A NEW WAY OF LOOKING AT COMPLEX NUMBERS

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Abstract

A new way to teach and to solve problems in complex numbers, as how to use the Moivre Formula, Newton's Binomial and Euler formula for the linearization. We will also share new results like The "Magic Formula" of the second root of any complex number, and the true method to solve the equation of second degree in general case (with complex coefficients) with the explicit formula of the solution and an algorithm to program it.

1. A SIMPLE TABLE:

In this section, we will introduce for the first time a simple table to simplify the calculations using Newton's binomial formula in the Moivre formula or Euler formula for the linearization.

1.1. The Moivre formula:

For all $n \in \mathbb{Z}$, the formula: $(\cos\,\theta + i\sin\,\theta)^n = \cos\,(n\theta) + i\sin\,(n\theta)$ is used in particular to express $\cos\,(n\theta)$ and $\sin\,(n\theta)$ as a function of $\cos\,\theta$ and $\sin\,\theta$.

Example-1:

Write $\cos\,(5x)$ as a function of $\cos\,(x)$ and $\sin\,(x)$

According to the Moivre formula, we have

$$\cos\,(n\theta) + i\sin\,(n\theta) = (\cos\,\theta + i\sin\,\theta)^n$$

For example, for $n = 5$ and $\theta = x$,

$$z = \cos\,(5x) + i\sin\,(5x) = (\cos\,x + i\sin\,x)^5$$

We notice that $\cos\,(5x)$ is exactly the real part of the first member of the equality (and of), so it is enough to expand the 5 th power (i.e. to distribute all) in the second member and extract its real part.

$$\begin{aligned}\cos\,(5x) &= \mathrm{Re}\,[\cos\,(5x) + i\sin\,(5x)] \\ &= \mathrm{Re}\,[(\cos\,x + i\sin\,x)^5]\end{aligned}$$

Remark 1. It is the same for the general case, we always need to develop the power n in $(\cos\,\theta + i\sin\,\theta)^n$ to extract the real part, hence the need for another very important formula called Newton's Binomial:

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^k b^{n-k}$$

With $C_n^k = \frac{n!}{k!(n-k)!}$, $n! = n \times (n-1)! = n \times (n-1) \times \dots \times 2 \times 1$

By convention we take $0! = 1$

$C_n^n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!0!} = 1$,

$C_n^0 = C_n^n$,

$C_n^1 = \dfrac{n!}{1!\,(n-1)!} = C_n^{n-1} = n$

In the general case:

$$C_n^{n-k} = C_n^k$$

Claim 1. To facilitate product calculations, powers and a summation should be done at the end, so, we draw up the following table:

Table-1: (the Simple table)

k	C_n^k	a^k	b^{n-k}	$\prod_{b^n}$
0	1	1	b^n	nab^{n-1}
1	n	a	b^{n-1}	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
k	$*$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\downarrow$	$\uparrow$	$\vdots$
$n-k$	$*$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$n-1$	n	a^{n-1}	b	$na^{n-1}b$
n	1	a^n	1	a^n
				result $=\sum$

Therefore, to calculate

$$(\cos x + i\sin x)^5 = \sum_{k=0}^{5} C_5^k \cos^k(x)(i\sin x)^{5-k}$$

we put the following table:

k	C_5^k	$\cos^k(x)$	$(i\sin x)^{5-k}$	$\prod$
0	1	1	$i\sin^5(x)$	$i\sin^5(x)$
1	5	$\cos x$	$\sin^4(x)$	$5\cos x\sin^4(x)$
2	10	$\cos^2(x)$	$-i\sin^3(x)$	$-10i\cos^2(x)\sin^3(x)$
3	10	$\cos^3(x)$	$-\sin^2(x)$	$-10\cos^3(x)\sin^2(x)$
4	5	$\cos^4(x)$	$i\sin x$	$5i\cos^4(x)\sin x$
5	1	$\cos^5(x)$	1	$\cos^5(x)$
				result $=\sum$

Consequently,

$$\begin{aligned}(\cos x + i\sin x)^5 = {} & i\sin^5(x) + 5\cos x\sin^4(x) + \\ & -10i\cos^2(x)\sin^3(x) + \\ & -10\cos^3(x)\sin^2(x) + \\ & +5i\cos^4(x)\sin x + \cos^5(x) \\ (\cos x + i\sin x)^5 = {} & [\cos^5(x) + 5\cos x\sin^4(x) + \\ & -10\cos^3(x)\sin^2(x)] + \\ & +i[\sin^5(x) + 5\cos^4(x)\sin x + \\ & -10\cos^2(x)\sin^3(x)] \\ = {} & \mathrm{Re}(z) + i\mathrm{Im}(z)\end{aligned}$$

Conclusion:

$$\cos(5x) = \cos^5(x) + 5\cos x\sin^4(x) - 10\cos^3(x)\sin^2(x)$$

2.1. Linearization:

It is about writing the product $\cos^p(\theta)\sin^q(\theta)$, with $p \in \mathbb{N}$ and $q \in \mathbb{N}$, as a sum of terms of the form $\cos(n\theta)$ or $\sin(m\theta)$, with $n \in \mathbb{N}$ and $m \in \mathbb{N}$.

We replace $\cos$ and sin using Euler's formulas:

Example-2:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

Linearize $\cos^3(x)\sin^2(x)$.

We have:

$$\begin{cases} \cos x & = \dfrac{e^{ix} + e^{-ix}}{2} \\ \sin\theta & = \dfrac{e^{ix} - e^{-i}}{2i} \end{cases}$$

So

$$\begin{aligned} \cos^3(x)\sin^2(x) & = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 \left(\frac{e^{ix} - e^{-i}}{2i}\right)^2 \\ & = -\frac{1}{32}\left(e^{ix} + e^{-ix}\right)^3\left(e^{ix} - e^{-ix}\right)^2 \end{aligned}$$

Although,

$$\begin{aligned} \left(e^{ix} - e^{-ix}\right)^2 & = \left(e^{ix}\right)^2 - 2e^{ix}e^{-ix} + \left(e^{-ix}\right)^2 \\ & = e^{i2x} - 2 + e^{-i2x} \end{aligned}$$

and

$$\left(e^{ix} + e^{-ix}\right)^3 = \sum_{k=0}^{3} C_3^k \left(e^{ix}\right)^k \left(e^{-i}\ \right)^{3-k}$$

Therefore, as we notice, we have to apply Newton's Binomial formula again and draw up "the Simple table" like we did for example-1.

k	C_3^k	$\left(e^{ix}\right)^k$	$\left(e^{-ix}\right)^{3-k}$	Π
0	1	1	e^{-i3x}	e^{-i3x}
1	3	e^{ix}	e^{-i2x}	$3e^{-ix}$
2	3	e^{i2x}	e^{-ix}	$3e^{ix}$
3	1	e^{i3x}	1	e^{i3x}
				$\sum(\Pi)$

Thus

$$\left(e^{ix} + e^{-ix}\right)^3 = e^{-i3x} + 3e^{-i}\ + 3e^{ix} + e^{i3x}$$

Consequently,

$$\cos^3(x)\sin^2(x) = -\frac{1}{32}\left(e^{-i3x} + 3e^{-ix} + 3e^{ix} + e^{i3x}\right) \times$$
$$\times\left(e^{i2x} - 2 + e^{-i2x}\right)$$
$$= -\frac{1}{32}\left(e^{-ix} - 2e^{-i3x} + e^{-i5x} + 3e^{ix} - 6e^{-ix} + 3e^{-i3x}\right) +$$
$$-\frac{1}{32}\left(3e^{i3x} - 6e^{ix} + 3e^{-ix} + e^{i5x} - 2e^{i3x} + e^{ix}\right)$$
$$= -\frac{1}{32}\left(e^{-ix} - 6e^{-ix} + 3e^{-ix} + 3e^{ix} - 6e^{ix} + e^{ix} - 2e^{-i3x}\right) +$$
$$-\frac{1}{32}\left(3e^{-i3x} + 3e^{i3x} - 2e^{i3x} + e^{i5x} + e^{-i5x}\right)$$
$$= -\frac{1}{32}\left(-2e^{-ix} + -2e^{ix} + e^{-i3x} + e^{i3x} + e^{i5x} + e^{-i5x}\right)$$
$$= -\frac{1}{32}\left(-4\left(\frac{e^{-ix} + e^{ix}}{2}\right) + 2\left(\frac{e^{-i3x} + e^{i3x}}{2}\right) + 2\left(\frac{e^{i5x} + e^{-i5x}}{2}\right)\right)$$
$$= -\frac{1}{32}\left(-4\cos x + 2\cos(3x) + 2\cos(5x)\right)$$

Conclusion:

$$\cos^3(x)\sin^2(x) = \frac{1}{8}\cos x - \frac{1}{16}\cos(3x) - \frac{1}{16}\cos(5x)$$

2. The Magic Formula of ARBAI:

We suppose that $z \in \mathbb{C} \setminus \mathbb{R}$ and we cannot write z explicitly in exponential or geometric form. Hence, $\exists(a,b) \in \mathbb{R}/z = a + ib$ (which means z is a complex not real with $b \neq 0$).

Thereafter, we write $z^{\frac{1}{2}}$ in the algebraic form $z^{\frac{1}{2}} = x + iy$ and we try to find a couple $(x,y) \in \mathbb{R}^2$ such that $(x+iy)^2 = a + ib$.

This step is a bit long, especially with the distressing calculations that repeats every time when practical work, especially when it comes to teach students how to find $z^{\frac{1}{2}}$ by hand, mostly when we are solving an equation of second degree with complex coefficients or a second-degree differential equations.

This article aims to find finally "the Magic Formula of Arbai" $z^{\frac{1}{2}}$ in function of z (and so $\Delta^{\frac{1}{2}}$ in function of Δ) and therefore result in a direct algorithm, exact and very fast for us to determine the solutions of equation of second degree with complex coefficients.

2.1. The second root:

2.1.1. $If\ z \in \mathbb{R}$:

$$z^{\frac{1}{2}} = \pm i^{\frac{1-\mathrm{sig}\ (z)1}{2}}\sqrt{|z|}$$

Where:

If $z \geq 0$ then $\mathbf{sign}\,(z) = +$ and $|z| = z$

If $z \leq 0$ then $\mathbf{sign}\,(z) = -$ and $|z| = -z$

2.1.2. $If\ z \notin \mathbb{R}$: which means that $z \in \mathbb{C} \setminus \mathbb{R}$

$$z^{\frac{1}{2}} = \pm \frac{\frac{|z| + z}{2}}{\sqrt{\frac{|z| + Re(z)}{2}}}$$

See [1], [2]

Where

$|z| = \sqrt{\mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2}$ (z 's modulus) or $|z| = \sqrt{z\bar{z}}$

$Re(z)$ (The real part of z)

$\mathrm{Im}\,(z)$ (The imaginary part of z).

Proof.

$$\left(\pm \frac{\frac{|z| + z}{2}}{\sqrt{\frac{|z| + \mathrm{Re}\,(z)}{2}}} \right)^2 = \frac{|z|^2 + 2|z|z + z^2}{2(|z| + \mathrm{Re}\,(z))} = \frac{z\bar{z} + 2|z|z + z^2}{2(|z| + \mathrm{Re}\,(z))}$$
$$= \frac{z}{2}\left(\frac{\bar{z} + 2|z| + z}{|z| + \mathrm{Re}\,(z)}\right) = \frac{z}{2}\left(\frac{2|z| + 2\mathrm{Re}\,(z)}{|z| + \mathrm{Re}\,(z)}\right)$$
$$= z$$

Because $\bar{z} + z = 2\mathrm{Re}\,(z)$ and $z\bar{z} = |z|^2$.

Claim2.

$$\text{If } z \in \mathbb{R} \text{ then } z^{\frac{1}{2}} = \pm i^{\frac{1-\mathrm{sig}\ \ (z)1}{2}}\sqrt{|z|}$$
$$\text{If } z \notin \mathbb{R} \text{ then } z^{\frac{1}{2}} = \pm \frac{\frac{|z| + z}{2}}{\sqrt{\frac{|z| + \mathrm{Re}\,(z)}{2}}}$$

2.2. A new way to solve an equation of second degree:

Let the equation:

$$az^2 + bz + c = 0$$
$$a \neq 0 \Longrightarrow z^2 + \frac{b}{a}z + \frac{c}{a} = 0$$
$$\Leftrightarrow z^2 + 2\frac{b}{2a}z + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$
$$\Leftrightarrow \left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Let $\Delta \in \mathbb{C}$ such that $\Delta = b^2 - 4ac$.

$$\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} = \left(\frac{\Delta^{\frac{1}{2}}}{2a}\right)^2$$

With $\Delta^{\frac{1}{2}}$ a second root of Δ (in $\mathbb{C}^*$ we have always two second root $\pm\Delta^{\frac{1}{2}}$).

So

$$z = \frac{-b \pm \Delta^{\frac{1}{2}}}{2a}$$

We have two cases:

(1) First case:

If $\Delta \in \mathbb{R}$

$$\Delta^{\frac{1}{2}} = \pm i^{\frac{1-\text{sign}\,(\Delta)1}{2}}\sqrt{|\Delta|}$$

Therefore, we can generalize:

$$z = \frac{-b \pm i^{\frac{1-\text{sig}\quad(\Delta)1}{2}}\sqrt{|\Delta|}}{2a} \text{ if } \Delta \in \mathbb{R}$$

Where:

If $\Delta \geq 0$ then $\mathbf{sign}\,(\Delta) = +$ and $|\Delta| = \Delta$

If $\Delta \leq 0$ then $\mathbf{sign}\,(\Delta) = -$ and $|\Delta| = -\Delta$

(2) Second case:

If $\Delta \notin \mathbb{R}$ (i.e. $\Delta \in \mathbb{C} \setminus \mathbb{R}$)

$$\Delta^{\frac{1}{2}} = \pm \frac{\frac{\parallel \Delta \parallel + \Delta}{2}}{\sqrt{\frac{\parallel \Delta \parallel + Re(\Delta)}{2}}} \text{ and } z = \frac{-b \pm \Delta^{\frac{1}{2}}}{2a}$$

So

$$z = \frac{-b \pm \frac{\frac{\parallel \Delta \parallel + \Delta}{2}}{\sqrt{\frac{\parallel \Delta \parallel + \mathrm{Re}\,(\Delta)}{2}}}}{2a}$$

With $\parallel \Delta \parallel = \sqrt{\mathrm{Re}\,(\Delta)^2 + \mathrm{Im}\,(\Delta)^2}$($\Delta$'s modulus) or $\parallel \Delta \parallel = \sqrt{\Delta\bar{\Delta}}$

$\mathbf{Re}\,(\Delta)$ (The real part of Δ)

$\mathbf{Im}\,(\Delta)$ (The imaginary part of Δ).

Claim 3.

First case: If $\Delta \in \mathbb{R}$ then $z = \dfrac{-b \pm i^{\frac{\frac{1-sign(\Delta)1}{2}}{2a}}\sqrt{|\Delta|}}{2a}$

Second case: If $\Delta \notin \mathbb{R}$ then $z = \dfrac{-b \pm \frac{\frac{\parallel \Delta \parallel + \Delta}{2}}{\sqrt{\frac{\parallel \Delta \parallel + Re(\Delta)}{2}}}}{2a}$

2.3. Algorithm: $\rightarrow \boldsymbol{a, b, c}$

$\rightarrow \Delta = b^2 - 4ac$

$\rightarrow$ If $\Delta \in \mathbb{R}$ So $z = \frac{-b \pm i^{\frac{\frac{1-sign(\Delta)1}{2}}{2a}}\sqrt{|\Delta|}}{2a}$ end

→ Else, $\rightarrow \delta = \frac{\frac{\|\Delta\|+\Delta}{2}}{\sqrt{\frac{\|\Delta\|+Re(\Delta)}{2}}}$

$\rightarrow z = \frac{-b\pm\delta}{2a}$ end

Example 1:

Calculate all second root of $z = 21 + 20i$:

It is impossible to write it explicitly in exponential or geometric form.

By applying our formula:

We have

$$z^{\frac{1}{2}} = \pm \frac{\frac{|z| + z}{2}}{\sqrt{\frac{|z| + \mathrm{Re}\,(z)}{2}}}$$

$|z| = \sqrt{(21)^2 + (20)^2} = \sqrt{441 + 400} = \sqrt{841}$

Then

$|z| = 29$

Therefore,

Then, $z^{\frac{1}{2}} = \pm \frac{\frac{29+21+20i}{2}}{\sqrt{\frac{29+21}{2}}} = \pm \frac{25+10i}{\sqrt{25}}$

$z^{\frac{1}{2}} = \pm(5 + 2i)$

Example 2:

Solving in $\mathbb{C}$ the equation:

$$z^2 - (2 + i)z + 2i = 0$$

We have

$$\Delta = (2 + i)^2 - 4 \times 2i = 4 - 1 + 4i - 8i$$

$\Longrightarrow$

$$\Delta = 3 - 4i$$

$$\Delta \in \mathbb{C} \setminus \mathbb{R} \Longrightarrow \Delta^{\frac{1}{2}} = \pm \frac{\frac{\| \Delta \| + \Delta}{2}}{\sqrt{\frac{\| \Delta \| + Re(\Delta)}{2}}} \text{ and } z = \frac{-b \pm \Delta^{\frac{1}{2}}}{2a}$$

$$\| \Delta \| = \sqrt{9 + 16}$$

$$\| \Delta \| = 5$$

So

$$\Delta^{\frac{1}{2}} = \pm \frac{\frac{\| \Delta \| + \Delta}{2}}{\sqrt{\frac{\| \Delta \| + \mathrm{Re}\,(\Delta)}{2}}} = \pm \frac{\frac{5 + 3 - 4i}{2}}{\sqrt{\frac{5 + 3}{2}}}$$

$$\Delta^{\frac{1}{2}} = \pm \frac{4 - 2i}{2} = \pm(2 - i)$$

$$z = \frac{(2 + i) \pm (2 - i)}{2}$$

$$S = \{i, 2\}$$

We can verify that

$$(z - i)(z - 2) = z^2 - (2 + i)z + 2i.$$

3. CONCLUSION:

Ultimately,

(1) To facilitate the calculations when applying Moivre formula, or Euler formula for Linearization using Newton's binomial and without involving Pascal's triangle formula (because there are at the same time products, powers and a summation to be done at the end), we draw up the table-1 (the Simple table).

(2) We give the general formula that gives us the second roots (square root) of a complex number.

So, let $\alpha \in \mathbb{C} \setminus \mathbb{R}$,

Then

$$X^2 = \alpha \Leftrightarrow X = \pm \frac{\frac{\| \alpha \| + \alpha}{2}}{\sqrt{\frac{\| \alpha \| + \mathrm{Re}\,(\alpha)}{2}}}$$

Or

$$\alpha^{\frac{1}{2}} = \pm \frac{\frac{\| \alpha \| + \alpha}{2}}{\sqrt{\frac{\| \alpha \| + \mathrm{Re}\,(\alpha)}{2}}}$$

(3) We have a new method and new way to resolve an equation of second degree with complex coefficients.

There are two cases:

$$\Delta \in \mathbb{R} \left(z = \frac{-b \pm i \frac{\frac{1 - \mathrm{sign}(\Delta)\, 1}{2}}{2a} \sqrt{|\Delta|}}{2a} \right) \text{ or } \mathbf{not} \left(z = \frac{-b \pm \frac{\frac{\| \Delta \| + \Delta}{2}}{\sqrt{\frac{\| \Delta \| + Re(\Delta)}{2}}}}{2a} \right)$$

with $\Delta = b^2 - 4ac.$

Therefore, we gave all the fast and efficient methods and formulas to solve an equation of second degree with complex coefficients.

References:

[1] A. Arbai: Principes d'algèbre et de Géométrie - 2ème édition 37 - 40 (2015).

[2] A. Arbai: A method to solve the equations of the second degree in the general case (with complex coefficients) - International Journal of Scientific Engineering and Applied Science Volume-2, Issue-12 (2016).