Quantum Measurement Problem as Loss of Adiabatic Invariance

Ervin Goldfain

Ronin Institute, Montclair, New Jersey 07043

Email: ervin.goldfain@ronininstitute.org

Abstract

The measurement problem of Quantum Mechanics reflects the tension between the deterministic evolution of wavefunctions and their random collapse caused by experimental observations. Here we argue that, in the Hamiltonian picture of quantum dynamics, wavefunction collapse follows from the destruction of *adiabatic invariance* on ultrashort time scales. Once adiabatic invariance is lost, Planck's constant becomes meaningless, and Quantum Mechanics breaks down. We also suggest that, in the long-time limit, action quantization is a result of *Arnold diffusion*, a process describing the instability of nearly integrable Hamiltonian systems with more than two degrees of freedom.

Key words: measurement problem, wavefunction collapse, adiabatic invariance, Hamiltonian dynamics, Arnold diffusion, action quantization.

1. Adiabatic invariants in Hamiltonian dynamics

Following [1-2], consider a one-dimensional system characterized by the Hamiltonian $H(q, p; \lambda)$, which is dependent on a time-varying parameter $\lambda = \lambda(t)$. Assume that the system undergoes a finite periodic motion with period T_0 and that the parameter λ is slowly varying during T_0, that is,

$$d\lambda/dt << \lambda/T_0 \tag{1}$$

If λ were constant, the motion of the field would be strictly periodic with a constant energy $E = E(T_0)$. Since λ is slowly varying, averaging the energy rate over T_0 yields the approximation,

$$\overline{\frac{dE}{dt}} = \frac{d\lambda}{dt}\overline{\frac{\partial H}{\partial \lambda}} \tag{2}$$

For fixed E and λ, the canonical action of the system is the integral taken over the closed path C in phase space, namely,

$$I = \int_C \frac{pdq}{2\pi} = \iint \frac{dpdq}{2\pi} \tag{3}$$

By (1) – (3), the rate of the action average is an adiabatic invariant defined by

$$\boxed{\frac{\overline{dI}}{dt}=0} \tag{4}$$

To fix ideas, consider a one-dimensional oscillator with parameter independent Hamiltonian,

$$H(p,q)=\frac{1}{2}(p^2+\omega^2 q^2)=E \tag{5}$$

The phase space trajectory of (5) is an ellipse, and the adiabatic invariant is simply,

$$\bar{I}=I=\frac{E}{\omega}=\frac{ET_0}{2\pi} \tag{6}$$

In this case, (1), (4), (6) are held by default, and the canonical action (6) recovers the *Lorentz invariant* of classical and quantum field theory. We next proceed with the following couple of assumptions:

A1) The evolution of (5) is monitored using a measurement signal $\lambda(t)$ playing the role of *external parameter*.

A2) $\lambda(t)$ represents a piecewise continuous function of period $2L$.

The overall Hamiltonian of the system oscillator plus signal can be presented as,

$$H = \frac{1}{2}(p^2 + \omega^2 q^2) + g f(q, p; \lambda) + h(\lambda) \tag{7}$$

where f and h are analytic functions and the oscillator is assumed to couple weakly to the signal, that is,

$$g << 1 \tag{8}$$

The Fourier series decomposition of the signal is given by

$$\lambda(t) \propto \frac{a_0}{2} + \sum_1^\infty (a_n \cos \nu_n t + b_n \sin \nu_n t) \tag{9}$$

in which

$$\nu_n = \frac{n\pi}{L} \tag{10a}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \lambda(t) \cos \nu_n t \, dt \tag{10b}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \lambda(t) \sin \nu_n t \, dt \tag{10c}$$

If time is counted in discrete intervals with $t_n = n t_0$, invoking the uncertainty relations of Fourier analysis gives

$$\Delta \nu_n \Delta t_n \geq 1/2 \tag{11}$$

where

$$\Delta \nu_n = (\pi / L) \Delta n \tag{12}$$

and

$$\Delta t_n = t_0 \Delta n \tag{13}$$

By (11), ultrashort time measurements ($t_0 \to 0$) imply an unbounded uncertainty in the number of terms entering (9), $\Delta n \to \infty$. Hence, taking $\Delta t_n = 0$, renders both the signal $\lambda(t)$ and its time rate undefined. As a result,

(1) turns into an ill-defined condition, ruins the adiabatic invariance mandated by (4) *along with the very meaning of Planck's constant*.

Although the above analysis has focused on classical Hamiltonian dynamics of periodic systems, its extrapolation to a quantum mechanical context is straightforward.

2. Arnold diffusion and action quantization

Before delving into the topic of Arnold diffusion, it is worthwhile recalling a few important concepts of classical Hamiltonian dynamics.

The formalism of action-angle variables consists of replacing the generalized coordinates and momenta through the transformation [1-2]

$$(q,p)\rightarrow(\theta,I) \tag{14}$$

Action-angle variables are canonically conjugate and introduced through the generating function

$$S(q,I)=\int_q p(q,H)dq \tag{15}$$

The action integral and its conjugate angle take the form,

$$I = \frac{1}{2\pi}\int_C p(q,H)\,dq = I(H) \tag{16a}$$

$$\theta = \frac{\partial S(q,I)}{\partial I} \tag{16b}$$

Since I is a cyclic variable, the corresponding drift of (15) per each period of I amounts to [1]

$$\Delta S = 2\pi I \tag{17}$$

Consider a nearly integrable periodic system with N degrees of freedom defined by the Hamiltonian [3-4]

$$H = H_0(I) + \varepsilon H_1(I,\theta,\varepsilon) \tag{18}$$

where $0 \le \varepsilon \le \varepsilon_0$ is a small perturbation and $H_0(I)$ is the unperturbed Hamiltonian, taken to be fully integrable in the limit $\varepsilon = 0$. The frequency of the unperturbed motion is determined by

$$\omega(I) = \frac{\partial H_0}{\partial I} \tag{19}$$

For $\varepsilon \le \varepsilon_0 << 1$, the equations of motion read

$$\dot{I} = -\frac{\partial H(\theta, I)}{\partial \theta} = 0 \tag{20a}$$

$$\dot{\theta} = \frac{\partial H(\theta, I)}{\partial I} = \omega(I) \tag{20b}$$

in which $I, \theta \in \mathbf{R}^N$. The solution of (20) lies on invariant N- tori residing in the phase-space of dimension $\mathbf{R}^{2N}$. For $N \le 2$, all solutions are stable since 2-tori confine trajectories on a 3-dimensional energy surface. This is no longer the case for $N \ge 3$ where, according to the *Arnold diffusion conjecture*, the action of nearly integrable systems changes by $O(1)$ over a sufficiently long time. Fig. 1 graphically explains why phase-space trajectories are confined by lines (so-called KAM surfaces or tori) in a 2-dimensional space but wander off in 3 or more dimensions [3].

Introducing the assumption

$$|H_0| < c_1, \ |H_1| < c_1 \tag{21}$$

where c_1 is a positive constant, and taking the unperturbed Hamiltonian to represent a quasi-convex function of the action variable, the following condition holds [4]

$$\delta I = \left|I(t) - I(0)\right| < C_1 \varepsilon^{1/(2N)} \tag{22}$$

over sufficiently long-times satisfying

$$0 \le t \le \exp\left(C_2^{-1} \big/ \varepsilon^{1/2N} \right) \tag{23}$$

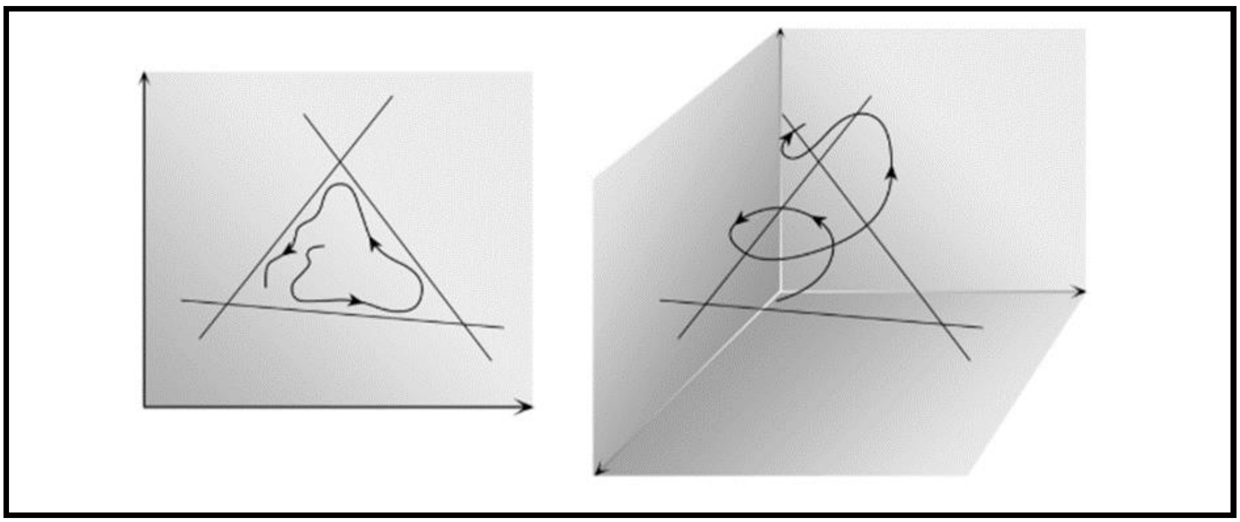

Fig. 1 Phase-space trajectories escaping KAM surfaces in 3 dimensions [3]

In (22) and (23), C_1, C_2 are also positive constants. By (17), the corresponding drift in action satisfies the inequality

$$\delta(\Delta S) = 2\pi\delta I < O(C_1 \varepsilon^{1/2N}) \tag{24}$$

Normalizing (24) to C_1 confirms that the drift in action is of $O(1)$, which naturally replicates the process of *action quantization* for $N >> 1$. The transition from classical to quantum behavior is expected to occur when

$$0 \leq t \leq \exp\left(C_2^{-1} / \varepsilon^{1/2N}\right);\ N >> 1 \tag{25}$$

leading to

$$\boxed{\delta(\Delta S) = 2\pi\delta I = O(1)} \tag{26}$$

It follows from these considerations that action quantization (and its associated Planck constant) may be mapped to the long-time behavior of (26), as applied to large ensembles of oscillators in near equilibrium conditions.

References

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