Clarifying an Early Step in Hardy's Transcendence of π Proof

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Abstract

We clarify and strengthen Hardy's footnote proof of an essential step in his proof of the transcendence of π. We show that ri is algebraic if and only if r is algebraic.

Introduction

On page 223 Hardy gives a proof that π is transcendental [1]. His proof shows that πi does not solve a integer polynomial, but technically this isn't showing π doesn't so solve an integer polynomial. He needs to show that the one implies the other. Here is his one line proof.

If $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ and $y = xi$ then

$$a_0y^n - a_2y^{n-2} + \cdots + i(a_1y^{n-1} - a_3y^{n-3} + \ldots) = 0$$

and so

$$(a_0y^n - a_2y^{n-2} + \ldots)^2 + (a_1y^{n-1} - a_3y^{n-3} + \ldots)^2 = 0.$$

This is very condensed and presupposes that $n \equiv 0$ mod(4) which he doesn't stipulate. As just about all proofs of π's transcendence require this step, we wish to remove this potential stumbling block.

The Idea

The idea is easily demonstrated. Consider $f(x) = a_0x^4+a_1x^3+a_2x^2+a_3x^1+a_4x^0$ and suppose $f(r) = 0$. We can find a new set of coefficients of the same ilk as a_i such that if $g(x)$ has this set and $g(ri) = 0$. This can be done as $i^k \in \{i^0, i^1, i^2, i^3\} = \{1, i, -1, -i\}$. These powers of i correspond to classes from modulo 4 (remainders on division by 4) and any natural number power (our exponents) is in one of these classes. So a_0x^4 with $x = ri$ is the same; a_1x^3 with ri is $a_1r^3i^3$ and this is $i(-a_1)r^3$. If we multiply this by i we get back to our original a_1r^3. Next $a_2r^2i^2 = -a_2r^2$ and if we multiply this by -1, we get back to the original. Next, a_1ri is the original times i. The constant is easy. So

$$g(ri) = a_0(ri)^4 - a_2(ri)^2 + a_0(ri)^0 + i(a_1(ri)^3 - a_3(ri)) = f(r) = 0.$$

We are almost there. The multiply of i in the odd powers sum makes the coefficients pure imaginary numbers, a no-no. But if a complex number is 0 then its absolute value is zero and

$$|g(x)| = (a_0(x)^4 - a_2(x)^2 + a_0(x)^0)^2 + (a_1(ri)^3 - a_3(ri))^2$$

is a polynomial with coefficients very much like our original $f(x)$. This $g(x)$ is such that $g(ri) = 0$, as needed.

Looking back at Hardy's proof(?), you see what he is up to and also how he really does have to assume his n is divisible by 4. Can we tighten the idea up to a real proof without this assumption. Next.

The Proof

Theorem 1. *A number ri is an algebraic number if and only if r is an algebraic number.*

Proof. Given any n degree polynomial $p(x)$, each term will be of the form $T_j(x) = a_jx^{n-j}$. The degree of each term will be in one of the four modulo 4 classes: $[0], [1], [2]$ or $[3]$. With one of multiply $m \in \{1, i, -1, -i\}$, $T_j(xi) = mT_j(x)$. Using these terms form $New(x) = E(x) + iO(x)$ where E are alternating evens and O are alternating odds. If either $p(ri)$ or $New(r)$ are zero the other will be too and $|new(x)|$ is a polynomial with integer coefficients if $p(x)$ is. □

Conclusion

There are places in Hardy's classic where he has an untoward step like this one. He leaves a lot to the reader. If the reader is steeped in techniques and can accept his word that a laborsome proof can be given, then all is well. But a novice reader might become forlorn at such fair. I hope this article helps such.

References

[1] G.H. Hardy and E.M.Wright, *An Introduction to the Theory of Numbers*, 6th ed., Oxford 2008.