A Modified Born-Infeld Model of Electrons Offering a Classical Analog to Heisenberg's Uncertainty Principle

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Abstract

Recently, the total momentum of a numerical field solution of a modified Born-Infeld model of electrons was found to rotate instead of being conserved as expected based on Noether's first theorem and Gauss's theorem. This work offers an explanation of this rotation, which is consistent with these theorems. Furthermore, the rotating momentum is interpreted as a classical analog to an electron's quantum-mechanical momentum, which is uncertain due to Heisenberg's uncertainty principle.

1 Introduction

Last year, a numerical field solution of a modified Born-Infeld field theory was presented [Kra23a]. More recently, the stress-energy tensor of this field solution was computed [Kra24]. Integrating specific elements of this tensor over all of space should provide the total momentum of the field solution. When it was numerically evaluated, it was found to rotate in synchrony with the rotating field solution. This result is consistent with a total intrinsic angular momentum that agrees with the spin of electrons, and conservation of the time average of momentum (instead of momentum itself) is consistent with quantum mechanics. However, the fact that the total momentum of a field solution is not conserved in a classical field theory with a Lagrangian density that does not explicitly depend on space coordinates appears to contradict Noether's first theorem. The primary purpose of this work is to resolve this apparent contradiction. Once this is achieved, the rotating momentum of the mentioned field solution may be considered a classical analog to the momentum of a quantum-mechanical electron that is uncertain due to Heisenberg's uncertainty principle.

Section 2 summarizes the mentioned modified Born-Infeld model of electrons and previous results for the stress-energy tensor of its field solution. Section 3 explains why the numerical result of a rotating momentum neither contradicts Noether's first theorem nor Gauss's theorem if there is a continuous exchange of momentum with a "momentum reservoir" that is not covered by the numerical simulation. Section 4 presents other, macroscopic systems with similar features. More importantly, it discusses the (quantitative) relation of the mentioned rotating momentum to Heisenberg's uncertainty principle. Section 5 concludes this work.

2 Previous Work

2.1 Modified Born-Infeld Model of Electrons

As in previous work [Kra24], the Lagrangian density $\mathscr{L}$ of the modified Born-Infeld field theory is defined in SI units as

$$\mathscr{L} \stackrel{\text{def}}{=} \frac{b^2}{\mu_0}\left(1-\sqrt{1-\frac{1}{b^2}(\partial^\mu A^\nu)(\partial_\mu A_\nu)}\right) \tag{1}$$

with the Born-Infeld parameter b specifying the maximum magnetic field strength, the vacuum permeability μ_0, and the electromagnetic four-potential $(A^0, A^1, A^2, A^3) = (\phi/c, A_x, A_y, A_z)$. Note that this work uses basic Ricci calculus including Einstein summation convention as well as the Minkowski metric tensor η in the form $\operatorname{diag}(+1, -1, -1, -1)$. More details about the notation are provided in previous work [Kra23b].

The corresponding Euler-Lagrange equations were solved numerically in previous work [Kra23a] resulting in a rotating field solution with a peak moving at the speed of light on a circular orbit with a radius equal to an electron's reduced Compton wavelength. While most features of electrons (electric charge, magnetic moment, Compton frequency) were imposed on the solution, the total field energy of the solution was matched to the rest mass energy of an electron by adjusting the Born-Infeld parameter [Kra24].

2.2 Stress-Energy Tensor

As in previous work [Kra24], the canonical stress-energy tensor $T^{\mu\nu}$ is defined as

$$T^{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial \mathscr{L}}{\partial(\partial_\mu A_\gamma)}(\partial^\nu A_\gamma) - \delta^{\mu\nu}\mathscr{L} \tag{2}$$

such that it satisfies the four continuity equations

$$\partial_\mu T^{\mu\nu} = 0, \tag{3}$$

which may be proven using Noether's first theorem for Lagrangian densities like $\mathscr{L}$ that do not explicitly depend on spacetime coordinates and, in particular, include no external sources of the field. Inserting the definition of the Lagrangian density $\mathscr{L}$ into the definition of $T^{\mu\nu}$ leads to this (manifestly symmetric) expression:

$$T^{\mu\nu} = \frac{b^2}{\mu_0}\left(\frac{\frac{1}{b^2}(\partial^\mu A^\gamma)(\partial^\nu A_\gamma) + \delta^{\mu\nu}\left(1 - \frac{1}{b^2}(\partial^\alpha A^\beta)(\partial_\alpha A_\beta)\right)}{\sqrt{1 - \frac{1}{b^2}(\partial^\alpha A^\beta)(\partial_\alpha A_\beta)}} - \delta^{\mu\nu}\right). \tag{4}$$

A (straightforward) method to numerically evaluate this expression at a specific point in spacetime has been described previously [Kra24].

Due to the mentioned continuity equations $\partial_\mu T^{\mu\nu} = 0$ and Gauss's theorem, integrals of T^{00}, T^{01}, T^{02}, T^{03}, T^{10}, T^{20}, and T^{30} over all of space at a constant time usually stay constant for all times. As Misner et al. [MTW73, page 145, Equation (5.30)] state:

$$\begin{aligned}\begin{pmatrix}\text{total 4-momentum in}\\ \text{all of space at time } t_1\end{pmatrix} &= \int_{\mathcal{S}_1} T^{\alpha 0}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ &= \begin{pmatrix}\text{total 4-momentum in}\\ \text{all of space at time } t_2\end{pmatrix} = \int_{\mathcal{S}_2} T^{\alpha 0}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\end{aligned}$$

for two spacetime slices $\mathcal{S}_1$ and $\mathcal{S}_2$ taken at constant times t_1 and t_2 of a specific Lorentz frame. Apart from the continuity equations, the main requirement is that "the stress-energy tensor dies out rapidly enough [at the timelike surfaces at 'infinity' that join the two slices together]" [MTW73, page 144].

The supposed conservation of total 4-momentum motivates the definition of total energy ΔE and total momentum $\mathbf{p}$ [Kra24]:

$$\Delta E \stackrel{\text{def}}{=} \int T^{00}\mathrm{d}^3\mathbf{x}, \tag{5}$$

$$\mathbf{p} \stackrel{\text{def}}{=} \frac{1}{c}\left(\int T^{01}\mathrm{d}^3\mathbf{x}, \int T^{02}\mathrm{d}^3\mathbf{x}, \int T^{03}\mathrm{d}^3\mathbf{x}\right). \tag{6}$$

Numerically evaluating ΔE and $\mathbf{p}$ for the rotating field solution representing an electron resulted in a conserved total energy ΔE and a rotating vector $\mathbf{p}$ pointing in the direction of the velocity of the peak of the rotating field solution with an absolute value of approximately 1.4×10^{-22} kg m/s, which is close to $m_0 c/2$ for the rest mass m_0 of electrons [Kra24]. Since $\mathbf{p}$ is rotating in synchrony with the rotating field solution, its time average approximates $\mathbf{0}$. However, the rotation of $\mathbf{p}$ stands in apparent contradiction to the supposed conservation of the total momentum $\mathbf{p}$. Resolving this contradiction is the purpose of the next section.

3 Explanation for Rotation of Momentum

While total energy ΔE is conserved as expected, the rotation of total momentum $\mathbf{p}$ is unexpected as discussed in Section 2.2 since $\mathbf{p}$ is constructed (using Noether's first theorem and Gauss's theorem) such that it is conserved for any Lagrangian density like $\mathscr{L}$ that does not explicitly depend on space coordinates.

The explanation proposed here is that *(at least) some of the stress components T^{ij} with $1 \leq i \leq 3$ and $1 \leq j \leq 3$ do not approach 0 fast enough for large distances such that integrals of them over timelike surface "at infinity" do not converge to 0.* Therefore, the continuity equations $\partial_\mu T^{\mu j} = 0$ do not necessarily imply that finite volume integrals of T^{0j} (specifically numerical approximations of components of $\mathbf{p}$) are conserved as would usually be the case (see Section 2.2).

Unfortunately, the existing numerical simulation [Kra24] does not provide any information about the field solution or T^{ij} at large distances. Thus, the explanation proposed here is based on these observations:

1. Many other potential explanations were checked and rejected; in particular because the numerical simulation provides plausible values for ΔE, $\mathbf{p}$, and the total intrinsic angular momentum [Kra24]. Thus, it appears appropriate to question the assumptions required for the conservation of total momentum.

2. Total energy is conserved, which suggests that the energy-flux/momentum-density components T^{10}, T^{20}, T^{30}, T^{01}, T^{02}, and T^{03} approach 0 fast enough for large distances. Thus, there should be a common feature of these components that sets them apart from the stress components T^{ij} with $1 \leq i \leq 3$ and $1 \leq j \leq 3$. In fact, at large distances $T^{\mu\nu}$ may be approximated by $\frac{1}{\mu_0}(\partial^\mu A^\gamma)(\partial^\nu A_\gamma)$, which includes exactly one time-derivative of A^γ for T^{10}, T^{20}, T^{30}, T^{01}, T^{02}, and T^{03} but none for T^{ij} with $1 \leq i \leq 3$ and $1 \leq j \leq 3$. Time-derivatives of A^γ that approach 0 fast enough for large distances could, therefore, explain why total energy is conserved while total momentum is not conserved.

3. One qualitative interpretation of the rotation of the field solution is a periodical deformation of the field with the peak of the field being pulled onto a circular orbit by the field equations [Kra23b]. The assumed behavior of T^{ij} at large distances suggests that the stress (i.e., momentum flux) that comes with this deformation is not limited to a small volume (possibly not even to a finite volume), while the energy flux/momentum density is limited to a very small volume. As Section 4.1 shows, mechanical systems that combine limited energy transfer with momentum transfer over large distances appear to be quite common in everyday physics on Earth.

Of course, a proof of the assumed behavior of T^{ij} at large distances would considerably strengthen the proposed explanation. Without such proof and without evidence to the contrary, the presented explanation is at least a plausible working hypothesis.

4 Discussion

4.1 Newtonian Examples

Section 3 presents an interpretation of the numerically simulated near-field of a rotating field solution as a subsystem that continuously exchanges momentum (but not energy) with the far-field (outside the numerical simulation) of the same field solution such that all continuity equations $\partial_\mu T^{\mu\nu} = 0$ hold but the momentum of the near-field is not conserved. Since this exchange of momentum over large distances without exchange of energy might appear rather counter-intuitive, this section presents some examples in Newtonian mechanics that illustrate how common this kind of momentum transfer is in everyday physics on Earth.

The first example is a single artificial satellite orbiting Earth as it is bound to Earth by Newton's gravitational force. For simplicity, friction forces and gravitational forces caused by other bodies are ignored. As the satellite is orbiting Earth, its momentum is periodically changing (in any inertial system). According to Newtonian mechanics, this changing momentum of the satellite is consistent with conservation of total momentum because momentum is exchanged between the satellite and Earth such that both are orbiting the center of their combined masses and the combined total momentum of satellite and Earth is conserved. In other words, regardless of how light and small the satellite is and regardless of how heavy and large Earth is, if the satellite is orbiting Earth, Earth is assumed to be continuously accelerated such that it orbits a common center of mass with the satellite. It is noteworthy that for a circular orbit the continuous exchange of momentum occurs without exchange of energy, and that the periodic change of the satellite's momentum is easily measured, but (except for very heavy satellites like the Moon) the periodic change of Earth's momentum is too small to be measured directly. Readers are encouraged to consider how counter-intuitive this explanation is.

Yet, the second example might be even less intuitive: Consider any falling body of constant mass on the surface of Earth. Again, friction is ignored—or eliminated by letting the body fall in an artificial vacuum. If the falling body is accelerated towards Earth, its momentum is changing. Newtonian mechanics explains this change of momentum by an equal but opposite change of Earth's momentum such that Earth is accelerated towards the falling body and the combined total momentum is conserved. Again, the change of Earth's momentum is usually too small to be measured directly, but it is required for conservation of total momentum. Thus, any tiny sand grain that accelerates while falling downwards accelerates Earth in the opposite direction.

There are many more examples of small mechanical systems that exchange momentum with Earth, which serves as a large reservoir of momentum for these systems. To name just one more example: a small ball (e.g., a marble) rolling (without friction) in circles inside a plate, bowl, or cup that is fixed to Earth by friction. (An interesting feature of this example is that conservation of total momentum may be demonstrated by removing the friction between Earth and the plate, bowl, or cup.)

These examples show that a rigid body of non-zero mass may exchange momentum with a large reservoir (e.g., Earth) over large distances regardless of how small or light the body is, regardless of energy exchange, and regardless of whether the change of momentum of the reservoir is directly measurable. The explanation proposed in Section 3 assumes that the peak of the field solution shows analogous features as it continuously exchanges momentum with the outer parts of the field solution.

4.2 Relation to Heisenberg's Uncertainty Principle

While the previous section compared the rotating momentum of the mentioned field solution to Newtonian systems that periodically exchange momentum with a momentum reservoir, this section explores the relation of the rotating momentum to Heisenberg's uncertainty principle. The relation may be expressed quantitatively as follows: The peak of the rotating field solution is orbiting on a circle with radius $\hbar/(m_0 c)$, i.e., an electron's reduced Compton wavelength. If we assume that the peak is somewhere on the circular orbit but that it is unknown where exactly the peak is at any specific time, we could express this limited knowledge as an uncertainty of position Δx equal to the radius of the orbit, i.e., $\Delta x \approx \hbar/(m_0 c)$. Similarly, an uncertainty of momentum could be defined as the absolute value of the rotating momentum (assuming that the whole orbit is at rest), i.e., $\Delta p \approx |\mathbf{p}| \approx m_0 c/2$ (see Section 2.2). The product of the two uncertainties is:

$$\Delta x \, \Delta p \approx \frac{\hbar}{m_0 c} \frac{m_0 c}{2} = \frac{\hbar}{2}, \tag{7}$$

which shows a strong resemblance to Heisenberg's uncertainty principle. This suggests that the feature of a continuously rotating momentum is in fact a classical analog to Heisenberg's uncertainty principle.

While there are several classical analogs to the uncertainty principle, this one might be particularly interesting because it appears in a field solution of a classical field theory, which is constructed as a model of an elementary particle. This might indicate that there is a (yet unknown) classical field theory that is the foundation not only of Heisenberg's uncertainty principle but of all of quantum mechanics. If this is true, quantum mechanics of point particles might be interpreted as an *incomplete description* of the behavior of field solutions of a classical field theory. Here, the term "incomplete description" is used in the same sense that Einstein described in a letter to Max Born in 1950:

> "Take a (macroscopic) body which can rotate freely about an axis. Its state is fully determined by an angle. Let the initial conditions (angle and angular momentum) be defined as precisely as the quantum theory allows. The Schroedinger equation then gives the ψ-function for any subsequent time interval. If this is sufficiently large, all angles become (in practice) equally probable. But if an observation is made (e.g. by flashing a torch), a definite angle is found (with sufficient accuracy). This does not prove that the angle

had a definite value before it was observed – but we believe this to be the case, because we are committed to the requirements of reality on the macroscopic scale. Thus, the ψ-function does not express the real state of affairs perfectly in this case. This is what I call 'incomplete description'." [EBB71]

5 Conclusion

This work set out to explain why the momentum of a previously computed rotating field solution is not conserved. The proposed explanation is based on the assumption of a continuous exchange of momentum between the near-field and the far-field of the field solution. The rotating momentum may also be understood (even quantitatively) as a classical analog to a quantum-mechanical momentum, which is uncertain due to Heisenberg's uncertainty principle. Thus, the rotating momentum is not a flaw of the modified Born-Infeld model of electrons, but a feature that resembles a quantum-mechanical property of electrons.

References

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A Revisions

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