WEIGHTED m-GENERALIZED GROUP INVERSE IN $*$-BANACH ALGEBRAS

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Abstract. Recently, Gao, Zuo and Wang introduced the W-weighted m-weak group inverse for complex matrices which generalized the (weighted) core-EP inverse and the WC inverse. The main purpose of this paper is to extend the concept of W-weighted m-weak group inverse for complex matrices to elements in a Banach *-algebra. This extension is called w-weighted m-generalized group inverse. We present various properties, presentations of such new weighted generalized inverse. Related (weighted) m-generalized core inverses are investigated as well. Many properties of the W-weighted m-weak group inverse are thereby extended to wider cases.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. An element $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such x is unique if exists, denoted by $a^{\#}$, and called the group inverse of a (see [14]). As is well known, a square complex matrix A has group inverse if and only if $rank(A) = rank(A^2)$.

A Banach algebra is called a Banach *-algebra if there exists an involution $* : x \to x^*$ satisfying $(x + y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda} x^*, (xy)^* = y^*x^*, (x^*)^* = x$. The involution $*$ is proper if $x^*x = 0 \Longrightarrow x = 0$ for any $x \in \mathcal{A}$, e.g., in a Rickart *-algebra, the involution is always proper. Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $*$ as the involution. Then the involution $*$ is proper. In [21], Zou et al. extended the notion of weak group inverse from complex matrices to elements in a ring with proper involution.

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Let $\mathcal{A}$ be a Banach algebra with a proper involution $*$. An element a in a $\mathcal{A}$ has weak group inverse if there exists $x\in \mathcal{A}$ such that

$$x=ax^2, (a^*a^2x)^*=a^*a^2x, a^n=xa^{n+1}$$

for som $n\in \mathbb{N}$. Such x is unique if it exists and is called the weak group inverse of a. We denote it by $a^{\textcircled{w}}$ (see [21, 22]). A square complex matrix A has weak group inverse X if it satisfies the system of equations:

$$AX^2=X, AX=A^{\textcircled{\dagger}}A.$$

Here, $A^{\textcircled{\dagger}}$ is the core-EP inverse of A (see [11, 23]). Weak group inverse was extensively studied by many authors, e.g., [8, 17, 20, 21, 22].

In [2], the authors extended weak group inverse and introduced generalized group inverse in a Banach algebra with proper involution. An element a in $\mathcal{A}$ has generalized group inverse if there exists $x\in \mathcal{A}$ such that

$$x=ax^2, (a^*a^2x)^*=a^*a^2x, \lim\limits_{n\to\infty}||a^n-xa^{n+1}||^{\frac{1}{n}}=0.$$

Such x is unique if it exists and is called the generalized group inverse of a. We denote it by $a^{\textcircled{g}}$. Many properties of generalized group inverse were presented in [2]. Mosić and Zhang introduced and studied weighted weak group inverse for a Hilbert space operator A in $\mathcal{B}(X)$ (see [17]). Furthermore, the weak group inverse was generalized to the m-weak group inverse (see [11, 18, 24]). Recently, Gao et al. further introduced and studied the W-weighted m-weak group inverse in [11].

The main purpose of this paper is to extend the concept of W-weighted m-weak group inverse for complex matrices to elements in a Banach *-algebra. This extension is called weighted m-generalized group inverse.

An element $a\in \mathcal{A}$ has generalized w-Drazin inverse x if there exists unique $x\in \mathcal{A}$ such that

$$awx=xwa, xwawx=x \text{ and } a-awxwa\in \mathcal{A}^{qnil}.$$

We denote x by $a^{d,w}$ (see [19]). Here, $\mathcal{A}^{qnil}=\{x\in \mathcal{A} \mid \lim\limits_{n\to\infty}\parallel x^n\parallel^{\frac{1}{n}}=0\}$. We denote $a^{d,1}$ by a^d. Evidently, $a^{d,w}=x$ if and only if $x=a[(wa)^d]^2$. We introduce a new weighted generalized inverse as follows:

Definition 1.1. *An element $a\in \mathcal{A}$ has w-weighted m-generalized group inverse if $a\in \mathcal{A}^{d,w}$ and there exists $x\in \mathcal{A}$ such that*

$$\begin{gathered}x=a(wx)^2, [(wa)^d]^*(wa)^{m+1}wx=[(wa)^d]^*wa,\\ \lim\limits_{n\to\infty}||(aw)^n-(xw)(aw)^{n+1}||^{\frac{1}{n}}=0.\end{gathered}$$

The preceding x is called the w-weighted m-generalized group inverse of a, and denoted by $a^{\textcircled{g}_m,w}$.

The w-weighted m-generalized group inverse is a natural generalization of the m-generalized group inverse which was introduced in [1]. Let $a^{\textcircled{g}_m}$ be the m-generalized group inverse of a. Evidently, $a^{\textcircled{g}_m} = a^{\textcircled{g}_m,1}$. We list some characterizations of m-generalized group inverse.

Theorem 1.2. *(see [1, Theorem 2.3, Theorem 3.1 and Theorem 4.1]) Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{g}_m}$.
(2) *There exist $x, y \in \mathcal{A}$ such that*
$$a = x + y, x^*a^{m-1}y = yx = 0, x \in \mathcal{A}^{\#}, y \in \mathcal{A}^{qnil}.$$
(3) *$a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that*
$$x = ax^2, (a^d)^*a^{m+1}x = (a^d)^*a^m, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$
(4) *$a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that*
$$x = ax^2, (a^d)^*a^{m+1}x = (a^d)^*a^m, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$
(5) *$a \in \mathcal{A}^d$ and there exists an idempotent $p \in \mathcal{A}$ such that*
$$a + p \in \mathcal{A}^{-1}, [(a^m)^*a^m p]^* = a^*ap \text{ and } pa = pap \in \mathcal{A}^{qnil}.$$
(6) *$a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that $(a^d)^*a^dx = (a^d)^*a^m$.*

In Section 2, we investigate elementary properties of w-weighted m-generalized group inverse in a Banach *-algebra. Many new properties of the weak group inverse for a complex matrix and Hilbert space operator are thereby obtained.

Following [3], an element a in $\mathcal{A}$ has generalized w-core-EP inverse if there exist $x \in \mathcal{A}$ such that
$$a(wx)^2 = x, (wawx)^* = wawx, \lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

The preceding x is unique if exists, and we denote it by $a^{\textcircled{d},w}$. We denote $a^{\textcircled{d},1}$ by $a^{\textcircled{d}}$. Evidently, $a^{\textcircled{d},w} = x$ if and only if $x = a[(wa)^{\textcircled{d}}]^2$ (see [3, Theorem 2.1]). In Section 3, we investigate the representations of m-generalized group inverse under weighted generalized core-EP invertibility.

Recall that an element $a \in \mathcal{A}$ has Moore-Penrose inverse if there exist $x \in \mathcal{A}$ such that $axa = a, xax = x, (ax)^* = ax, (xa)^* = xa$. The preceding x is unique if it exists, and we denote it by $a^{\dagger}$. An element a in $\mathcal{A}$ has weak core inverse provided that $a \in \mathcal{A}^{\textcircled{w}} \bigcap \mathcal{A}^{\dagger}$ (see [16, 23]). In [4], the authors introduced and studied the generalized core inverse. The m-weak core inverse and weighted weak core inverse were investigated in [10, 15]. Recently, Ferreyra and Mosić introduced the W-weighted m-weak core inverse for complex matrices which generalized the (weighted) core-EP inverse, the weak group inverse and m-weak core inverse (see [7]). A square complex matrix A has W-weighted m-weak core inverse X if

$$X = A^{\textcircled{w}_m,W}(WA)^m[(WA)^m]^{\dagger}.$$

Here, $A^{\textcircled{w}_m,W}$ is the W-weighted m-weak group inverse of A, i.e., $(WA)^m$ has weak group inverse (see [20]). Let $a, w \in \mathcal{A}, m \in \mathbb{N}$. Set $a \in \mathcal{A}^{\dagger_m,w}$ if $(wa)^m \in \mathcal{A}^{\dagger}$. We have

Definition 1.3. *An element $a \in \mathcal{A}$ has w-weighted m-generalized core inverse if $a \in \mathcal{A}^{\textcircled{g}_m,w} \bigcap \mathcal{A}^{\dagger_m,w}$.*

In Section 4, We present various properties, presentations of such weighted generalized group inverse combined with weighted Moore-Penrose inverse. We extend the properties of generalized core inverse in Banach *-algebra to the general case(see [4]). Many properties of the W-weighted m-weak core inverse are thereby extended to wider cases, e.g. Hilbert operators over an infinitely dimensional space.

Finally, in Section 5, we give the applications of the w-weighted m-generalized group (core) inverse in solving the matrix equations.

Throughout the paper, all Banach algebras are complex with a proper involution $*$. We use $\mathcal{A}^{\dagger}, \mathcal{A}^{d,w}, \mathcal{A}^{\textcircled{d}}, \mathcal{A}^{\textcircled{g}}$ and $\mathcal{A}^{\textcircled{w}}$ to denote the sets of all Moore-Penrose invertible, weighted generalized Drazin invertible, generalized core-EP invertible, generalized group invertible and weak group invertible elements in $\mathcal{A}$, respectively.

2. WEIGHTED m-GENERALIZED GROUP INVERSE

In this section we introduce and establish elementary properties of weighted m-generalized group inverse which will be used in the next section. This also extend the concept of w-weighted m-weak group inverse from complex matrices to elements in a Banach algebra (see [11]). We begin with

Theorem 2.1. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{g}_m,w}$.
(2) $wa \in \mathcal{A}^{\textcircled{g}_m}$.

In this case, $a^{\textcircled{g}_m,w} = a[(wa)^{\textcircled{g}_m}]^2$.

Proof. (1) ⇒ (2) By hypothesis, we can find $x \in \mathcal{A}$ such that

$$x = a(wx)^2, [(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*wa,$$
$$\lim_{n\to\infty} ||(aw)^{n-1} - (xw)(aw)^n||^{\frac{1}{n-1}} = 0.$$

Furthermore, we have

$$\begin{array}{ll} & ||(wa)^n - (wx)(wa)^{n+1}||^{\frac{1}{n}} \\ = & ||w(aw)^{n-1}a - wxw(aw)^n a||^{\frac{1}{n}} \\ = & ||w[(aw)^{n-1} - xw(aw)^n]a||^{\frac{1}{n}} \\ \leq & ||w||^{\frac{1}{n}}[||(aw)^{n-1} - xw(aw)^n||^{\frac{1}{n-1}}]^{\frac{n-1}{n}}||a||^{\frac{1}{n}}. \end{array}$$

Therefore

$$\lim_{n\to\infty} ||(wa)^n - (wx)(wa)^{n+1}||^{\frac{1}{n}} = 0.$$

Obviously, $wx = (wa)(wx)^2$. Hence,

$$wa \in \mathcal{A}^{\circledg_m} \text{ and } (wa)^{\circledg_m} = wx.$$

Accordingly,

$$x = a(wx)^2 = a[(wa)^{\circledg_m}]^2,$$

as desired.

(2) ⇒ (1) Let $x = a[(wa)^{\circledg_m}]^2$. Then $a \in \mathcal{A}^{d,w}$ and we verify that

$$\begin{array}{rcl} a(wx)^2 & = & awa[(wa)^{\circledg_m}]^2wa[(wa)^{\circledg_m}]^2 \\ & = & a[(wa)^{\circledg_m}]^2 = x. \end{array}$$

One easily checks that

$$\begin{array}{rcl} [(wa)^d]^*(wa)^{m+1}wx & = & [(wa)^d]^*(wa)^{m+1}wa[(wa)^{\circledg_m}]^2 \\ & = & [(wa)^d]^*(wa)^{m+1}(wa)^{\circledg_m} \\ & = & [(wa)^d]^*wa. \end{array}$$

Since

$$\begin{array}{rcl} (xw)(aw)^{n+1} & = & a[(wa)^{\circledg_m}]^2w(aw)^{n+1} \\ & = & (aw)^n - a[(wa)^{n-1} - (wa)^{\circledg_m}(wa)^n]w \\ & - & a(wa)^{\circledg_m}[(wa)^n - (wa)^{\circledg_m}(wa)^{n+1}]w, \end{array}$$

we have

$$\begin{array}{ll} & ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} \\ \leq & ||a||^{\frac{1}{n}}||(wa)^{n-1} - (wa)^{\circledg_m}(wa)^n||^{\frac{1}{n}}||w||^{\frac{1}{n}} \\ + & ||a(wa)^{\circledg_m}||^{\frac{1}{n}}||(wa)^n - (wa)^{\circledg_m}(wa)^{n+1}||^{\frac{1}{n}}||w||^{\frac{1}{n}}. \end{array}$$

Therefore

$$\lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0,$$

the result follows. □

The preceding unique solution x is called the w-weighted generalized m-group inverse of a, and denote it by $a^{ⓖ_m,w}$. That is, $a^{ⓖ_m,w} = a[(wa)^{ⓖ_m}]^2$. We use $\mathcal{A}^{ⓖ_m,w}$ to denote the set of all w-weighted generalized m-group invertible elements in $\mathcal{A}$. By the argument above, we have

Corollary 2.2. *Let $a, w \in \mathcal{A}$. Then*

(1) $a^{ⓖ_m,w} = x$.
(2) $wa \in \mathcal{A}^{ⓖ_m}$ and $(wa)^{ⓖ_m} = wx$.

Corollary 2.3. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{ⓖ_m,w}$ if and only if*

(1) $a \in \mathcal{A}^{d,w}$;
(2) There exists $x \in \mathcal{A}$ such that

$$x = a[wx]^2, [(wa)^*(wa)^{m+1}wx]^* = (wa)^*(wa)^{m+1}wx,$$
$$\lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Proof. $\Longrightarrow$ Obviously, $a \in \mathcal{A}^{d,w}$. By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = a[wx]^2, [(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*wa,$$
$$\lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

In this case, $x = a[(wa)^{ⓖ_m}]^2$. Then

$$\begin{array}{rcl} (wa)^*(wa)^{m+1}wx & = & (wa)^*(wa)^{m+1}wa[(wa)^{ⓖ_m}]^2 \\ & = & (wa)^*(wa)^{m+1}(wa)^{ⓖ_m}, \\ \big((wa)^*(wa)^{m+1}wx\big)^* & = & (wa)^*(wa)^{m+1}wx. \end{array}$$

$\Longleftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = a[wx]^2, [(wa)^*(wa)^{m+1}wx]^* = (wa)^*(wa)^{m+1}wx,$$
$$\lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Clearly, $wx = (wa)[wx]^2$. Observing that

$$\begin{array}{rcl} ||(wa)^{n+1} - (wx)(wa)^{n+2}|| & = & ||w(aw)^n - (w(xw)(aw)^{n+1}a|| \\ & \leq & ||w||||(aw)^n - (xw)(aw)^{n+1}||||a||, \end{array}$$

we see that

$$\lim_{n\to\infty} ||(wa)^n - (wx)(wa)^{n+1}||^{\frac{1}{n}} = 0.$$

This implies that $wa \in \mathcal{A}^{ⓖ_m}$. According to Theorem 2.1, $a \in \mathcal{A}^{ⓖ_m,w}$, as asserted. □

Theorem 2.4. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{ⓖ_m,w}$ if and only if*

(1) $a \in \mathcal{A}^{d,w}$;
(2) There exists $x \in \mathcal{A}$ such that
$$x = a[wx]^2, [((wa)^m)^*(wa)^{m+1}wx]^* = ((wa)^m)^*(wa)^{m+1}wx,$$
$$\lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Proof. $\Longrightarrow$ Clearly, $a \in \mathcal{A}^{d,w}$. In view of Theorem 2.1, $wa \in \mathcal{A}^{ⓖ_m}$. According to Theorem 1.2, There exists $z \in \mathcal{A}$ such that
$$z = (wa)z^2, [((wa)^m)^*(wa)^{m+1}z]^* = ((wa)^m)^*(wa)^{m+1}z,$$
$$\lim_{n\to\infty} ||(wa)^n - z(wa)^{n+1}||^{\frac{1}{n}} = 0.$$
Here, $z = (wa)^{ⓖ_m} = wa[(wa)^{ⓖ_m}]^2$. Set $x = a[(wa)^{ⓖ_m}]^2$. Then
$$[((wa)^m)^*(wa)^{m+1}wz]^* = ((wa)^m)^*(wa)^{m+1}wz,$$
$$\lim_{n\to\infty} ||(aw)^n - (zw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$
Moreover, we have
$$wx = wa[(wa)^{ⓖ_m}]^2 = (wa)^{ⓖ_m},$$
and then
$$a(wx)^2 = a[(wa)^{ⓖ_m}]^2 = x.$$
In this case, $a^{ⓖ_m,w} = x$, as desired.

$\Longleftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that
$$x = a(wx)^2, [((wa)^m)^*(wa)^{m+1}wx]^* = ((wa)^m)^*(wa)^{m+1}wx,$$
$$\lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$
Then $wx = wa(wx)^2$. In view of Theorem 1.2, $wa \in \mathcal{A}^{ⓖ_m}$. According to Theorem 2.1, $a \in \mathcal{A}^{ⓖ_m,w}$, as asserted. □

Corollary 2.5. *Let $a, w \in \mathcal{A}$. Then $a \in \mathcal{A}^{ⓦ,w}$ if and only if*

(1) $a \in \mathcal{A}^{D,w}$;
(2) There exists $x \in \mathcal{A}$ such that
$$x = a[wx]^2, [(wa)^*(wa)^2wx]^* = (wa)^*(wa)^2wx,$$
$$\lim_{n\to\infty} ||(aw)^n - (xw)(aw)^{n+1}||^{\frac{1}{n}} = 0.$$

Proof. This is obvious by Theorem 2.4. □

Set $im(x) = \{xr \mid r \in \mathcal{A}\}$. We are ready to prove:

Theorem 2.6. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a^{ⓖ_m,w} = x$.

(2) $awx = a(wa)^{\textcircled{g}_m}, a(wx)^2 = x$.
(3) $wawx = wa(wa)^{\textcircled{g}_m}, im(x) \subseteq im(aw)^d$.
(4) $awx = a(wa)^{\textcircled{g}_m}, im(x) \subseteq im(aw)^d$.

Proof. (1) $\Rightarrow$ (2) In view of Theorem 2.1, $x = a[(wa)^{\textcircled{g}_m}]^2$. Then $a(wx)^2 = x$ and

$$\begin{array}{rcl} awx & = & (aw)a[(wa)^{\textcircled{g}_m}]^2 \\ & = & a(wa)[(wa)^{\textcircled{g}_m}]^2 \\ & = & a(wa)^{\textcircled{g}_m}. \end{array}$$

(2) $\Rightarrow$ (3) Obviously, $wawx = w(awx) = w[a(wa)^{\textcircled{g}_m}] = wa(wa)^{\textcircled{g}_m}$. Moreover, we have

$$\begin{array}{rcl} x & = & a(wx)^2 = (awx)wx = a(wa)^{\textcircled{g}_m}wx \\ & = & a(wa)^d(wa)(wa)^{\textcircled{g}_m}wx \\ & = & a[(wa)^d]^2w(awa)(wa)^{\textcircled{g}_m}wx \\ & = & (aw)^d(awa)(wa)^{\textcircled{g}_m}wx. \end{array}$$

Therefore $im(x) \subseteq im(aw)^d$, as desired.

(3) $\Rightarrow$ (4) Since $im(x) \subseteq im(aw)^d$, we see that

$$\begin{array}{rcl} awx & = & aw[(aw)(aw)^dx] = (aw)^da[wawx] \\ & = & (aw)^da[wa(wa)^{\textcircled{g}_m}] \\ & = & a[(wa)^d]^2(wa)^2(wa)^{\textcircled{g}_m} \\ & = & awa)(wa)^d(wa)^{\textcircled{g}_m} \\ & = & a(wa)^{\textcircled{g}_m}, \end{array}$$

as desired.

(4) $\Rightarrow$ (1) Write $x = (aw)^dz$ for some $z \in R$. Then

$$\begin{array}{rcl} x & = & aw(aw)^dx = (aw)^d(awx) \\ & = & (aw)^d[a(wa)^{\textcircled{g}_m}] \\ & = & (aw)^d(aw)a[(wa)^{\textcircled{g}_m}]^2 \\ & = & a[(wa)^{\textcircled{g}_m}]^2. \end{array}$$

This completes the proof by Theorem 2.1. □

Corollary 2.7. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a^{\textcircled{g}_m} = x$.
(2) $ax = aa^{\textcircled{g}_m}, ax^2 = x$.
(3) $ax = aa^{\textcircled{g}_m}, im(x) \subseteq im(a^d)$.

Proof. This is a direct consequence of Theorem 2.6. □

We are ready to prove:

Theorem 2.8. *Let $a \in \mathcal{A}^{\textcircled{g}_m,w}$. Then $wawa^{\textcircled{g}_{m+1},w} = wa^{\textcircled{g}_m,w}wa$.*

Proof. In view of Theorem 2.1, we see that

$$\begin{array}{rcl} wawa^{\textcircled{g}_{m+1},w} & = & wawa[(wa)^{\textcircled{g}_{m+1}}]^2 \\ & = & wa(wa)^{\textcircled{g}_{m+1}} \\ wa^{\textcircled{g}_m,w}wa & = & wa[(wa)^{\textcircled{g}_m}]^2wa \\ & = & (wa)^{\textcircled{g}_m}wa. \end{array}$$

In view of [1, Corollary 2.4], we have

$$(wa)^{\textcircled{g}_{m+1}} = [(wa)^{\textcircled{g}_m}]^2wa.$$

Therefore

$$\begin{array}{rcl} wawa^{\textcircled{g}_{m+1},w} & = & wa(wa)^{\textcircled{g}_{m+1}} \\ & = & wa[(wa)^{\textcircled{g}_m}]^2wa \\ & = & (wa)^{\textcircled{g}_m}wa \\ & = & wa^{\textcircled{g}_m,w}wa. \end{array}$$

This completes the proof. □

Corollary 2.9. *Let $a \in \mathcal{A}^{\textcircled{g}_m}$. Then $aa^{\textcircled{g}_{m+1}} = a^{\textcircled{g}_m}a$.*

Proof. This is obvious by choosing $w = 1$ in Theorem 2.8. □

3. REPRESENTATIONS OF m-GENERALIZED GROUP INVERSE

In this section, we present the representations of m-generalized group inverse under weighted generalized core-EP invertibility.

Theorem 3.1. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then $a \in \mathcal{A}^{\textcircled{g}_m,w}$ and*

$$a^{\textcircled{g}_m,w} = [a^{\textcircled{d},w}w]^{m+1}(aw)^{m-1}a.$$

Proof. In view of [3, Theorem 2.1], $a^{\textcircled{d},w} = a[(wa)^{\textcircled{d}}]^2$; hence, $wa^{\textcircled{d},w} = (wa)^{\textcircled{d}}$. Then we easily check that

$$\begin{array}{rcl} [a^{\textcircled{d},w}w]^{m+1}(aw)^{m-1}a & = & a^{\textcircled{d},w}[wa^{\textcircled{d},w}]^m w(aw)^{m-1}a \\ & = & a^{\textcircled{d},w}[(wa)^{\textcircled{d}}]^m(wa)^m \\ & = & a[(wa)^{\textcircled{d}}]^2[(wa)^{\textcircled{d}}]^m(wa)^m \\ & = & a[(wa)^{\textcircled{d}}]^{m+2}(wa)^m. \end{array}$$

Thus,

$$w[a^{\textcircled{d},w}w]^{m+1}(aw)^{m-1}a = wa[(wa)^{\textcircled{d}}]^{m+2}(wa)^m = [(wa)^{\textcircled{d}}]^{m+1}(wa)^m.$$

Set $x = [(wa)^{\textcircled{d}}]^{m+1}(wa)^m$. Then

$$\begin{array}{rcl} (wa)x^2 & = & (wa)[(wa)^{ⓓ}]^{m+1}(wa)^m[(wa)^{ⓓ}]^{m+1}(wa)^m \\ & = & (wa)[(wa)^{ⓓ}]^{m+1}(wa)^{ⓓ}(wa)^m \\ & = & [(wa)^{ⓓ}]^{m+1}(wa)^m \\ & = & x, \\ ((wa)^d)^*(wa)^{m+1}x & = & ((wa)^d)^*(wa)^{m+1}[(wa)^{ⓓ}]^{m+1}(wa)^m \\ & = & ((wa)^d)^*(wa)(wa)^{ⓓ}(wa)^m \\ & = & ((wa)^d)^*[(wa)(wa)^{ⓓ}]^*(wa)^m \\ & = & [(wa)((wa)^{ⓓ})^2]^*(wa)^m \\ & = & ((wa)^d)^*(wa)^m, \\ \lim\limits_{n\to\infty}||(wa)^n - x(wa)^{n+1}||^{\frac{1}{n}} & = & 0. \end{array}$$

This implies that

$$(wa)^{ⓖ_m} = [(wa)^{ⓓ}]^{m+1}(wa)^m.$$

According to Theorem 2.1, we prove that $a \in \mathcal{A}^{ⓖ_m,w}$ and

$$\begin{array}{rcl} a^{ⓖ_m,w} & = & a[(wa)^{ⓖ_m}]^2 \\ & = & a[((wa)^{ⓓ})^{m+1}(wa)^m]^2 \\ & = & a[((wa)^{ⓓ})^{m+1}(wa)^m][((wa)^{ⓓ})^{m+1}(wa)^m] \\ & = & a((wa)^{ⓓ})^{m+1}(wa)^{ⓓ}(wa)^m \\ & = & a[(wa)^{ⓓ}]^{m+2}(wa)^m \\ & = & [a^{ⓓ,w}w]^{m+1}(aw)^{m-1}a, \end{array}$$

as required. □

Corollary 3.2. *Let $a \in \mathcal{A}^{ⓓ}$. Then $a \in \mathcal{A}^{ⓖ_m}$ and*

$$a^{ⓖ_m} = (a^{ⓓ})^{m+1}a^m.$$

Proof. This is obvious by choosing $w = 1$ in Theorem 3.1. □

We call x is the $(1,3)$-inverse of a if x satisfies the equations $axa = a$ and $(ax)^* = ax$. We use $\mathcal{A}^{(1,3)}$ to denote the set of all $(1,3)$-invertible elements in $\mathcal{A}$. Let $a \in \mathcal{A}^{ⓓ,w}$ and $a(wa)^{ⓓ}w \in \mathcal{A}^{(1,3)}$. By using [3, Theorem 2.5], $aw, wa \in \mathcal{A}^{ⓓ}$. Let $p = (aw)(aw)^{ⓓ}, q = (wa)(wa)^{ⓓ}$. Then $p, q \in \mathcal{A}$ are projections.

Lemma 3.3. *Let $a \in \mathcal{A}^{ⓓ,w}$ and $a(wa)^{ⓓ}w \in \mathcal{A}^{(1,3)}$. Then*

$$a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}_{p,q}, w = \begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}_{q,p},$$

where $a_1 \in [p\mathcal{A}q]^{-1}, w_1 \in [q\mathcal{A}p]^{-1}$ and a_3w_3 and w_3a_3 are quasinilpotent.

Proof. We easily verify that

$$\begin{array}{rcl}(1-p)aq & = & [1-(aw)(aw)^{ⓓ}]a(wa)(wa)^{ⓓ}\\ & = & [1-(aw)(aw)^{ⓓ}]awa(wa)^n[(wa)^{ⓓ}]^{n+1}\\ & = & [1-(aw)(aw)^{ⓓ}](aw)^{n+1}a[(wa)^{ⓓ}]^{n+1}\\ & = & aw[(aw)^n-(aw)^{ⓓ}(aw)^{n+1}]a[(wa)^{ⓓ}]^{n+1}.\end{array}$$

Then

$$||(1-p)aq||^{\frac{1}{n}} \leq ||aw||^{\frac{1}{n}}||(aw)^n-(aw)^{ⓓ}(aw)^{n+1}||^{\frac{1}{n}}||a[(wa)^{ⓓ}]^{n+1}||^{\frac{1}{n}}.$$

Since $\lim\limits_{n\to\infty}||(aw)^n-(aw)^{ⓓ}(aw)^{n+1}||^{\frac{1}{n}}=0$, we see that $\lim\limits_{n\to\infty}||(1-p)aq||^{\frac{1}{n}}=0$. This implies that $(1-p)aq=0$. Likewise, we prove that

$$(1-q)wp=[1-(wa)(wa)^{ⓓ}]w(aw)(aw)^{ⓓ}=0.$$

Moreover, we have

$$\begin{array}{rl} & [(aw)(aw)^{ⓓ}a(wa)(wa)^{ⓓ}][(wa)(wa)^{ⓓ}w(aw)^{ⓓ}(aw)(aw)^{ⓓ}]\\ = & (aw)(aw)^{ⓓ}(aw)a(wa)^{ⓓ}w(aw)^{ⓓ}\\ = & (aw)a(wa)^{ⓓ}w(aw)^{ⓓ}\\ = & (aw)(aw)^{ⓓ},\\ & [(wa)(wa)^{ⓓ}w(aw)^{ⓓ}(aw)(aw)^{ⓓ}][(aw)(aw)^{ⓓ}a(wa)(wa)^{ⓓ}]\\ = & (wa)(wa)^{ⓓ}w(aw)^{ⓓ}a(wa)(wa)^{ⓓ}\\ = & (wa)(wa)^{ⓓ}wa(wa)^{ⓓ}\\ = & (wa)(wa)^{ⓓ}.\end{array}$$

Then $a_1=paq\in[p\mathcal{A}q]^{-1}$. Similarly, $w_1=qwp\in[q\mathcal{A}p]^{-1}$.

Also we easily see that

$$\begin{array}{rcl}a_3w_3 & = & [1-(aw)(aw)^{ⓓ}]a[1-wa(wa)^{ⓓ}]w[1-(aw)(aw)^{ⓓ}]\\ & \in & \mathcal{A}^{qnil}.\end{array}$$

Thus, a_3w_3 is quasinilpotent. By using Cline's formula, w_3a_3 is quasinilpotent. This completes the proof. □

Lemma 3.4. *Let $a\in\mathcal{A}^{ⓓ,w}$ and $a(wa)^{ⓓ}w\in\mathcal{A}^{(1,3)}$. Then*

$$a^{ⓓ,w}=\begin{pmatrix}(w_1a_1w_1)^{-1} & 0\\ 0 & 0\end{pmatrix}_{p,q}.$$

Proof. In view of [3, Theorem 2.1], $a^{ⓓ,w} = a[(wa)^{ⓓ}]^2$. One easily checks that

$$\begin{array}{rcl} pa^{ⓓ,w}(1-q) & = & (aw)(aw)^{ⓓ}a[(wa)^{ⓓ}]^2[1-(wa)(wa)^{ⓓ}] \\ & = & (aw)(aw)^{ⓓ}a(wa)^{ⓓ}[(wa)^{ⓓ}-(wa)^{ⓓ}(wa)(wa)^{ⓓ}] = 0, \\ (1-p)a^{ⓓ,w}q & = & [1-(aw)(aw)^{ⓓ}]a[(wa)^{ⓓ}]^2(wa)(wa)^{ⓓ} \\ & = & [1-(aw)(aw)^{ⓓ}]awa[(wa)^{ⓓ}]^3(wa)(wa)^{ⓓ} \\ & = & aw[1-(aw)^{ⓓ}aw]a[(wa)^{ⓓ}]^3(wa)(wa)^{ⓓ} = 0, \\ (1-p)a^{ⓓ,w}(1-q) & = & [1-(aw)(aw)^{ⓓ}]a[(wa)^{ⓓ}]^2[1-(wa)(wa)^{ⓓ}] = 0. \end{array}$$

Moreover, we see that

$$\begin{array}{rcl} pa^{ⓓ,w}q & = & (aw)(aw)^{ⓓ}a[(wa)^{ⓓ}]^2(wa)(wa)^{ⓓ} \\ & = & (aw)(aw)^{ⓓ}a[(wa)^{ⓓ}]^2 \\ & = & a[(wa)^{ⓓ}]^2 \\ & = & w_1a_1w_1 \in \big(p\mathcal{A}q\big)^{-1}, \end{array}$$

thus yielding the result. □

Theorem 3.5. *Let $a \in \mathcal{A}^{ⓓ,w}$ and $a(wa)^{ⓓ}w \in \mathcal{A}^{(1,3)}$. Then*

$$a^{ⓖ_m,w} = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}_{p,q},$$

where

$$\begin{array}{rcl} \alpha & = & (w_1a_1w_1)^{-1}, \\ \beta & = & (w_1a_1w_1)^{-1}a_2 + [(w_1a_1w_1)^{-1}w_1]^{m+1}c_{m-1}a_3 + b_{m+1}(a_3w_3)^{m-1}a_3; \\ b_1 & = & (w_1a_1w_1)^{-1}w_2, b_{n+1} = (w_1a_1w_1)^{-1}w_1b_n, \\ c_1 & = & a_1w_2 + a_2w_3, c_{n+1} = a_1w_1c_n + (a_1w_2+a_2w_3)(a_3w_3)^m. \end{array}$$

Proof. Construct two series $\{b_n\}$ and $\{c_n\}$ by the equalities: Here,

$$b_1 = (w_1a_1w_1)^{-1}w_2, b_{n+1} = (w_1a_1w_1)^{-1}w_1b_n,$$
$$c_1 = a_1w_2 + a_2w_3, c_{n+1} = a_1w_1c_n + (a_1w_2+a_2w_3)(a_3w_3)^m.$$

Then we compute that

$$[\begin{pmatrix} (w_1a_1w_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}]^{m+1} = \begin{pmatrix} [(w_1a_1w_1)^{-1}w_1]^{m+1} & b_{m+1} \\ 0 & 0 \end{pmatrix},$$
$$[\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}\begin{pmatrix} w_1 & w_2 \\ 0 & w_3 \end{pmatrix}]^{m-1} = \begin{pmatrix} (a_1w_1)^{m-1} & c_{m-1} \\ 0 & (a_3w_3)^{m-1} \end{pmatrix}.$$

According to Theorem 3.1 and Lemma 3.4, we derive

$$\begin{aligned} & a^{\textcircled{g}_m,w} = [a^{\textcircled{d},w}w]^{m+1}(aw)^{m-1}a \\ = & \begin{pmatrix} [(w_1a_1w_1)^{-1}w_1]^{m+1} & b_{m+1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (a_1w_1)^{m-1} & c_{m-1} \\ 0 & (a_3w_3)^{m-1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \\ = & \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= (w_1a_1w_1)^{-1}, \\ \beta &= (w_1a_1w_1)^{-1}a_2 + [(w_1a_1w_1)^{-1}w_1]^{m+1}c_{m-1}a_3 + b_{m+1}(a_3w_3)^{m-1}a_3. \end{aligned}$$

This completes the proof. □

Corollary 3.6. *Let $a \in \mathcal{A}^{\textcircled{d}}$. Then*

$$a^{\textcircled{g}_m} = \begin{pmatrix} a_1^{-1} & (a_1)^{-(m+1)}b_m \\ 0 & 0 \end{pmatrix}_{s,t},$$

where $b_1 = a_2, b_{m+1} = a_1b_m + a_2a_3^m, s = aa^{\textcircled{d}}$ and $t = a^{\textcircled{d}}a$.

Proof. This is immediate by choosing $w = 1$ in Theorem 3.5. □

4. WEIGHTED m-GENERALIZED CORE INVERSE

The aim of this section is to investigate weighted m-generalized group inverse with weighted Moore-Penrose inverse. We introduce and study weighted m-generalized core inverse in a Banach $*$-algebra. Let $p_{(wa)^m} = (wa)^m[(wa)^m]^\dagger$ be the projection on $(wa)^m$. The following theorem is crucial.

Theorem 4.1. *Let $a \in \mathcal{A}^{\textcircled{c}_m,w}$. Then there exists a unique $x \in \mathcal{A}$ such that*

$$xwawx = x, awx = awa^{\textcircled{g}_m,w}p_{(wa)^m}, x(wa)^m = a^{\textcircled{g}_m,w}(wa)^m.$$

Proof. Taking $x = a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger$. Then

$$\begin{aligned} xwawx &= a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger wawa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger \\ &= a^{\textcircled{g}_m,w}wawa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger \\ &= a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger \\ &= x, \\ awx &= awa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger = awa[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^\dagger \\ &= a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^\dagger \\ &= awa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger, \\ x(wa)^m &= a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger(wa)^m = a^{\textcircled{g}_m,w}(wa)^m. \end{aligned}$$

Suppose that x' satisfies the preceding equations. Then one checks that

$$\begin{array}{rcl} x' & = & x'wawx' = a^{\textcircled{g}_m,w}wawx' \\ & = & a^{\textcircled{g}_m,w}wawa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger} \\ & = & a^{\textcircled{g}_m,w}wawa[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger} \\ & = & a[(wa)^{\textcircled{g}_m}]^2wa(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger} \\ & = & a[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger} \\ & = & a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger} \\ & = & x, \end{array}$$

as required. □

We denote the preceding unique x by $a^{\textcircled{c}_m,w}$.

Corollary 4.2. *Let $a \in \mathcal{A}^{\textcircled{c}_m,w} (m \geq 2)$. Then the following are equivalent:*

(1) $a^{\textcircled{c}_m,w} = x$.
(2) The equation system

$$awx = a(wa)^{\textcircled{g}_m}p_{(wa)^m}, a(wx)^2 = x$$

is consistent and its unique solution $x = a^{\textcircled{c}_m,w}$.

Proof. (1) ⇒ (2) In view of Theorem 4.1, we have

$$\begin{array}{rcl} awx & = & awa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger} \\ & = & awa[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger} \\ & = & a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger}. \end{array}$$

Moreover, we have

$$\begin{array}{rcl} a(wx)^2 & = & awa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger}wa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger} \\ & = & awa[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger}wa[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger} \\ & = & awa[(wa)^{\textcircled{g}_m}]^2(wa)[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger} \\ & = & a[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger} \\ & = & a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger} \\ & = & x. \end{array}$$

(2) ⇒ (1) Suppose that the equation system

$$awx = a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger}, a(wx)^2 = x$$

is consistent. In view of [1, Corollary 2.4], we have

$$\begin{aligned}
x &= (awx)wx = (a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger})wx\\
&= a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger})wa(wx)^2\\
&= a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger})(wa)^m(wx)^{m+1}\\
&= a(wa)^{\textcircled{g}_m}(wa)^m(wx)^{m+1}\\
&= a(wa)^{\textcircled{g}_m}wx\\
&= a[(wa)^{\textcircled{g}_{m-1}}]^2w(awx)\\
&= a[(wa)^{\textcircled{g}_{m-1}}]^2w[a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger}]\\
&= a[(wa)^{\textcircled{g}_{m-1}}]^2(wa)[(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger}]\\
&= a(wa)^{\textcircled{g}_m}[(wa)^{\textcircled{g}_m}(wa)^m((wa)^m)^{\dagger}]\\
&= a[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger}\\
&= a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger}\\
&= a^{\textcircled{c}_m,w},
\end{aligned}$$

as asserted. □

Let $a \in \mathcal{A}^{\textcircled{c}_m,w}$. In view of Theorem 4.1, $a^{\textcircled{c}_m,w} = a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger}$. Set $c = a(wa)^{\textcircled{g}_m}(wa)^m$. We now establish necessary and sufficient conditions under which a has weighted m-generalized core inverse.

Theorem 4.3. *Let $a \in \mathcal{A}^{\textcircled{c}_m,w}$. The following are equivalent:*

(1) $a^{\textcircled{c}_m,w} = x$.
(2) $awx = c[(wa)^m]^{\dagger}$ and $x\mathcal{A} \subseteq a^{d,w}\mathcal{A}$.
(3) $awx = c[(wa)^m]^{\dagger}$ and $a(wx)^2 = x$.

Proof. (1) ⇒ (2) In view of Theorem 4.1, we have

$$\begin{aligned}
awx &= awa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger}\\
&= c[(wa)^m]^{\dagger}.
\end{aligned}$$

By virtue of Theorem 2.1, we have

$$\begin{aligned}
x\mathcal{A} &= a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger}\mathcal{A}\\
&\subseteq a^{\textcircled{g}_m,w}\mathcal{A}\\
&= a[(wa)^{\textcircled{g}_m}]^2\mathcal{A}\\
&\subseteq a(wa)^d\mathcal{A}\\
&\subseteq a[(wa)^d]^2\mathcal{A}\\
&\subseteq a^{d,w}\mathcal{A}.
\end{aligned}$$

(2) ⇒ (1) Since $awx = c[(wa)^m]^{\dagger}$, we have $awx = a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^{\dagger} = awa[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^{\dagger} = awa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^{\dagger}$.

Since $x\mathcal{A} \subseteq a^{d,w}\mathcal{A}$, we derive that $a^{d,w}wawa^{d,w} = a^{d,w}$. Hence, $a^{d,w}wawx = x$. In view of Theorem 2.1, $a(wa)^{\textcircled{g}_m} \subseteq a[(wa)^d]^2w\mathcal{A} = (aw)^d\mathcal{A}$. Then $(aw)^dawa(wa)^{\textcircled{g}_m} = a(wa)^{\textcircled{g}_m}$. We deduce that

$$\begin{aligned} x &= a^{d,w}w(awx) = a[(wa)^d]^2w(awx) \\ &= a[(wa)^d]^2wawa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger \\ &= (aw)^dawa^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger \\ &= (aw)^dawa[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^\dagger \\ &= a[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^\dagger \\ &= a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger. \end{aligned}$$

Therefore $a^{\copyright_m,w} = x$, as desired.

(1) $\Rightarrow$ (3) By the argument above, we have $awx = c[(wa)^m]^\dagger$. In view of Theorem 2.1, $x = a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger$. By using Corollary 4.2, we have $a(wx)^2 = x$, as required.

(3) $\Rightarrow$ (1) Since $awx = c[(wa)^m]^\dagger$, we see that $awx = a(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^\dagger$. As $a(wx)^2 = x$, by virtue of Corollary 4.2, $x = a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger = a^{\copyright_m,w}$, as required. □

Let $X \in \mathbb{C}^{n\times n}$. The symbol $\mathcal{R}(X)$ denote the range space of X. We now derive

Corollary 4.4. *Let $A \in \mathbb{C}^{n\times n}$. The following are equivalent:*

(1) $A^{\textcircled{W},\dagger} = X$.
(2) $AWX = AA^{\textcircled{W}}AA^\dagger$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A^D)$.
(3) $AWX = A(WA)^{\textcircled{W}}WAA^\dagger$ and $A(WX)^2 = X$.

Proof. Since $A \in \mathbb{C}^{n\times n}$, we easily see that $A^{\copyright_1,w} = A^{\textcircled{W},\dagger}$. Therefore we complete the proof by Theorem 4.3. □

We are now ready to prove the following.

Theorem 4.5. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $a^{\copyright_m,w} = x$.
(2) $xwcwx = x, awx = c[(wa)^m]^\dagger$ and $xwc = (aw)^dc$.

Proof. (1) ⇒ (2) In view of Theorem 4.1, $x = a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger$. By virtue of Theorem 4.3, $awx = c[(wa)^m]^\dagger$. Moreover, we verify that

$$\begin{aligned}
& xwcwx \\
= \; & a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger[wa(wa)^{\textcircled{g}_m}(wa)^m w]a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger \\
= \; & a[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^\dagger[wa(wa)^{\textcircled{g}_m}(wa)^m w][a(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^\dagger \\
= \; & a[(wa)^{\textcircled{g}_m}]^2 wa(wa)^{\textcircled{g}_m}(wa)^m[(wa)^m]^\dagger \\
= \; & a[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^\dagger \\
= \; & a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger \\
= \; & x, \\
& xwc \\
= \; & a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger w[a(wa)^{\textcircled{g}_m}(wa)^m] \\
= \; & a[(wa)^{\textcircled{g}_m}]^2(wa)^m[(wa)^m]^\dagger wa(wa)^{\textcircled{g}_m}(wa)^m \\
= \; & a[(wa)^{\textcircled{g}_m}]^2 wa(wa)^{\textcircled{g}_m}(wa)^m \\
= \; & a[(wa)^{\textcircled{g}_m}]^2(wa)^m \\
= \; & a(wa)^d(wa)[(wa)^{\textcircled{g}_m}]^2(wa)^m \\
= \; & a(wa)^d(wa)^{\textcircled{g}_m}(wa)^m \\
= \; & a[(wa)^d]^2 wa(wa)^{\textcircled{g}_m}(wa)^m \\
= \; & (aw)^d c,
\end{aligned}$$

as required.

(2) ⇒ (1) By hypothesis, we check that

$$\begin{aligned}
x &= xwcwx = (xwc)wx = [(aw)^d c]wx \\
&= a[(wa)^d]^2(wcwx) \\
&\in a^{d,w}\mathcal{A}.
\end{aligned}$$

According to Theorem 4.3, we complete the proof. □

Corollary 4.6. *Let $A \in \mathbb{C}^{n\times n}$ and $C = A(WA)^{\textcircled{W}}WA$. The following are equivalent:*

(1) $A^{\textcircled{W},\dagger} = X$.
(2) $XWCWX = X, AWX = C(WA)^\dagger$ and $XWC = (AW)^D C$.

Proof. It is immediate by Theorem 4.5 by choosing $m = 1$. □

5. APPLICATIONS

The purpose of this section is to give the applications of the w-weighted m-generalized group (core) inverse in solving the matrix equations. We consider the following equation in $\mathcal{A}$:

$$[(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*(wa)^m b, \quad (5.1)$$

where $a, w, b \in \mathcal{A}$ and $m \in \mathbb{N}$.

Theorem 5.1. *Let $a \in \mathcal{A}^{\textcircled{g}_m,w}$. Then Eq. (5.1) has solution*

$$x = a^{\textcircled{g}_m,w}b + [1 - a^{\textcircled{g}_m,w}waw]y,$$

where $y \in \mathcal{A}$ is arbitrary.

Proof. Let $x = a^{\textcircled{g}_m,w}b + [1 - a^{\textcircled{g}_m,w}waw]y$, where $y \in \mathcal{A}$. Then

$$\begin{array}{rcl} wx & = & wa[(wa)^{\textcircled{g}_m}]^2b + w[1 - a((wa)^{\textcircled{g}_m})^2waw]y \\ & = & (wa)^{\textcircled{g}_m}b + [w - (wa)^{\textcircled{g}_m}waw]y. \end{array}$$

Since $[(wa)^d]^*(wa)^{m+1}(wa)^{\textcircled{g}_m} = (wa)^m$, we verify that

$$\begin{array}{rl} & [(wa)^d]^*(wa)^{m+1}wx \\ = & [(wa)^d]^*(wa)^{m+1}(wa)^{\textcircled{g}_m}b + [(wa)^d]^*(wa)^{m+1}[w - (wa)^{\textcircled{g}_m}waw]y \\ = & (wa)^d]^*(wa)^m b + [(wa)^d]^*(wa)^{m+1}w - (wa)^d]^*(wa)^m waw]y \\ = & [(wa)^d]^*(wa)^m b, \end{array}$$

as asserted. □

Corollary 5.2. *Let $a \in \mathcal{A}^{\textcircled{d},w}$. Then the general solution of Eq. (5.1) is*

$$x = a^{\textcircled{g}_m,w}b + [1 - a^{\textcircled{g}_m,w}waw]y,$$

where $y \in \mathcal{A}$ is arbitrary.

Proof. Let x be the solution of the Eq. (5.1). Then

$$[(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*(wa)^m b.$$

In view of Theorem 3.1, $a^{\textcircled{g}_m,w} = [a^{\textcircled{d},w}w]^{m+1}(aw)^{m-1}a$. Then

$$\begin{array}{rcl}
a^{\textcircled{g}_m,w}wawx & = & [a^{\textcircled{d},w}w]^{m+1}(aw)^{m-1}awawx \\
& = & [a^{\textcircled{d},w}w]^{m+1}(aw)^{m+1}x \\
& = & [a^{\textcircled{d},w}w]^{m}[a((wa)^{\textcircled{d}})^2w](aw)^{m+1}x \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}][(wa)^{\textcircled{d}}wa(wa)^{\textcircled{d}}](wa)^{m+1}wx \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}](wa)^{\textcircled{d}}[wa(wa)^{\textcircled{d}}](wa)^{m+1}wx \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}](wa)^{\textcircled{d}}[wa(wa)^{\textcircled{d}}]^*(wa)^{m+1}wx \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}](wa)^{\textcircled{d}}[(wa)^d(wa)^2(wa)^{\textcircled{d}}]^*(wa)^{m+1}wx \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}](wa)^{\textcircled{d}}[(wa)^2(wa)^{\textcircled{d}}]^*[((wa)^d)^*(wa)^{m+1}wx] \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}](wa)^{\textcircled{d}}[(wa)^2(wa)^{\textcircled{d}}]^*[((wa)^d)^*(wa)^{m}b] \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}](wa)^{\textcircled{d}}[(wa)^d(wa)^2(wa)^{\textcircled{d}}]^*(wa)^{m}b \\
& = & [a^{\textcircled{d},w}w]^{m}[a(wa)^{\textcircled{d}}][(wa)^{\textcircled{d}}wa(wa)^{\textcircled{d}}](wa)^{m}b \\
& = & [a^{\textcircled{d},w}w]^{m}[a((wa)^{\textcircled{d}})^2](wa)^{m}b \\
& = & [a^{\textcircled{d},w}w]^{m}[a((wa)^{\textcircled{d}})^2w](aw)^{m-1}ab \\
& = & [a^{\textcircled{d},w}w]^{m+1}(aw)^{m-1}ab \\
& = & a^{\textcircled{g}_m,w}b.
\end{array}$$

Accordingly,

$$x = a^{\textcircled{g}_m,w}b + [1 - a^{\textcircled{g}_m,w}waw]x.$$

By using Theorem 5.1, we complete the proof. ☐

Corollary 5.3. *Let $a \in \mathcal{A}^{\textcircled{g}_m,w}$. If x is the solution of Eq. (5.1) and $im(x) \subseteq im\big((aw)^d\big)$, then*

$$x = a^{\textcircled{g}_m,w}b.$$

Proof. By virtue of Theorem 5.1, $a^{\textcircled{g}_m,w}b$ is a solution of Eq. (5.1). Let $x_1, x_2 \in \mathcal{A}$ be the solutions of Eq. (5.1) and satisfy $im(x_i) \subseteq im\big((aw)^d\big)$. Write $x_1 = (aw)^d y_1$ and $x_2 = (aw)^d y_2$. Then $x_1 - x_2 = [(aw)^d]^2 a(waw)(x_1 - x_2)$. Hence, $im(x_1 - x_2) \subseteq im\big((aw)^d\big)$. By hypothesis, we have

$$[(wa)^d]^*(wa)^{m+1}wx_i = [(wa)^d]^*(wa)^m b$$

for $i = 1, 2$. Then $[(wa)^d]^*(wa)^{m+1}w(x_1 - x_2) = 0$; and so

$$[(wa)^d]^*(wa)^{m+1}w[(aw)^d]^2 a(waw)(x_1 - x_2) = 0.$$

By using Cline's formula, we have $w[(aw)^d]^2 a = (wa)^d$, and then

$$[(wa)^d]^*(wa)^d(wa)^{m+2}w(x_1 - x_2) = 0.$$

Since the involution is proper, we have $(wa)^d(wa)^{m+2}w(x_1 - x_2) = 0$; whence, $(aw)(aw)^d(x_1 - x_2) = 0$. Thus, $x_1 = aw(aw)^d x_1 = aw(aw)^d x_2 = x_2$. Therefore $x = a^{\textcircled{g}_m,w}b$ is the unique solution of Eq. (5.1). ☐

Consider the following matrix equation:

$$[(WA)^D]^*(WA)^{m+1}WX = [(WA)^D]^*(WA)^m B, \quad (5.2)$$

where $A \in \mathbb{C}^{q\times n}, W \in \mathbb{C}^{n\times q}, B \in \mathbb{C}^{n\times p}$ and $m \in \mathbb{N}$.

Corollary 5.4. (1) *The general solution of Eq. (5.2) is*

$$X = A^{\textcircled{W}_m,W}B + [I_n - A^{\textcircled{W}_m,W}WAW]Y,$$

where $Y \in \mathbb{C}^{n\times p}$ is arbitrary.

(2) *If X is the solution of Eq. (5.2) and $\mathcal{R}(X) \subseteq \mathcal{R}\big((AW)^D\big)$, then*

$$X = A^{\textcircled{W}_m,W}B.$$

Proof. This is obvious by Corollary 5.2 and Corollary 5.3. □

Let $a \in \mathcal{A}^{\textcircled{c}_m,w}$. We now come to consider the following equation in $\mathcal{A}$:

$$[(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*(wa)^{2m}[(wa)^m]^\dagger b, \quad (5.3)$$

where $a, w, b \in \mathcal{A}$ and $m \in \mathbb{N}$. The following lemma is crucial.

Lemma 5.5. *Let $a \in \mathcal{A}^{\textcircled{c}_m,w}$. Then $a \in \mathcal{A}^{\textcircled{d},w}$.*

Proof. By hypothesis, $a \in \mathcal{A}^{\textcircled{g}_m,w} \bigcap \mathcal{A}^{\dagger_m,w}$. In light of [1, Theorem 2.1], $(wa)^m \in \mathcal{A}^{\textcircled{g}} \bigcap \mathcal{A}^\dagger$. By virtue of [4, Theorem 3.1], $(wa)^m \in \mathcal{A}^{\textcircled{d}}$. Then $wa \in \mathcal{A}^{\textcircled{d}}$. Evidently, $(wa)^{\textcircled{d}} = (wa)^{m-1}[(wa)^m]^{\textcircled{d}}$. Accordingly, $a \in \mathcal{A}^{\textcircled{d},w}$ by [3, Theorem 2.1]. □

We are ready to prove:

Theorem 5.6. *Let $a \in \mathcal{A}^{\textcircled{c}_m,w}$. Then the general solution of Eq. (5.3) is*

$$x = a^{\textcircled{c}_m,w}b + [1 - a^{\textcircled{g}_m,w}waw]y,$$

where $y \in \mathcal{A}$ is arbitrary.

Proof. Let $x = a^{\textcircled{c}_m,w}b + [1 - a^{\textcircled{g}_m,w}waw]y$, where $y \in \mathcal{A}$. In view of Theorem 4.1, $a^{\textcircled{c}_m,w} = a^{\textcircled{g}_m,w}(wa)^m[(wa)^m]^\dagger$. Then

$$x = a^{\textcircled{g}_m,w}[(wa)^m[(wa)^m]^\dagger b] + [1 - a^{\textcircled{g}_m,w}waw]y.$$

By virtue of Theorem 5.1, x is the solution of Eq. (5.3).

In light of Lemma 5.5, $a \in \mathcal{A}^{\textcircled{d},w}$. By using Corollary 5.2,

$$x = a^{\textcircled{g}_m,w}[(wa)^m[(wa)^m]^\dagger b] + [1 - a^{\textcircled{g}_m,w}waw]y$$

is the general solution of Eq. (5.3), as required. □

Corollary 5.7. *Let $a \in \mathcal{A}^{©_m,w}$. If x is the solution of Eq. (5.3) and $im(x) \subseteq im\big((aw)^d\big)$, then*

$$x = a^{©_m,w}b.$$

Proof. By virtue of Theorem 5.6, $a^{©_m,w}b$ is a solution of Eq. (5.3). Let $x_1, x_2 \in \mathcal{A}$ be the solutions of Eq. (5.3) and satisfy $im(x_i) \subseteq im\big((aw)^d\big)$. Then they are solutions of the equation:

$$[(wa)^d]^*(wa)^{m+1}wx = [(wa)^d]^*(wa)^m[(wa)^m[(wa)^m]^\dagger b],$$

as desired. □

Consider the following matrix equation:

$$[(WA)^D]^*(WA)^{m+1}WX = [(WA)^D]^*(WA)^{2m}[(WA)^m]^\dagger B, \quad (5.4)$$

where $A \in \mathbb{C}^{q\times n}, W \in \mathbb{C}^{n\times q}, B \in \mathbb{C}^{n\times p}$ and $m \in \mathbb{N}$.

Corollary 5.8. (1) *The general solution of Eq. (5.4) is*

$$X = A^{\oplus_m,W}B + [I_n - A^{\oplus_m,W}WAW]Y,$$

where $Y \in \mathbb{C}^{n\times p}$ is arbitrary.

(2) *If X is the solution of Eq. (5.4) and $\mathcal{R}(X) \subseteq \mathcal{R}\big((AW)^D\big)$, then*

$$X = A^{\oplus_m,W}B.$$

Proof. This is obvious by Theorem 5.5 and Corollary 5.6. □

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