n + w^n, n = 2, 3, 4$ have been considered by many mathematicians. Euler[3] showed the parametric solution for $n = 4$. Choudhry[4] considered the equation $x^4 + hy^4 = z^4 + hw^4$. Izadi[5] and Nabardi showed that there are infinitely many integer solutions of $x^4 + y^4 = 2(z^4 + w^4)$. Janfada[6] and Nabardi considered the equation $x^4 + y^4 = h(z^4 + w^4)$. Shabani-Solt[8], Yusefnejad, and Janfada considered the equation $x^6 + ky^3 = z^6 + kw^3$ with $k < 500$. Muthuvel[7] and Venkatraman showed that there are infinitely many integer solutions of $X^4 - Y^4 = R^2 - S^2$.

Izadi[1] and Baghalaghdam proved that $\sum_{i=1}^{n} a_i x_i^6 + \sum_{i=1}^{m} b_i y_i^3 = \sum_{i=1}^{n} a_i X_i^6 + \sum_{i=1}^{m} b_i Y_i^3$ has infinitely many integer solutions.

They[2] also proved that $a(X_1'^5 + X_2'^5) + \sum_{i=0}^{n} a_i X_i^5 = b(Y_1'^3 + Y_2'^3) + \sum_{i=0}^{m} b_i Y_i^3$ has infinitely many integer solutions.

This author[10] proved that there are infinitely many integer solutions of $X^6 - Y^6 = W^n - Z^n,\ n = 2, 3, 4$ last time.

The objective of this work is to prove that there are infinitely many integer solutions of $X^6 + Y^6 = W^n + Z^n,\ n = 2, 3, 4$.

2. Solving the diophantine equation $X^6 + Y^6 = W^2 + Z^2$

$$X^6 + Y^6 = W^2 + Z^2 \tag{2.1}$$

We give three methods to prove that there are infinitely many integer solutions of $X^6 + Y^6 = W^2 + Z^2$.

2.1. Method-1

Let $X = a$ and $Y = b$ where $a, b \in \mathbb{Z}$.

We use the factorization of

$$X^6 - Z^2 = W^2 - Y^6 \implies (a^3 - Z)(a^3 + Z) = (W - b^3)(W + b^3).$$

Consider the simultaneous equations,

$$\begin{cases} a^3 + Z = (b^3 + W)t \\ a^3 - Z = \dfrac{W - b^3}{t} \end{cases}$$

We have $(W, Z) = \left(\dfrac{2a^3t - t^2b^3 + b^3}{t^2 + 1},\ \dfrac{a^3t^2 - a^3 + 2tb^3}{t^2 + 1}\right)$ where $t \in \mathbb{Z}$.

After canceling the denominators, we get

$$(X, Y, Z, W) = (a(t^2 + 1),\ b(t^2 + 1),\ (a^3t^2 - a^3 + 2tb^3)(t^2 + 1)^2,\ (2a^3t - t^2b^3 + b^3)(t^2 + 1)^2).$$

2.2. Method-2

Let $X = a,\ Y = b,\ W = pt + a^3,\ Z = t + b^3$ where $a, b, p, t \in \mathbb{Z}$.

We obtain

$$(1 + p^2)t + (2a^3p + 2b^3) = 0.$$

Hence, put $t = -\dfrac{2(a^3p + b^3)}{1 + p^2}$,

then we acquire a parametric solution as follows.

$$(X, Y, Z, W) = (a(1 + p^2),\ b(1 + p^2),\ (1 + p^2)^2(-2a^3p - b^3 + b^3p^2),\ (1 + p^2)^2(a^3p^2 + 2pb^3 - a^3)).$$

2.3. Method-3

Let $X = at,\ Y = bt,\ W = ct,\ z = \dfrac{Z}{t}$ where $a, b, c, t \in \mathbb{Z}$.

We get

$$(a^6 + b^6)t^4 = c^2 + z^2 = 0. \tag{2.2}$$

Hence, we have

$$z^2 = (a^6 + b^6)t^4 - c^2.$$

Take $(a, b, c) = (2, 1, 1)$, we get

$$Q : z^2 = 65t^4 - 1.$$

The quartic equation is birationally equivalent to an elliptic curve E,

$$E : N^2 = M^3 + 260M.$$

We know E has rank 2 and generators are $P(M, N) = (10, 60)$ and $(64, -528)$ using software **SAGE**[9].

The point $P(M, N)$ is of infinite order, and the multiples $mP, m = 2, 3, \cdots$ give infinitely many points on E.

For instance, we obtain two quartic points corresponding to $P(M, N)$,

$$Q(t, z) = \Big(-\frac{5}{14}, \frac{47}{196}\Big), \Big(\frac{1153}{3071}, \frac{5092328}{9431041}\Big).$$

We give two solutions of equation (2.1) using the two points $Q(t, z)$.

$$(X, Y, Z, W) = (10,\ 5,\ 235,\ 980),$$

$$(X, Y, Z, W) = (2306,\ 1153,\ 5871454184,\ 10873990273).$$

3. Solving the diophantine equation $X^6 + Y^6 = W^3 + Z^3$

$$X^6 + Y^6 = W^3 + Z^3 \tag{3.1}$$

We give two methods to prove that there are infinitely many integer solutions of $X^6 + Y^6 = W^3 + Z^3$.

3.1. Method-1

Let $X = a,\ Y = b,\ Z = pt + a^2,\ W = t + b^2$ with $a, b, p, t \in \mathbb{Z}$

We obtain

$$((p^3+1)t^2 + (3a^2p^2 + 3b^2)t + (3a^4p + 3b^4) = 0.$$

Hence, take $p = -\dfrac{b^4}{a^4}$ and $t = -\dfrac{3b^2a^6}{a^6 - b^6}$,

then we acquire a parametric solution as follows.

$$\begin{aligned}
X &= a(a^6 - b^6),\\
Y &= b(a^6 - b^6),\\
Z &= -(a^6 - b^6)b^2(2a^6 + b^6),\\
W &= (a^6 - b^6)a^2(2b^6 + a^6).
\end{aligned}$$

3.2. Method-2

First, we obtain a parametric solution of equation (3.2).

$$U^3 + V^3 = Z^3 + W^3 \tag{3.2}$$

Let $U = pt + a, V = qt + b, W = pt + c, Z = qt + d$ with $a, b, c, d, p, q, t \in \mathbb{Z}$ where $a^3 + b^3 = c^3 + d^3$.

$$(3ap^2 + 3bq^2 - 3dq^2 - 3cp^2)t^2 + (3a^2p + 3b^2q - 3d^2q - 3c^2p)t = 0.$$

Take

$$t = -\frac{a^2p + b^2q - d^2q - c^2p}{ap^2 + bq^2 - dq^2 - cp^2},$$

then we acquire a parametric solution as follows.

$$\begin{aligned}
U &= (c^2 - ac)p^2 + (d^2 - b^2)qp + (ba - da)q^2,\\
V &= (ba - bc)p^2 + (c^2 - a^2)qp + (-bd + d^2)q^2,\\
Z &= (da - dc)p^2 + (c^2 - a^2)qp + (-b^2 + bd)q^2,\\
W &= (-a^2 + ac)p^2 + (d^2 - b^2)qp + (bc - dc)q^2.
\end{aligned}$$

Hence, we consider the simultaneous equations (3), (4),

$$u^2 = (c^2 - ac)p^2 + (d^2 - b^2)qp + (ba - da)q^2, \tag{3.3}$$
$$v^2 = (ba - bc)p^2 + (c^2 - a^2)qp + (-bd + d^2)q^2. \tag{3.4}$$

Put $(a, b, c, d) = (2,\ 16,\ 9,\ 15)$ then we obtain,

$$u^2 = -63p^2 + 31qp - 2q^2, \tag{3.5}$$
$$v^2 = 112p^2 - 77qp + 15q^2. \tag{3.6}$$

Take $x = \dfrac{p}{q}$ and parametrization of equation (3.5) with $(x,\ u) = (\dfrac{2}{9},\ \dfrac{4}{3})$. Substitute $x = \dfrac{2k^2 + 153 - 24k}{9(k^2 + 63)}$ into equation (3.6), then we have

$$v^2 = \frac{277k^4 + 92799k^2 + 764316 + 5880k^3 + 225288k}{81(k^2 + 63)^2}.$$

Put $s = 9v(k^2 + 63)$, hence we consider the quartic equation

$$Q: s^2 = 277k^4 + 5880k^3 + 92799k^2 + 225288k + 764316.$$

The quartic equation is birationally equivalent to an elliptic curve.

$$E: N^2 = M^3 - M^2 - 364688M + 11999856.$$

We know E has rank 2 and generators are $P(M,N) = (-415, -9576)$ and $(\frac{2957}{4}, \frac{96615}{8})$ using software **SAGE**[9].

The point $P(M,N)$ is of infinite order, and the multiples $mP, m = 2, 3, \cdots$ give infinitely many points on E.

Thus, we can obtain the quartic points corresponding to $P(M,N)$,

$$Q(k,s) = (-\frac{15}{2}, -\frac{6561}{4}), (-\frac{5865}{134}, \frac{472421889}{17956}).$$

We give two solutions of equation (1) using $Q(k,s)$.

$$(X,Y,Z,W) = (6,\ 81,\ 5982,\ 4089).$$

$$(X,Y,Z,W) = (5190414,\ 5832369,\ 40880961934782,\ -21110638986951).$$

4. Solving the diophantine equation $X^6 + Y^6 = W^4 + Z^4$

$$X^6 + Y^6 = Z^4 + W^4. \tag{4.1}$$

We prove that there are infinitely many integer solutions of $X^6 + Y^6 = W^4 + Z^4$. Let $X = at,\ Y = bt,\ W = ct,\ z = \frac{Z}{t}$ where $a, b, c, t \in \mathbb{Z}$.

We get

$$(a^6 + b^6)t^2 = c^4 + z^4. \tag{4.2}$$

Put $v = (a^6 + b^6)t$ and $(a, b, c) = (4, 1, 1)$, we have

$$Q: v^2 = 4097z^4 + 4097.$$

The quartic equation is birationally equivalent to an elliptic curve E.

$$E: N^2 = M^3 - 67141636M.$$

We know E has rank 2 and generators are $P(M,N) = (-256,\ -131040)$ and $\left(\frac{21167269313}{614656}, \frac{2991175443874143}{481890304}\right)$ using software **SAGE**[9].

The point $P(M,N)$ is of infinite order, and the multiples $mP, m = 2, 3, \cdots$ give infinitely many points on E.

Thus, we can obtain the quartic points corresponding to $P(M,N)$,

$$Q(z,v) = \left(-\frac{134021096}{50356223}, -\frac{1161086615964209153}{2535749194825729}\right), \left(-\frac{188034968}{108091793}, \frac{2383498299788538913}{11683835713954849}\right).$$

We give two solutions of equation (1) using Q.

$$(X,Y,Z,W) = (5606823091076,\ 1401705772769,\ 11919242440871550296,\ 10215742406696520751).$$

$$(X,Y,Z,W) = (1133596891348996,\ 283399222837249,\ 37981474450196340604904,\ 14270914463219203350527).$$

5. Concluding Remarks

We proved that there are infinitely many integer solutions of $x^6 + y^6 = z^n + w^n, n = 2, 3, 4$.
However, finding the primitive solutions of $x^6 + y^6 = z^n + w^n, n = 2, 3, 4$ remains an open problem.

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