Proof of Collatz conjecture

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Abstract

In this working paper we try to prove the Collatz conjecture also known the 3x+1 problem.

The conjecture :

let a_0 be a strictly positive integer and consider the recursive sequence $(\mathrm{a_n})_{\mathrm{n}\geq 0}$

$$\forall \mathrm{n} \in \mathbb{N} \qquad a_{(n+1)} = \begin{cases} \frac{a_n}{2} & ,\mathrm{if}\ a_n = 2\mathrm{k}\ ,\ \ k \in \mathbb{N} \\ 3a_n + 1 & ,\mathrm{if}\ a_n = 2\mathrm{k} + 1\ ,\ k \in \mathbb{N} \end{cases}$$

So there exist $n_0 \in \mathbb{N}$ such that $a_{n_0} = 1$.

$$\text{Therfore } \ \forall k \in \mathbb{N} \begin{cases} a_{(n_0+3k)} = 1 \\ a_{(n_0+3k+1)} = 4 \\ a_{(n_0+3k+2)} = 2 \end{cases}$$

The sequence $(a_n)_{n\geq n_0}$ is the cycle$(1{,}4{,}2)$

Let $(\mathrm{u_n})_{\mathrm{n}\geq 0}$ be the subsequence of $(\mathrm{a_n})_{\mathrm{n}\geq 0}$ such that :

$u_0 = a_0$

$$\forall \mathrm{n} \in \mathbb{N} \qquad u_{(n+1)} = \begin{cases} \frac{u_n}{4} & ,if\ u_n = 4k\ ,\ \ k \in \mathbb{N} \\ \frac{u_n}{2} & ,if\ u_n = 4k + 2\ ,\ \ k \in \mathbb{N} \\ 3u_n + 1 & ,if\ u_n = 4k + 1 or\ u_n = 4k + 3\ ,\ k \in \mathbb{N} \end{cases}$$

Remark :the terms of the sequences $(\mathrm{a_n})_{\mathrm{n}\geq 0}$ and $(\mathrm{u_n})_{\mathrm{n}\geq 0}$ are strictly positive integers because a_0 is a strictly positive integer.

Lemma1 :

There exist no integer n_0 such that $\forall n \geq n_0\ \ u_n\ is\ a\ multiple\ of\ 5.$

Proof :

Suppose that there exist an integer n_0 such that $\forall n \geq n_0\ \ u_n\ is\ a\ multiple\ of\ 5$

take $u_{n_0}$and write $u_{n_0} = 5 \times 2^p q\ with\ p \in \mathbb{N}, q \in \mathbb{N}\ and\ \ q\ odd$

if $\mathrm{p} = 2\mathrm{k}$ we have $u_{(n_0+k)} = 5q$ then $u_{(n_0+k+1)} = 15q + 1$wich is not multiple of 5

if $\mathrm{p} = 2\mathrm{k} + 1$ we have $u_{(n_0+k+1)} = 5q$ then $u_{(n_0+k+2)} = 15q + 1$wich is not multiple of 5

this is a contradiction

so there exist no integer n_0 such that $\forall n \geq n_0\ \ u_n\ is\ a\ multiple\ of\ 5$

let the sequence $(v_n)_{n\geq 0}$ such that :

$v_0 = u_0 = a_0$

$$\forall n \in \mathbb{N} \;\; v_{(n+1)} = \begin{cases} \frac{v_n}{4} & ,if\ v_n=4k\ ,\ k\in\mathbb{N} \\ 3v_n+10 & ,if\ v_n=4k+2\ ,\ k\in\mathbb{N} \\ 3v_n+1 & ,if\ v_n=4k+1\ ,\ k\in\mathbb{N} \\ 3v_n+11 & ,if\ v_n=4k+3\ ,\ k\in\mathbb{N} \end{cases}$$

Lemma2 :

there exist an integer n_0 such that $\forall n \geq n_0$ The sequence $(v_n)_{n\geq n_0}$is the cycle $(1,4)$ or the cycle$(10,40)$

or the cycle$(11,44)$.

Proof :

Let's consider the subsequence $(w_n)_{n\geq 0}$ of $(v_n)_{n\geq 0}$ such that :

$w_0 = v_0 = a_0$

$$\forall n \in \mathbb{N} \;\; w_{(n+1)} = \begin{cases} \frac{w_n}{4} & ,if\ w_n=4k\ ,\ k\in\mathbb{N} \\ \frac{3w_n+10}{4} & ,if\ w_n=4k+2\ ,\ k\in\mathbb{N} \\ \frac{3w_n+1}{4} & ,if\ w_n=4k+1\ ,\ k\in\mathbb{N} \\ \frac{3w_n+11}{4} & ,if\ w_n=4k+3\ ,\ k\in\mathbb{N} \end{cases}$$

Remark :the terms of the sequences $(v_n)_{n\geq 0}$ and $(w_n)_{n\geq 0}$ are strictly positive integers because a_0 is a strictly positive integer.

We have $\forall n \in \mathbb{N} \quad w_{(n+1)} \leq \frac{3w_n+11}{4}$

So $\forall n \in \mathbb{N} \;\; 4w_{(n+1)} \leq 3w_n + 11$

So $\forall n \in \mathbb{N} \;\; 4(w_{(n+1)} - 11) \leq 3(w_n - 11)$

So $\forall n \in \mathbb{N} \;\; (w_{(n+1)} - 11) \leq \frac{3(w_n-11)}{4}$

So if $\forall n \in \mathbb{N} \;\; w_n > 11$ the sequence $(z_n)_{n\geq 0}$ where $z_n = w_n - 11$ satisfies :

$\forall n \in \mathbb{N} \;\; z_n \geq 1$ and $\forall n \in \mathbb{N} \;\; z_{(n+1)} < z_n$ (because $\frac{3(w_n-11)}{4} < w_n - 11$)

the sequence of integers $(z_n)_{n\geq 0}$ is decreasing strictly so it will reach 0 .this is a contradiction

we deduce that There exist n_0 such that $w_{n_0} \leq 11$

(we can also use $\forall n \in \mathbb{N} \;\; (w_n - 11) \leq \left(\frac{3}{4}\right)^n (w_0 - 11)$)

1) If $w_{n_0} = 1$ The sequence $(w_n)_{n\geq n_0}$ is 1,1,1 … so $\forall n \in \mathbb{N}\; n \geq n_0 \;\; w_n = 1$
2) If $w_{n_0} = 2$ The sequence $(w_n)_{n\geq n_0}$ is 2,4,1,1,1 … so $\forall n \in \mathbb{N}\; n \geq (n_0 + 2) \;\; w_n = 1$
3) If $w_{n_0} = 3$ The sequence $(w_n)_{n\geq n_0}$ is 3,5,4,1,1,1 … so $\forall n \in \mathbb{N}\; n \geq (n_0 + 3) \;\; w_n = 1$
4) If $w_{n_0} = 4$ The sequence $(w_n)_{n\geq n_0}$ is 4,1,1,1 … so $\forall n \in \mathbb{N}\; n \geq (n_0 + 1) \;\; w_n = 1$
5) If $w_{n_0} = 5$ The sequence $(w_n)_{n\geq n_0}$ is 5,4,1,1,1 … so $\forall n \in \mathbb{N}\; n \geq (n_0 + 2) \;\; w_n = 1$
6) If $w_{n_0} = 6$ The sequence $(w_n)_{n\geq n_0}$ is 6,7,32,8,2,4,1,1,1 … so $\forall n \in \mathbb{N}\; n \geq (n_0 + 6) \;\; w_n = 1$
7) If $w_{n_0} = 7$ The sequence $(w_n)_{n\geq n_0}$ is 7,32,8,2,4,1,1,1 … so $\forall n \in \mathbb{N}\; n \geq \; (n_0 + 5)\; w_n = 1$
8) If $w_{n_0} = 8$ The sequence $(w_n)_{n\geq n_0}$ is 8,2,4,1,1,1 … so $\forall n \in \mathbb{N}\; n \geq (n_0 + 3) \;\; w_n = 1$

9) If $w_{n_0} = 9$ The sequence $(w_n)_{n\geq n_0}$ is 9,7,32,8,2,4,1,1,1 … so $\forall n \in \mathbb{N}\ n \geq (n_0+6)\ \ w_n = 1$
10) If $w_{n_0} = 10$ The sequence $(w_n)_{n\geq n_0}$ is 10,10,10 … so $\forall n \in \mathbb{N}\ n \geq n_0\ \ w_n = 10$
11) If $w_{n_0} = 11$ The sequence $(w_n)_{n\geq n_0}$ is 11,11,11 … so $\forall n \in \mathbb{N}\ n \geq n_0\ \ w_n = 11$

Let $m_0 = n_0 + 6$

So we have $\forall n \in \mathbb{N}\ \ n \geq m_0\ \ w_n = 1$ or $\forall n \in \mathbb{N}\ \ n \geq m_0\ \ w_n = 10$ or $\forall n \in \mathbb{N}\ \ n \geq m_0\ \ w_n = 11$

Since $(w_n)_{n\geq 0}$ is a subsequence of $(v_n)_{n\geq 0}$ We deduce that there exist an integer p_0 such that $v_{p_0} = w_{m_0}$

Since $w_{m_0} = 1$ or $w_{m_0} = 10$ or $w_{m_0} = 11$ we have $v_{p_0} = 1$ or $v_{p_0} = 10$ or $v_{p_0} = 11$

So The sequence $(v_n)_{n\geq p_0}$ is the cycle $(1,4)$ or the cycle$(10,40)$ or the cycle$(11,44)$.

So there exist an integer p_0 such that The sequence $(v_n)_{n\geq p_0}$ is the cycle $(1,4)$ or the cycle$(10,40)$

or the cycle$(11,44)$.

<u>Consequence</u> :

there exist an integer p'_0 such that The sequence $(v_n)_{n\geq p'_0}$ is the cycle $(4,1)$ or the cycle$(40,10)$

or the cycle$(44,11)$.

(take $p'_0 = p_0 + 1$)

<u>Lemma3</u> :

in $\mathbb{Z}/5\mathbb{Z}$ $\quad \forall n \in \mathbb{N} \quad \overline{u_n} = \overline{v_n}$

<u>Proof</u> :

$$\forall n \in \mathbb{N} \qquad 4u_{(n+1)} = \begin{cases} u_n & ,if\ u_n = 4k\ ,\ \ k \in \mathbb{N} \\ 2\,u_n & ,if\ u_n = 4k+2\ ,\ \ k \in \mathbb{N} \\ 4(3u_n+1) & ,if\ u_n = 4k+1 or\ u_n = 4k+3\ ,\ k \in \mathbb{N} \end{cases}$$

$$\text{So } \forall n \in \mathbb{N} \qquad \overline{4u_{(n+1)}} = \begin{cases} \overline{u_n} & ,if\ u_n = 4k\ ,\ \ k \in \mathbb{N} \\ \overline{2\,u_n} & ,if\ u_n = 4k+2\ ,\ \ k \in \mathbb{N} \\ \overline{4(3u_n+1)} & ,if\ u_n = 4k+1 or\ u_n = 4k+3\ ,\ k \in \mathbb{N} \end{cases}$$

We multiply by $\overline{4}$

$$\text{So } \forall n \in \mathbb{N} \qquad \overline{16u_{(n+1)}} = \begin{cases} \overline{4u_n} & ,if\ u_n = 4k\ ,\ \ k \in \mathbb{N} \\ \overline{8\,u_n} & ,if\ u_n = 4k+2\ ,\ \ k \in \mathbb{N} \\ \overline{16(3u_n+1)} & ,if\ u_n = 4k+1 or\ u_n = 4k+3\ ,\ k \in \mathbb{N} \end{cases}$$

$$\text{So } \forall n \in \mathbb{N} \qquad \overline{u_{(n+1)}} = \begin{cases} \overline{4u_n} & ,if\ u_n = 4k\ ,\ \ k \in \mathbb{N} \\ \overline{3\,u_n} & ,if\ u_n = 4k+2\ ,\ \ k \in \mathbb{N} \\ \overline{(3u_n+1)} & ,if\ u_n = 4k+1 or\ u_n = 4k+3\ ,\ k \in \mathbb{N} \end{cases} \qquad (1)$$

$$\forall n \in \mathbb{N}\ \ v_{(n+1)} = \begin{cases} \frac{v_n}{4} & ,if\ v_n=4k\ ,\ k\in\mathbb{N} \\ 3v_n+10 & ,if\ v_n=4k+2\ ,\ k\in\mathbb{N} \\ 3v_n+1 & ,if\ v_n=4k+1\ ,\ k\in\mathbb{N} \\ 3v_n+11 & ,if\ v_n=4k+3\ ,\ k\in\mathbb{N} \end{cases}$$

$$\text{So } \forall n \in \mathbb{N}\ \ 4v_{(n+1)} = \begin{cases} v_n & ,if\ v_n=4k\ ,\ k\in\mathbb{N} \\ 4(3v_n+10) & ,if\ v_n=4k+2\ ,\ k\in\mathbb{N} \\ 4(3v_n+1) & ,if\ v_n=4k+1\ ,\ k\in\mathbb{N} \\ 4(3v_n+11) & ,if\ v_n=4k+3\ ,\ k\in\mathbb{N} \end{cases}$$

$$\text{So } \forall n \in \mathbb{N}\ \ \overline{4v_{(n+1)}} = \begin{cases} \overline{v_n} & ,if\ v_n=4k\ ,\ k\in\mathbb{N} \\ \overline{4(3v_n+10)} & ,if\ v_n=4k+2\ ,\ k\in\mathbb{N} \\ \overline{4(3v_n+1)} & ,if\ v_n=4k+1\ ,\ k\in\mathbb{N} \\ \overline{4(3v_n+11)} & ,if\ v_n=4k+3\ ,\ k\in\mathbb{N} \end{cases}$$

We multiply by $\overline{4}$

$$\text{So } \forall n \in \mathbb{N}\ \ \overline{16v_{(n+1)}} = \begin{cases} \overline{4v_n} & ,if\ v_n=4k\ ,\ k\in\mathbb{N} \\ \overline{16(3v_n+10)} & ,if\ v_n=4k+2\ ,\ k\in\mathbb{N} \\ \overline{16(3v_n+1)} & ,if\ v_n=4k+1\ ,\ k\in\mathbb{N} \\ \overline{16(3v_n+11)} & ,if\ v_n=4k+3\ ,\ k\in\mathbb{N} \end{cases}$$

$$\text{So } \forall n \in \mathbb{N}\ \ \overline{v_{(n+1)}} = \begin{cases} \overline{4v_n} & ,if\ v_n=4k\ ,\ k\in\mathbb{N} \\ \overline{3v_n} & ,if\ v_n=4k+2\ ,\ k\in\mathbb{N} \\ \overline{(3v_n+1)} & ,if\ v_n=4k+1\ ,\ k\in\mathbb{N} \\ \overline{(3v_n+1)} & ,if\ v_n=4k+3\ ,\ k\in\mathbb{N} \end{cases} \qquad (2)$$

Since $\overline{u_0} = \overline{v_0}$ we deduce from (1) and (2) that the sequences $(\overline{u_n})_{n\geq 0}$ and $(\overline{v_n})_{n\geq 0}$ are equal

So in $\mathbb{Z}/5\mathbb{Z}$ $\forall n \in \mathbb{N}$ $\overline{u_n} = \overline{v_n}$

So there exist a sequence $(s_n)_{n\geq 0}$ with $s_n \in \mathbb{Z}$ such that $\forall n \in \mathbb{N}$ $u_n = v_n + 5s_n$

1) if The sequence $(v_n)_{n\geq n_0}$is the cycle $(4,1)$

we have $\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 4 + 5q_p \\ u_{(n_0+2p+1)} = 1 + 5Q_p \end{cases}$ where $q_p and\ Q_p \in \mathbb{Z}$

2) if The sequence $(v_n)_{n\geq n_0}$is the cycle $(40,10)$

we have $\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 40 + 5q_p \\ u_{(n_0+2p+1)} = 10 + 5Q_p \end{cases}$ where $q_p and\ Q_p \in \mathbb{Z}$

3) if The sequence $(v_n)_{n\geq n_0}$is the cycle $(44,11)$

we have $\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 44 + 5q_p \\ u_{(n_0+2p+1)} = 11 + 5Q_p \end{cases}$ where $q_p and\ Q_p \in \mathbb{Z}$

in the case 2) we have $\forall n \geq n_0\ \ u_n\ is\ a\ multiple\ of$ 5 so it is not possible.

in the case 3) we have $\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 4 + 5(q_p + 8) \\ u_{(n_0+2p+1)} = 1 + 5(Q_p + 2) \end{cases}$ where $q_p\ and\ Q_p \in \mathbb{Z}$

So $\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 4 + 5q'_p \\ u_{(n_0+2p+1)} = 1 + 5Q'_p \end{cases}$ where $q'_p and\ Q'_p \in \mathbb{Z}$

So we are in the case 1)

we deduce that :

$\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 4+5q_p \\ u_{(n_0+2p+1)} = 1+5Q_p \end{cases}$ where $q_p and Q_p \in \mathbb{Z}$

the sequence $(u_n)_{n\geq n_0}$is in the form :

$4+5q_0, 1+5Q_0, 4+5q_1, 1+5Q_1, 4+5q_2, 1+5Q_2, \dots \dots \dots.$

1) if $q_p = 4k_p$, $k_p \in \mathbb{Z}$

$u_{(n_0+2p)} = 4+5q_p = 4+20k_p$

So $u_{(n_0+2p+1)} = \frac{u_{(n_0+2p)}}{4} = 1+5k_p$ wich is of the form $1+5Q_p$

2) if $q_p = 4k_p + 1$, $k_p \in \mathbb{Z}$

$u_{(n_0+2p)} = 4+5q_p = 4+5(4k_p+1) = 9+20k_p$

So $u_{(n_0+2p+1)} = 3u_{(n_0+2p)} + 1 = 3(9+20k_p)+1 = 28+60k_p = 3+5(5+12k_p)$

wich is not of the form $1+5Q_p$

So $q_p \neq 4k_p + 1$

3) if $q_p = 4k_p + 2$, $k_p \in \mathbb{Z}$

$u_{(n_0+2p)} = 4+5q_p = 4+5(4k_p+2) = 14+20k_p$

So $u_{(n_0+2p+1)} = \frac{u_{(n_0+2p)}}{2} = 7+10k_p = 2+5(1+2k_p)$ wich is not of the form $1+5Q_p$

So $q_p \neq 4k_p + 2$

4) if $q_p = 4k_p + 3$, $k_p \in \mathbb{Z}$

$u_{(n_0+2p)} = 4+5q_p = 4+5(4k_p+3) = 19+20k_p$

So $u_{(n_0+2p+1)} = 3u_{(n_0+2p)} + 1 = 3(19+20k_p)+1 = 58+60k_p = 3+5(11+12k_p)$

wich is not of the form $1+5Q_p$

So $q_p \neq 4k_p + 3$

we deduce that $\forall p \in \mathbb{N}$ $q_p = 4k_p$ $k_p \in \mathbb{Z}$

So

$\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 4+20k_p \\ u_{(n_0+2p+1)} = 1+5k_p \end{cases}$ where $k_p \in \mathbb{Z}$

the sequence $(u_n)_{n\geq n_0}$is in the form :

$4+20k_0, 1+5k_0, 4+20k_1, 1+5k_1, 4+20k_2, 1+5k_2, \dots \dots \dots.$

By the same way

1) if $k_p = 4k'_p$, $k'_p \in \mathbb{Z}$

$u_{(n_0+2p+1)} = 1 + 5k_p = 1 + 20k'_p$

So $u_{(n_0+2p+2)} = 3u_{(n_0+2p+1)} + 1 = 3\big(1 + 20k'_p\big) + 1 = 4 + 60k'_p$ wich is of the form $4 + 20k_{(p+1)}$

2) if $k_p = 4k'_p + 1$, $k'_p \in \mathbb{Z}$

$u_{(n_0+2p+1)} = 1 + 5k_p = 1 + 5(4k'_p + 1) = 6 + 20k'_p$

So $u_{(n_0+2p+2)} = \frac{u_{(n_0+2p+1)}}{2} = 3 + 10k'_p$ wich is not of the form $4 + 20k_{(p+1)}$ (because it is odd)

So $k_p \neq 4k'_p + 1$

3) if $k_p = 4k'_p + 2$, $k'_p \in \mathbb{Z}$

$u_{(n_0+2p+1)} = 1 + 5k_p = 1 + 5(4k'_p + 2) = 11 + 20k'_p$

So $u_{(n_0+2p+2)} = 3u_{(n_0+2p+1)} + 1 = 3\big(11 + 20k'_p\big) + 1 = 34 + 60k'_p = 14 + 20(1 + 3k'_p)$

wich is not of the form $4 + 20k_{(p+1)}$

So $k_p \neq 4k'_p + 2$

4) if $k_p = 4k'_p + 3$, $k'_p \in \mathbb{Z}$

$u_{(n_0+2p+1)} = 1 + 5k_p = 1 + 5(4k'_p + 3) = 16 + 20k'_p$

So $u_{(n_0+2p+2)} = \frac{u_{(n_0+2p+1)}}{4} = 4 + 5k'_p$

Since $u_{(n_0+2p+2)}$ is in the form $4 + 20k_{(p+1)}$ we have $k'_p = 4k''_p$, $k''_p \in \mathbb{Z}$ (where $k''_p = k_{(p+1)}$)

So $k_p = 16k''_p + 3$

Since $\forall p \in \mathbb{N}$ $\begin{cases} u_{(n_0+2p)} = 4 + 20k_p \\ u_{(n_0+2p+1)} = 1 + 5k_p \end{cases}$ where $k_p \in \mathbb{Z}$

We deduce that for each $p \in \mathbb{N}$ we have the following two cases :

1) if $k_p = 4k'_p$ we have:

$$\begin{cases} u_{(n_0+2p)} = 4 + 80k'_p \\ u_{(n_0+2p+1)} = 1 + 20k'_p \end{cases} \quad \text{where } k'_p \in \mathbb{Z}$$

2) if $k_p = 16k''_p + 3$ we have :

$$\begin{cases} u_{(n_0+2p)} = 64 + 320k''_p \\ u_{(n_0+2p+1)} = 16 + 80k''_p \end{cases} \quad \text{where } k''_p \in \mathbb{Z}$$

So $\forall p \in \mathbb{N}$ we have :

$$\begin{cases} u_{(n_0+2p)} = 4 + 80k'_p \\ u_{(n_0+2p+1)} = 1 + 20k'_p \end{cases} \quad \text{where } k'_p \in \mathbb{Z} \qquad \text{or} \qquad \begin{cases} u_{(n_0+2p)} = 64 + 320k''_p \\ u_{(n_0+2p+1)} = 16 + 80k''_p \end{cases} \quad \text{where } k''_p \in \mathbb{Z}$$

Since $(u_n)_{n\geq 0}$ is a sequence of strictly positive integers we have :

$$\forall p \in \mathbb{N} \quad \begin{cases} u_{(n_0+2p)} = 4 + 80k'_p \\ u_{(n_0+2p+1)} = 1 + 20k'_p \end{cases} \text{ where } k'_p \in \mathbb{N} \qquad \text{or} \qquad \begin{cases} u_{(n_0+2p)} = 64 + 320k''_p \\ u_{(n_0+2p+1)} = 16 + 80k''_p \end{cases} \text{ where } k''_p \in \mathbb{N}$$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ u_n \ is\ multiple\ of\ 4\ or\ odd\ of\ the\ form\ 1 + 20k'_p$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ u_{(n+1)} = \frac{u_n}{4} \ \ or\ u_{(n+1)} = 3u_n + 1$ where $(3u_n + 1)$ is multiple of 4

Let $(x_n)_{n\geq n_0}$ the subsequence of the sequence $(u_n)_{n\geq n_0}$ where we keep all terms of $(u_n)_{n\geq n_0}$ except the terms following the odd terms of $(u_n)_{n\geq n_0}$ (the terms $(3u_n + 1)$ where u_n in the form $1 + 20k'_p$)

We skip the term $(3u_n + 1)$ and we keep the next term $\frac{3u_n+1}{4}$

$\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ x_n$ is multiple of 4 or odd of the the form $(1 + 20k'_p) \ \ k'_p \in \mathbb{N}$

the sequence $(x_n)_{n\geq n_0}$ satisfies $x_{n_0} = u_{n_0}$ and $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ x_{(n+1)} = \begin{cases} \frac{x_n}{4} & if\ x_n\ is\ multiple\ 0f\ 4 \\ \frac{(3x_n+1)}{4} & if\ x_n\ is\ odd \end{cases}$

$(x_n)_{n\geq n_0}$ is a sequence of strictly positive integers

We have $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ x_{(n+1)} = \frac{x_n}{4} \ \ or \ \ x_{(n+1)} = \frac{(3x_n+1)}{4}$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ x_{(n+1)} \leq \frac{(3x_n+1)}{4}$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ 4\, x_{(n+1)} \leq 3x_n + 1$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ 4\big(\, x_{(n+1)} - 1\big) \leq 3(x_n - 1)$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ \big(\, x_{(n+1)} - 1\big) \leq \frac{3}{4}(x_n - 1)$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ (\, x_n - 1) \leq \left(\frac{3}{4}\right)^{(n-n_0)} (x_{n_0} - 1)$

So $\forall n \in \mathbb{N} \ \ n \geq n_0 \ \ 0 \leq (\, x_n - 1) \leq \left(\frac{3}{4}\right)^{(n-n_0)} (x_{n_0} - 1)$

So $\lim_{n\to+\infty} x_n = 1$

Since $(x_n)_{n\geq n_0}$ is a sequence of strictly positive integers we deduce that :

There exist $n'_0 \in \mathbb{N} \ \ n'_0 \geq n_0$ such that $\forall n \in \mathbb{N} \ \ n \geq n'_0 \ \ x_n = 1$

Since $(x_n)_{n\geq n_0}$ is a subsequence of the sequence $(u_n)_{n\geq n_0}$

So also $(x_n)_{n\geq n_0}$ is a subsequence of the sequence $(a_n)_{n\geq 0}$

we deduce that there exist $N_0 \in \mathbb{N}$ such that $a_{N_0} = 1$

So The sequence $(a_n)_{n\geq N_0}$ is the cycle$(1,4,2)$.