A unified theory of gravity and inertia:

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21.6.2024

abstract:

In this paper, we want to show how the phenomenon of inertia can be explained in classical mechanics in a unified theory of gravity and inertia. This is achieved by correctly implementing Mach's principle and the idea of inertia being of gravitational origin. As a basis, we use the inertia-free mechanics of H.J. Treder. We want to show that it realises Mach's principle in the sense that the inertial frames of reference are completely determined by the relative motion of all particles in the universe. The theory is valid in arbitrary frames of reference and yields the exact Newtonian fictitious forces for translational and rotational motion of any non-inertial frame. Inertial mass and fictitious forces can be explained as of gravitational origin while the former at the same time remains isotropic, as demanded by experiment. We will show how inertial and gravitational mass are related to each other, providing an explanation for the weak equivalence principle. In the lowest order v/c of the theory, Newtonian mechanics is obtained, but including the fictitious forces. As correction in the next order, the theory yields Gravitoelectromagnetism. We show, that a Lorentz-type force equation valid in arbitrary accelerated frames can be derived. Ultimately, it is possible to formulate classical mechanics without a priori introducing the gravitational constant. Instead, an expression for it can be derived from the theory itself, allowing for an explanation of the strength of gravity.

1. Introduction:

The origin of inertia is still unknown. Neither classical mechanics nor general relativity provide a satisfactory explanation for the inertial properties of matter. A possible approach to this problem was proposed by Mach, as what became later known as Mach's principle [1]. He argued, that only relative quantities are determined by the dynamical laws of the universe, and in turn, only this relative quantities must enter the dynamical laws of the universe. Mach wrote that " […] The universe is not twice given, with an earth at rest and an earth in motion; but only once, with its relative motions, alone determinable." According to Mach, the inertial frames of reference, the frames in which Newton's laws of motion hold, should be completely determined by the relative motion of all particles in the universe. And not like in Newtonian theory, by a postulated absolute space, which is unobservable. Fictitious forces should therefore arise when a body is accelerated relative to the other masses in the universe, instead of absolute space. Criticizing Newton in his bucket experiment, with which he had intended to demonstrate the role of absolute space in the occurrence of the fictitious forces, Mach said that "it (the bucket experiment) only informs us, that the relative motion of the water with respect to the sides of the vessel produces no noticeable centrifugal forces, but that such forces are produced by its relative rotation with respect to the mass of the Earth and the other celestial bodies. " and that "no one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass till they were ultimately several leagues thick" [1, p. 216 f]. This last statement already hints at the influence of not just the motion, but also the mass of other particles in the universe on the inertia of a body.
Mach himself did not attach any particular importance to the explanation of inertial mass. In his opinion, it is just empirically defined by Newton's third law: If two bodies act on each other, they experience accelerations in opposite directions and of magnitude $\frac{a_1}{a_2}=\frac{m_2}{m_1}$. Inertial mass is then

just empirically defined as the inverse of the ratio of accelerations. Mach held the opinion that "every venture beyond this will only be productive in obscurity".
However, Mach's demand that only relative quantities enter the dynamical laws of the universe, implies that inertial mass is not an intrinsic property of matter, but results from an interaction with all other particles in the universe. This is an immediate consequence of the fact that a Lagrange function satisfying Mach's principle must necessarily depend on purely relative quantities

$$L=\sum_{i>j} m_i m_j f(r_{ij}, \dot{r}_{ij}, \vec{v}_{ij}) \quad .$$

First Friedländer [2, p. 17], and later Einstein suggested, based on the equivalence principle, that this interaction should be gravity. In his definition of Mach's principle [3] he argued that "the G-field (the metric tensor) is completely defined by the masses of the bodies (of the universe)". Since the metric tensor determines the inertial mass of a body in special and general relativity, his definition implied that inertial mass is of gravitational origin. His general theory of relativity was intended to incorporate this idea, but, according to his own words, failed to do so: A particle in an empty universe, which corresponds to flat Minkowski space, does have a non-vanishing inertial mass. If it were indeed, according to his definition, completely determined by the gravitational interaction with other masses, this could not be the case [4]). As we shall see though, it is this idea of inertia being of gravitational origin, together with Mach's principle, which, if implemented correctly, allows for a theory correctly accounting for the inertial properties of matter. Ultimately, this will allow to explain the strength of gravity and explain Newton's constant G.
Historically, there were many attempts to build theories realising both ideas. As was proposed by Barbour & Bertotti [5], a non-relativistic theory realising Mach's principle should be invariant under transformations of the form

$$\vec{r} \rightarrow \underline{A}(t) \cdot \vec{r} + \vec{g}(t) \quad , \tag{1.1}$$

with $\underline{A}$ an orthogonal matrix and g a displacement vector. This invariance ensures the dependence of the theory on purely relative quantities, as demanded by Mach's principle. First, Barbour & Bertotti [6] and later also Lynden-Bell & Katz [7] developed a mathematically equivalent non-relativistic theory invariant under (1.1). However, they didn't incorporate the idea of inertia being of gravitational origin and therefore, were unable to obtain the correct fictitious forces. As we will see later, it is for this reason they were not able to provide an explanation for the gravitational constant.
Many, especially earlier, attempts to incorporate both ideas resulted in theories only depending on relative distances $r_{ij}=|\vec{r}_i-\vec{r}_j|$ between particles, and their derivatives [5, 8-11]. Most were for example built on the velocity dependent Weber potential [9-11]:

$$V_{Weber}=-\frac{f m_1 m_2}{r_{12}} \cdot \left(1-\frac{\dot{r}_{ij}^2}{2c^2}\right) \quad , \tag{1.2}$$

with f the gravitational constant and c the speed of light. This potential then takes the role of both kinetic and potential energy. Those theories do indeed explain inertia as of gravitational origin, since the kinetic energy is part of the gravitational potential. At the same time are invariant under (1.1). However, they lead to an anisotropic inertial mass, which is ruled out experimentally[1]). This also has led to a refutation of theories built on such velocity dependent potentials.
Another remarkable attempt to explain inertia as of gravitational origin was made by Sciama [12]. In a model using the gravitoelectromagnetic equations he postulated that a particle always moves in a way that in its rest frame the total gravitational field is zero. He could then show how an inertial term $m a$ could be derived from what appeared as purely a vector potential in the particles rest frame. As we will see later, his idea is precisely the way how the inertia term arises from the gravitational field in a theory built on a velocity dependent gravitational potential. Although it is well

1) The relative anisotropy of inertia expected by such potentials due to the contribution of e.g. the Milky Way to a particle's inertia is roughly 10^{-9} , while the latest upper bound from experiment is 10^{-34} [16]

known that Gravitoelectromagnetism is even present in linearised general relativity, Sciama's ideas have never been built into a complete theory, neither non-relativistic nor relativistic.
Consequently, until today there exists no accepted theory successfully incorporating both Mach's principle and Einstein's idea of inertial mass resulting from gravitational interaction. But, as we will see, it is exactly those theories which are able to successfully explain inertia. It makes sense, to first look for a non-relativistic theory, correctly implementing both ideas, and later contemplate how they can be extended to a relativistic theory. In this paper we therefore only deal with a non-relativistic theory, a relativistic generalisation will be discussed in a subsequent paper.
A largely unknown theory which is capable of implementing both Mach's principle and Einstein's idea is the inertia free mechanics of H.J. Treder [13, 14], on which we want to draw attention. It is built on the Riemann-potential, which was originally used by Riemann in his theory of electromagnetism [15, p. 325 f]

$$V_{Riemann} = -\frac{f\, m_1 m_2}{r_{12}} \cdot \left(1 - \frac{\vec{v}_{ij}^{\,2}}{c^2}\right) \quad , \tag{1.3}$$

with $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, which again takes the role of both kinetic and potential energy. In this paper, we want to present and further develop this theory. It is capable of implementing both Mach's principle and the idea of inertia having a gravitational origin, without predicting it to be anisotropic. As a consequence, it yields a unified description of gravity and inertia. It can explain inertial mass as well as all the Newtonian fictitious forces as of gravitational origin, allowing us to derive the weak equivalence principle. The inertial frames of reference are determined by all other particles in the universe and the exact Newtonian fictitious forces arise in any non-inertial frame, for translational and rotational acceleration. Ultimately, we will show that this allows to formulate classical mechanics without a priori introducing a gravitational constant. Instead, it can be derived from the theory itself, allowing for an explanation of the strength of gravity.

2. The inertia free mechanics:

In this section, we first want to present the inertia-free mechanics of H.J. Treder [13, 14]. It is built on the Lagrange function:

$$L = T - V = \sum_{i>j} \frac{f\, m_i m_j}{r_{ij}} \left(1 + b \cdot \frac{\vec{v}_{ij}^{\,2}}{c^2}\right) \quad , \tag{2.1}$$

with the kinetic and potential energy

$$T = b \sum_{i>j} \frac{f\, m_i m_j}{r_{ij}} \left(\frac{\vec{v}_{ij}^{\,2}}{c^2}\right) \tag{2.2}$$

$$V = -\sum_{i>j} \frac{f\, m_i m_j}{r_{ij}} \quad . \tag{2.3}$$

Here $\vec{v}_{ij} = \vec{v}_i - \vec{v}_j$, $r_{ij} = |\vec{r}_i - \vec{r}_j|$, $\vec{\beta} = \frac{\vec{v}}{c}$. f is the gravitational constant, c the speed of light. b is a dimensionless constant. This Lagrangian is invariant under any transformation $\vec{r} \to \vec{r} + \vec{g}(t)$. We will later (section 5) show how it can be extended to also be invariant under the full transformation (1.1) and as a consequence will completely satisfy Mach's principle.
It was shown by Treder that the energy corresponding to this Lagrangian is E=T+V with T and V given by (2.2) and (2.3), respectively. This quantity denotes the energy of the universe and is conserved, it holds

$$\frac{dE}{dt} = 0 \quad .$$

The generalised momentum of some particle k following from the Lagrange function (2.1) is

$$\vec{P}_k = \frac{\partial L}{\partial \vec{v}_k} = m_k^* \cdot \vec{v}_k - b \cdot \frac{2 m_k f}{c} \vec{A}_k \quad , \tag{2.4}$$

with the potential and vector potential:

$$\varphi_k := \sum_{j \neq k} \frac{m_j}{r_{kj}} \tag{2.5}$$

$$\vec{A}_k := \sum_{j \neq k} \frac{m_j}{r_{kj}} \vec{\beta}_j \tag{2.6}$$

and the inertial mass:

$$m_k^* = \frac{2 b f \varphi_k}{c^2} \cdot m_k \quad . \tag{2.7}$$

This equation provides a relation between the inertial mass and the gravitational mass m_k . It shows, that the inertial mass (2.7) is induced by the gravity of all other masses in the universe. At the same time, it is isotropic, as demanded by experiment. This can be seen by its scalar character[2]). By demanding the strict equivalence of inertial and gravitational mass $m_k^* = m_k$, Treder obtained as self-consistency condition of the theory

$$\frac{2 b f \varphi_k}{c^2} = 1 \quad . \tag{2.8}$$

He interpreted this equation in the way that it determined the average gravitational potential of the universe for a *given* gravitational constant. Apart from leading to problems with the equations of motion and ambiguities, we believe that the true value of the theory lies in not having to postulate any gravitational constant at all, as will be shown in the next section. Further, by *demanding* the validity of (2.9), Treder applied the weak equivalence principle. As we will also see in the next section, this requirement is unnecessary. It will come out of the theory by itself, as a result of it correctly describing inertial mass as of gravitational origin.

Further, equation (2.4) implies that the total momentum of the universe is zero

$$\vec{P} = \sum_k \vec{P}_k = 0 \quad . \tag{2.9}$$

Treder also derived an equation of motion from the Lagrangian for a simplified model of two particles moving in front of a distant background consisting of the other particles. In the next section, we will derive the exact equations of motion, therefore we don't present it here.

3. Equation of motion, the equivalence principle and Gravitoelectromagnetism:

In this section, we want to derive the equations of motion following from the Lagrangian (2.1). We will show that it is possible to formulate classical mechanics without a priori postulating the equivalence principle or the gravitational constant. Instead, we will derive both from the theory. We show that Newtonian mechanics is re-obtained in the lowest order β. As correction, Gravitoelectromagnetism arises in the higher orders and a Lorentz-type force equation can be obtained.

In the Lagrangian (2.1)[3])

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2) If one used a Lagrangian based on the Weber potential (1.2) instead of the Riemann potential (1.3), then (2.7) would have tensorial character and thus be anisotropic.

$$L=\sum_{i>j}\frac{f\,m_i m_j}{r_{ij}}\left(1+\vec{\beta}_{ij}^{\,2}\right)\quad , \tag{3.1}$$

the gravitational constant appears in both terms, kinetic and potential energy. It is therefore nothing more than a constant factor, which doesn't change the equations of motion. One can drop it and write

$$L=\sum_{i>j}\frac{m_i m_j}{r_{ij}}\left(1+\vec{\beta}_{ij}^{\,2}\right)\quad . \tag{3.2}$$

Consequently, it is not necessary to a priori introduce any gravitational constant; it will come out naturally later.
By using $\vec{\beta}_{kj}^{\,2}=\vec{\beta}_k^{\,2}+\vec{\beta}_j^{\,2}-2\langle\vec{\beta}_k,\vec{\beta}_j\rangle$ and gathering together all terms involving the k th particle, one obtains for its Lagrangian

$$L_k=\frac{1}{2}m_k^*\cdot v_k^2+m_k\varphi_k-2m_k\langle\vec{A}_k,\vec{\beta}_k\rangle+\sum_{j\neq k}\frac{m_k m_j}{r_{kj}}\cdot\beta_j^2\quad . \tag{3.3}$$

The first three terms in this expression are the Lagrangian for a particle in a gravitoelectromagnetic field with a factor of 2 at the magnetic term (we will see later that the correct relativistic value of 4 from 1st PN of GR comes out in the equations of motion). The only difference is that the inertial "mass" is given by[4]):

$$m_k^*=m_k\frac{2\varphi_k}{c^2} \tag{3.4}$$

It is interesting to notice that the gravitomagnetic contribution to (3.3) arises due to the dependence of the Lagrangian (3.2) on the *relative* velocities. Expression (3.4) is a scalar, showing again that inertial mass is isotropic. Also, this equation provides a relation between gravitational and inertial mass, showing that the latter is a derived quantity. As we will see in a moment, this will allow us to explain the weak equivalence principle, instead of having to postulate it.
Applying the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial\vec{v}_k}=\frac{\partial L}{\partial\vec{r}_k} \tag{3.5}$$

to (3.3), one obtains the equation of motion

$$m_k^*\frac{\partial\vec{v}_k}{\partial t}=m_k\vec{E}_k-2m_k\cdot\sum_{j\neq k}\vec{\beta}_{kj}\times\vec{B}_{kj}\quad . \tag{3.6}$$

Here, the gravitoelectric and magnetic fields are given by

$$\vec{E}_k:=-\sum_{j\neq k}\frac{m_j}{r_{kj}^3}\vec{r}_{kj}\cdot\left(1-\vec{\beta}_{kj}^{\,2}\right)+\frac{2}{c}\frac{\partial\vec{A}_k}{\partial t} \tag{3.7}$$

$$\vec{B}_{kj}:=\vec{\nabla}_k\times\vec{A}_{kj}+\vec{\beta}_k\times\vec{\nabla}_k\varphi_{kj}\quad . \tag{3.8}$$

The partial time derivative in (3.7) means that only the velocity in A is to be differentiated in time. If one divides eq. (3.6) by the inertial mass m_k^*, one obtains

$$\frac{\partial\vec{v}_k}{\partial t}=\frac{c^2}{2\varphi_k}\left(\vec{E}_k-2\sum_{j\neq k}\vec{\beta}_{kj}\times\vec{B}_{kj}\right)\quad . \tag{3.9}$$

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3) For simplicity and it being the natural choice, we set b=1. Treder used the value of b=3/2 to get the correct value for the perihelion shift of Mercury. Since we only have a non-relativistic theory which is to be generalised relativistically, we don't bother with getting the correct value here.
4) The unit of this expression is not the one of a mass since we dropped f in the Lagrange function (3.1). If we kept it, the units would be correct, but f will cancel out in the equations of motion anyway. Consequently, nothing of what is said about the inertial mass in the following is altered by this "wrong" units.

In the lowest order $\frac{v}{c}$ this reduces to

$$\frac{\partial \vec{v}_k}{\partial t} = \frac{c^2}{2\varphi_k} \cdot \vec{\nabla}_k \varphi_k \quad , \tag{3.10}$$

which is the Newtonian law of gravity with the gravitational constant given by[5]):

$$G_k = \frac{c^2}{2\varphi_k} \quad . \tag{3.11}$$

It comes out naturally and does not have to be put in by hand. And so does the weak equivalence principle, as can be seen by equation (3.9, 3.10): No inertial mass appears in it, implying the universality of free fall. Both are a direct consequence of the inertial mass being induced by gravity, according to (3.4). We will discuss this in more detail in section 7.

As correction to Newton's law of gravity, in the next order $\frac{v}{c}$ we get Gravitoelectromagnetism, as can be seen in equation (3.9). It can also be written as:

$$m_k \cdot \frac{\partial \vec{v}_k}{\partial t} = \vec{F}_k \quad , \tag{3.12}$$

with the gravitoelectromagnetic force:

$$\vec{F}_k = m_k \cdot G_k \cdot \left(\vec{E}_k - 2 \sum_{j \neq k} \vec{\beta}_{kj} \times \vec{B}_{kj} \right) \quad . \tag{3.13}$$

Unlike in the conventional Lorentz force, the magnetic part of the force here depends on the relative velocities. The equation of motion (3.13) is invariant under any transformation $\vec{r} \rightarrow \vec{r} + \vec{g}(t)$, as was the Lagrangian (3.2) it was derived from[6]). Therefore, it is indeed valid in arbitrary linear accelerated frames of reference. This can be seen from the fact that the potentials (2.5 & 2.6) behave under such a transformation as

$$\vec{A}_k \rightarrow \varphi_k \cdot \frac{\vec{V}}{c} + \vec{A}_k \tag{3.14}$$

$$\varphi_k \rightarrow \varphi_k \quad , \tag{3.15}$$

and therefore the fields (3.7 & 3.8) as:

$$\vec{E}_k \rightarrow \vec{E}_k + \frac{2\varphi_k}{c^2} \frac{\partial \vec{V}}{\partial t} \tag{3.16}$$

$$\vec{B}_k \rightarrow \vec{B}_k \tag{3.17}$$

with $\vec{V} = \frac{d\vec{g}}{dt}$.

5) An equation like (3.15) has been obtained in various works on Mach's principle [9, 12].
6) This of course also applies to (3.6) and (3.9)

4. The relativity of linear acceleration and the origin of inertia:

In this section, we want to show how the inertial frames of reference are defined by the motion of all particles in the universe, as demanded by Mach's principle. Also, we show that any linear acceleration relative to those frames yields the exact Newtonian fictitious forces and further, that these forces are of gravitational origin. Since our Lagrangian is by now only invariant under transformations $\vec{r} \to \vec{r} + \vec{g}(t)$, we will restrict ourselves to linear accelerations here. In the next section, we will show how it can be made invariant under the complete set of transformations (1.1) and then discuss rotational accelerations.

In the force (3.13) acting on the particle k, we can identify the fictitious force as that dependent on the accelerations of the other particles

$$\vec{F}_k^{fic} = \frac{2 m_k G_k}{c} \frac{\partial \vec{A}_k}{\partial t} \quad . \tag{4.1}$$

Indeed, according to the transformation law (3.14), this force behaves under a transformation $\vec{v} \to \vec{v} + \vec{V}$ as

$$\vec{F}_k^{fic} \to \vec{F}_k^{fic} + m_k \cdot \frac{d\vec{V}}{dt} \quad . \tag{4.2}$$

It therefore automatically yields the additional Newtonian inertial force when transformed into a new frame, accelerating relative to the old one. The inertial frames are those in which this force vanishes, which is equivalent to the condition:

$$\frac{\partial \vec{A}_k}{\partial t} = 0 \quad . \tag{4.3}$$

The left side is dependent on the acceleration of all other particles in the universe. Therefore, the inertial frames are determined by the motion of all other particles in the universe, as demanded by Mach's principle. Especially, any frame accelerated relative to the one defined by (4.3) will experience a fictitious force

$$\vec{F}_k^{fic} = m_k \cdot \frac{d\vec{V}}{dt} \quad , \tag{4.4}$$

which means that the fictitious force arise in any frame accelerated relative to the rest-frame defined by the universe via (4.2).

The fictitious force (4.1) also gives rise to a "dragging" effect: If some particle j accelerates, this induces a drag force on particle k equal to:

$$\vec{F}_{kj}^{fic} = \frac{2 G_k m_k m_j}{c^2 r_{kj}} \frac{\partial \vec{v}_j}{\partial t} \quad .$$

If now the whole universe would be accelerating uniformly with $\frac{d\vec{V}}{dt}$, then the whole fictitious force on the particle k would be again equal to (4.4). The universe drags the particle with it. Even though the particle is resting in the rest-frame of absolute space, it experiences the same fictitious force as if it were in a non-inertial frame accelerating with $-\frac{d\vec{V}}{dt}$. In Newtonian mechanics, the particles acceleration in this case would be zero: It would not recognise the motion of the universe. The inertial frame of reference is again defined by (4.2), which means that we have to transform into a frame accelerating with $\frac{d\vec{V}}{dt}$ in order to satisfy the condition. The inertial frame is the one co-accelerating with the universe. It can thus be seen, that the role of absolute space as the universal inertial frame has been removed. As demanded by Mach's critique, it no longer plays any role in

when fictitious forces arise, but only the accelerations relative to the other particles in the universe matter.
The above mentioned is a manifestation of the relativity of linear acceleration, which is realised in the theory. It is dynamically equivalent if particle is accelerating, or the rest of the universe is accelerating in the opposite direction. The reason for this is that only accelerations relative to the universe enter the equation of motion (3.6). This was already shown by Treder for a simplified model.
If we bring the fictitious force (4.1) to the other side in (3.6), we can write the left side as

$$m_k^* \frac{\partial \vec{v}_k}{\partial t} - \frac{2 \cdot m_k}{c} \cdot \frac{\partial \vec{A}_k}{\partial t} = \sum_{j \neq k} \frac{2 m_k m_j}{c^2 r_{kj}} \dot{\vec{v}}_{kj} \quad . \tag{4.5}$$

Indeed, this expression contains only accelerations relative to the other particles.
From (4.1), it can also clearly be seen that the fictitious force is, as is the inertial mass, of gravitational origin. It is just the gravitoelectric induction force. We can also show how the whole inertia term of a particle

$$m_k^* \frac{\partial \vec{v}_k}{\partial t}$$

can be derived from what appears as purely a vector potential in his rest frame. The mechanism is the same that had been proposed by Sciama [11]. The term (4.5) is just the vector potential $\vec{A}_k^{\,'}$ as seen in the particles rest frame:

$$\sum_{j \neq k} \frac{2 m_k m_j}{c^2 r_{kj}} \dot{\vec{v}}_{kj} = \frac{-2 \cdot m_k}{c} \cdot \frac{\partial \vec{A}_k^{\,'}}{\partial t} \quad . \tag{4.6}$$

In this frame, the equation of motion therefore reads:

$$m_k \vec{E}_k^{\,'} - 2 m_k \cdot \sum_{j \neq k} \vec{\beta}_{kj}^{\,'} \times \vec{B}_{kj}^{\,'} = 0 \quad , \tag{4.7}$$

whereas the generalised momentum is

$$\vec{P}_k^{\,'} = -2 m_k \vec{A}_k^{\,'} \quad . \tag{4.8}$$

Indeed, no inertia term is present. Equation (4.7) also expresses that the total gravitational field in the particle’s rest frame is zero; this is what had been postulated by Sciama. If we now transform into an arbitrary moving frame, the particle’s velocity in this system is $\vec{v}_k$. We get according to the transformation law (3.14):

$$\vec{P}_k = \frac{2 m_k \varphi_k}{c^2} \cdot \vec{v}_k - \frac{2 m_k}{c} \vec{A}_k = m_k^* \cdot \vec{v}_k - \frac{2 m_k}{c} \vec{A}_k \tag{4.9}$$

and for the equation of motion (4.7) again:

$$m_k^* \frac{\partial \vec{v}_k}{\partial t} = m_k \vec{E}_k - 2 m_k \cdot \sum_{j \neq k} \vec{\beta}_{kj} \times \vec{B}_{kj} \quad . \tag{4.10}$$

One can see how the inertia term arises from what appears as purely the vector potential in the particle’s rest frame.

<u>5. The relativity of rotation:</u>

As was stated in the beginning, the Lagrangian (2.1) is only invariant under translations $\vec{r} \rightarrow \vec{r} + \vec{g}(t)$, but not under rotations, and therefore not under the full transformation (1.1). In this section, we want to show how the theory can be extended to be invariant under arbitrary transform-

ations (1.1). We will then be able to obtain the same results that were obtained for linear acceleration in the previous section, also for rotational acceleration.
To extend the Lagrangian, we adopt an idea used by Lynden-Bell & Katz in their approach to Mach's principle [7]. We write

$$T'=\sum_{i>j}\frac{m_i m_j}{c^2 r_{ij}}\cdot(\vec{v}_{ij}-\vec{\Omega}\times\vec{r}_{ij})^2 \tag{5.1}$$

and minimise with respect to Ω

$$\vec{M}=\frac{dL}{d\vec{\Omega}}=0 \quad . \tag{5.2}$$

This yields

$$\vec{J}=\underline{I}\cdot\vec{\Omega} \tag{5.3}$$

$$\vec{J}=\sum_{i>j}\frac{m_i m_j}{c^2 r_{ij}}\cdot\vec{r}_{ij}\times\vec{v}_{ij} \tag{5.4}$$

$$\underline{I}=\sum_{i>j}\frac{m_i m_j}{c^2 r_{ij}}\cdot(\underline{\delta}\cdot\vec{r}_{ij}^{\,2}-\vec{r}_{ij}\otimes\vec{r}_{ij}) \quad . \tag{5.5}$$

As in the paper of Lynden-Bell & Katz, J is the angular momentum of the universe around its centre of mass, I its moment of inertia around the centre of mass[7]). Equation (5.2) expresses that the total angular momentum M of the universe is zero, just like the regular momentum P according to (2.12). The presence of the additional r_{ij} terms doesn't change the behaviour of (5.1) under rotations compared to the expression obtained by Lynden-Bell & Katz

$$T'=\frac{1}{2}\cdot\sum_{i>j}\frac{m_i m_j}{M}\cdot(\vec{v}_{ij}-\vec{\Omega}\times\vec{r}_{ij})^2 \quad , \tag{5.6}$$

since distances remain invariant. Therefore, the Lagrangian

$$L=T'-V=\sum_{i>j}\frac{m_i m_j}{r_{ij}}\cdot\left(1+\left(\vec{\beta}_{ij}-\frac{1}{c}\vec{\Omega}\times\vec{r}_{ij}\right)^2\right) \tag{5.7}$$

is also invariant under arbitrary rotations.
With the definitions

$$\vec{\beta}':=\vec{\beta}-\frac{\vec{\Omega}}{c}\times\vec{r} \tag{5.8}$$

$$\vec{A}_k':=\sum_{j\neq k}\frac{m_j}{r_{kj}}\vec{\beta}_j' \quad , \tag{5.9}$$

one obtains for the equation of motion for some particle k

$$m_k^*\cdot\frac{\partial\vec{v}_k}{\partial t}=m_k^*\cdot\left(\frac{d\vec{\Omega}}{dt}\times\vec{r}_k+2\cdot\vec{\Omega}\times\vec{v}_k+(\vec{\Omega}\times\vec{r}_k)\times\vec{\Omega}\right)+m_k\vec{E}_k'-2m_k\cdot\sum_{j\neq k}\vec{\beta}_{kj}'\times\vec{B}_{kj}' \quad .\tag{5.10}$$

The gravitoelectric and magnetic fields are the ones introduced in (3.7 & 3.8), as seen in a frame rotating with an angular velocity $\vec{\Omega}$

$$\vec{E}_k'=-\sum_{j\neq k}\frac{m_j}{r_{kj}^3}\vec{r}_{kj}\cdot(1-\vec{\beta}_{kj}'^2)+\frac{2}{c}\left(\frac{\partial\vec{A}_k'}{\partial t}-\vec{\Omega}\times\vec{A}_k'\right) \tag{5.11}$$

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7) Since the gravitational constant had been dropped in the Lagrangian (3.2) and consequently the kinetic energy (4.1), the units are different from the usual units of J and I.

$$\vec{B}_{kj}{}' := \vec{\nabla}_k \times \vec{A}_{kj}{}' + \vec{\beta}_k{}' \times \vec{\nabla}_k \varphi_{kj} \quad . \tag{5.12}$$

The inertial “mass” is again given by

$$m_k^* = m_k \frac{2\varphi_k}{c^2} \quad . \tag{5.13}$$

If one divides by $\frac{2\varphi_k}{c^2}$, one can write eq. (5.10) in the form

$$m_k \frac{\partial \vec{v}_k}{\partial t} = \vec{F}_k^{\text{fic}} + \vec{F}_k^{\text{GEM}} \tag{5.14}$$

$$\vec{F}_k^{\text{fic}} = m_k \cdot \left(\frac{d\vec{\Omega}}{dt} \times \vec{r}_k + 2 \cdot \vec{\Omega} \times \vec{v}_k + (\vec{\Omega} \times \vec{r}_k) \times \vec{\Omega} \right) \tag{5.15}$$

$$\vec{F}_k^{\text{GEM}} = m_k G_k \left(-\sum_{j \neq k} \frac{m_j}{r_{kj}^3} \vec{r}_{kj} \cdot (1 - \vec{\beta}_{kj}{}'^2) + \frac{2}{c} \left(\frac{\partial \vec{A}_k{}'}{\partial t} - \vec{\Omega} \times \vec{A}_k{}' \right) - 2 \cdot \sum_{j \neq k} \vec{\beta}_{kj}{}' \times \vec{B}_{kj}{}' \right) \quad . \tag{5.16}$$

Here, $\vec{F}_k^{\text{GEM}}$ is the gravitoelectromagnetic force introduced in (3.13), as seen in a frame rotating with an angular velocity $\vec{\Omega}$. $\vec{F}_k^{\text{fic}}$ is again an additional fictitious force, caused by the rotation of the universe. It exactly agrees with the Newtonian expression. G_k is again given by (3.11). We can see again, that the inertial frames of reference are determined by all other particles in the universe, as demanded by Mach’s principle: The fictitious force (5.15) vanishes in exactly that frame of reference where $\vec{\Omega} = 0$. This is according to (5.3) equivalent to the condition

$$I^{-1} \cdot \vec{J} = 0 \quad . \tag{5.17}$$

The left side of this equation is purely dependent on relative quantities between all particles in the universe, cf. equations (5.4 & 5.5).
Since the Lagrangian (5.7) is independent of the rotation of the frame chosen to write it in, so is the equation of motion (5.14). $\vec{\Omega}$ is the angular frequency of the universe perceived in this frame. Consequently, the exact Newtonian fictitious forces arise automatically in any frame rotating relative to the one defined by (5.17), just like it was the case for linear acceleration (cf. eq. 4.4). This also again implies that a particle at rest in the rest-frame of absolute space would experience the same centrifugal forces if the rest of the universe were rotating, then it would experience if itself would be rotating with the same angular frequency. In Newtonian mechanics, this is not the case: The particle would experience no force if just the universe were rotating. This is a manifestation of the relativity of rotation. It is dynamically equivalent if the particle is rotating or the universe is rotating in the other direction. This is mathematically expressed by the fact that only rotations relative to the universe enter the Lagrange function (5.7).
This also answers Mach’s criticism to Newton’s bucket: The Newtonian fictitious forces arise in any frame rotating relative to the one defined by all other masses in the universe via (5.17). The water in the bucket is pushed upwards because it is rotating relative to the universe, not absolute space like in Newtonian theory. If now the walls of the bucket would hypothetically become thicker and thicker, ultimately consisting of the matter of the entire universe, then the water wouldn’t be pushed up against the walls at all anymore, because the inertial frame would be that co-rotating with the universe, which is in this case the walls if the bucket. But this is exactly the one, in which the water rests.

6. Frame-dragging as consequence of Mach's principle:

Like in the linear case, the relativity of rotation gives rise to a dragging effect, which agrees qualitatively with the frame-dragging effect predicted by General relativity. Assume therefore some mass distribution around the origin, rotating with a velocity $\vec{v}=\vec{\omega}\times\vec{r}$. According to (5.9), this gives rise to a vector potential

$$\vec{A}_k{}'=\sum_{j\in V}\frac{m_j}{r_{kj}}\vec{\omega}'\times\vec{r}_j\approx\int_V\rho(\vec{r}_j)\frac{\vec{\omega}'\times\vec{r}_j}{r_{kj}}d^3\vec{r}_j \quad , \tag{6.1}$$

where $\vec{\omega}'=\vec{\omega}-\vec{\Omega}$ is the angular frequency of the mass distribution relative to the universe, V is its volume. We assume the distribution to be continuous and therefore replaced the sum by an integral in the second step. Assuming our mass distribution to be small compared to the universe, it won't influence the angular frequency of the universe $\vec{\Omega}$ much. We can treat it as constant and pull it out of the integral to write (6.1) as

$$\vec{A}_k{}'=\vec{\omega}'\times\int_V\rho(\vec{r}_j)\frac{\vec{r}_j}{r_{kj}}d^3\vec{r}_j \quad . \tag{6.2}$$

The far field expression for (6.1) is well known to be

$$\vec{A}_k{}'=\frac{\vec{\omega}'}{c\,r_k^3}\times\int_V\rho\,\vec{r}\,'\langle\vec{r},\vec{r}\,'\rangle d^3\vec{r}\,' \quad . \tag{6.3}$$

This expression agrees with the one obtained in linearised gravity for the same situation, except for the dependence on the *relative* angular frequency, instead of the absolute one. Therefore, from (6.2) all the well known results can be derived for any mass distribution.
For the equation of motion we neglect anything beyond first order in the particle velocity $\vec{v}_k$ and the source velocities $\vec{v}_j$, as well as the fictitious forces caused by the rotation of the universe (which we assume to be small). Then we obtain

$$\frac{\partial\vec{v}_k}{\partial t}=G_k\cdot(\vec{E}_k-4\vec{\beta}_k{}'\times\vec{B}_k'+\delta\vec{E}_k) \tag{6.4}$$

$$\vec{E}_k=\vec{\nabla}\varphi_k+\frac{2}{c}(\frac{\partial\vec{A}_k{}'}{\partial t}-\vec{\Omega}\times\vec{A}_k{}') \tag{6.5}$$

$$\vec{B}_k'=\vec{\nabla}\times\vec{A}_k' \tag{6.6}$$

$$\delta\vec{E}_k=2\vec{\nabla}_k\int_V\rho(\vec{r}_j)\frac{\langle\vec{\beta}_k{}',\vec{\omega}'\times\vec{r}_j\rangle}{r_{kj}}d^3\vec{r}_j-2\vec{\nabla}_k\times\int_V\rho(\vec{r}_j)\frac{(\vec{\beta}_k{}'\times(\vec{\omega}'\times\vec{r}_j))}{r_{kj}}d^3\vec{r}_j \quad . \tag{6.7}$$

The equation of motion is again independent of the rotation of the frame chosen to write it in, as was already pointed out in the previous section. It agrees with the one obtained from linearised GR in the same order considered, apart from the last term in (6.4) (an additional acceleration which comes out of this theory) and a factor of 2 at the gravitomagnetic induction term in (6.5). The potentials also agree with the ones obtained in linearised GR. There is, however, one crucial difference: The vector potential and the equation of motion, and therefore also the whole dragging effect, depend on the *relative* angular velocity $\vec{\omega}'=\vec{\omega}-\vec{\Omega}$ of the mass distribution with respect to the universe, as well as the velocity $\vec{\beta}_k'=\vec{\beta}_k-\frac{\vec{\Omega}}{c}\times\vec{r}_k$. In GR, it depends on the absolute angular velocity $\vec{\omega}$. This is again a manifestation of Mach's principle and the aforementioned relativity of rotation.

We have thus shown that already a classical theory implementing Mach's principle and Einstein's idea of inertia being a gravitational effect gives rise to the frame-dragging effect. No curved space-time or relativistic physics is necessary for it. As we have seen, it is just a consequence of the inertial frames of reference being defined by all other masses in the universe, as demanded by Mach's principle. If a mass distribution is rotating, the inertial frame for a particle close to it is one slightly co-rotating, depending on the distance and the mass of the distribution.

7. The gravitational constant:

We have shown in section 3 that the kinetic and potential energy being proportional to the gravitational potential, and therefore the gravitational constant, allows for its elimination from the Lagrangian (3.1). This is not possible with the Newtonian kinetic energy or even the Machian one obtained by Lynden-Bell & Katz

$$T=\frac{1}{2}\cdot\sum_{i>j}\frac{m_i m_j}{M}\cdot\vec{v}_{ij}^{\,2} \quad . \tag{7.1}$$

This kinetic energy is not proportional to the gravitational potential and therefore does not depend on f. We can conclude that what finally allowed us to obtain an expression for the gravitational constant was the ability of the theory to correctly explain inertial mass as of gravitational origin. Equation (3.11)

$$G=\frac{c^2}{2\varphi(\vec{r})} \tag{7.2}$$

also allows us to shed light on the physical reason for the existence of the constant G. It is not the gravitational field itself, which has G built into it, but the inertial masses (3.4)

$$m_k^*=m_k\frac{2\varphi_k}{c^2} \tag{7.3}$$

have built in the factor 1/G, which is a direct result of their gravitational origin. Indeed, the mass on the right side is the gravitational mass, the one on the left side is the inertial mass. 1/G is the factor relating the two. Its value represents how large the inertial mass caused by the entire universe is. The gravitational constant is the inverse of this value (cf. the derivation of eq. (3.9 & 3.10) from (3.6)).
This also explains why the gravity appears to be such a weak force. One can approximately calculate the value of (7.2). For an approximately homogeneous universe, we have

$$\varphi=\int_{K_{R_u}\subset\mathbb{R}^3}\frac{\rho_j}{r_{ij}}d^3\vec{r}_j\approx 4\pi\ \rho_0\int_0^{R_u} r\,dr=\frac{3}{2}\frac{M_u}{R_u} \quad , \tag{7.4}$$

where M_u, R_u are the mass and radius of the observable universe. For the gravitational constant, this yields

$$G\approx\frac{1}{3}\frac{R_u c^2}{M_u} \quad . \tag{7.5}$$

Since there is such a huge amount of matter in the universe, bodies have a very large inertial mass, and consequently, only experience very small gravitational accelerations. If there was considerably less matter in the universe, gravity would be predicted to be much stronger. E.g. if the universe consisted only of the Milky Way, then gravity would be roughly 10^7 times stronger than it is in our universe, at least if c would keep its known value in such a situation. This conclusion was also reached by Sciama & Treder for similar expressions for G as (7.5) [12, 14].

It is well known that expression (7.5) is confirmed by observation. Plugging in the observed values $M_u \approx 10^{53}\ kg$, $R_u \approx 4 \cdot 10^{26}\ m$ and $c \approx 3 \cdot 10^{8}\ m s^{-1}$ one gets

$$G \approx 8 \cdot 10^{-11}\ m^3 kg^{-1} s^{-2} \quad . \tag{7.6}$$

Indeed, $\frac{G M_u}{R_u c^2} \sim 1$ is one of the unexplained "cosmological coincidences". In common theories, it is a coincidence; in the theory presented here, it is a confirmed prediction. If the matter content of the universe or its density were considerably different from what is observed, common theories would remain valid, whereagainst the theory presented here would be disproved[8]).
Equation (7.6) is an agreement to a very good accuracy since the mass and the radius of the universe are only known by orders of magnitude, and a homogeneous universe is only a rough approximation. Also, this is the result of a non-relativistic theory. The main contribution to (7.4) comes from the most distant masses in the universe, for which retardation effects are expected to be non-negligible. Those can only be treated properly in a relativistic theory of what was presented here. It is therefore very unlikely that this relation is just a coincidence, rather than being anchored in an underlying theory like the one presented here. As was argued in the previous section, it is to be expected that a correct relativistic theory must also provide a relation like (7.2).

8. Conclusion:

We have shown how the phenomenon of inertia in non-relativistic mechanics can be explained in a unified theory of gravity and inertia. This was achieved by incorporating both Mach's principle and the idea of inertia being a gravitational effect. As a basis, we have used H.J. Treder's inertia-free mechanics. Mach's principle is fulfilled in the way that the inertial frames of reference are completely determined by the relative motion of all particles in the universe. The theory is valid in arbitrary frames of reference and yields the correct Newtonian fictitious forces for translational and rotational motion of any non-inertial frame. The fictitious forces, as well as the inertial mass, were explained as being of gravitational origin, which also allowed us to derive the weak equivalence principle. In the lowest order β, Newtonian theory has been re-obtained; the corrections in the next orders have been shown to be Gravitoelectromagnetism. Due to the theory's ability to explain the phenomenon of inertia, it does not need to postulate the gravitational constant. Instead, it allows to derive it from the theory itself, explaining the strength of gravity. This leads us to believe that this is the correct formulation of classical non-relativistic mechanics. Also, it shows that the appearance of the unexplained constant G in common theories is tied to their inability to correctly explain inertia. Consequently, a correct relativistic theory should be a generalisation of what was presented here. It should explicitly incorporate Mach's principle and inertia being of gravitational origin. Like for the non-relativistic version, it should not contain any gravitational constant a priori, but instead allow for a derivation of it from the theory itself. Further, the unification of gravity and inertia allows to draw some important conclusions on the nature of mass and the elementary particles themselves, showing directly a road towards a relativistic generalisation of what was presented here. This will be discussed in a subsequent paper.

Acknowledgements:

8) If the universe indeed consisted only of the Milky Way we had $M_u \approx 10^{41}\ kg$ and $R_u \approx 10^{21}\ m$, and therefore roughly $G \approx 6 \cdot 10^{-4}\ m^3 kg^{-1} s^{-2}$, 10^7 times the known value.

We want to thank Dr. Julian Barbour and Dr. Tim Koslowski for reading and commenting on this work. Further, we want to thank Dr. Julian Barbour for extensive and helpful discussions.

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