Pi's Irrationality Using Maclaurin Polynomials

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Abstract

After reviewing Maclaurin series and the Alternating Series Estimation Theorem, we show how these can be combined with some arithmetic and algebraic observations to prove that π is irrational.

Introduction

There are many proofs of the irrationality of π [2, 4], but beginning calculus books tend not to use them [6, 9]. Niven's proof [2, 7, 5] is a top contender for inclusion; it is short, but difficult. Even analysis books tend not to use it [8] and, if they do, they don't prove it in the text proper. In Apostol's *Mathematical Analysis* [1] it's relegated to an exercise. Here is a new proof that we claim is easy enough for a calculus course. It seems at the level of e's irrationality proofs that are generally in beginning calculus and analysis books [1, 6, 8, 9].

The hope is to make this article readable by calculus students; we start with a review of the pre-requisites.

Review

We use the Maclaurin series

$$\sin(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}. \tag{1}$$

This is easily derived using the formula for a Taylor series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

The Maclaurin series is just a Taylor series with $c = 0$. The derivatives of $f(x) = \sin(x)$ at 0 are $\sin(0) = 0$, $\cos(0) = 1$, $-\sin(0) = 0$ and $-\cos(0) = -1$. With a little reflection this becomes (1).

To calculate the value of $\sin(x)$ at a particular point, approximations must be used and these give rise to Taylor and Maclaurin polynomials. When a value of x is substituted into (1) it becomes an alternating series and these polynomials become partial sums of this series. Alternating series have a key property we will use: the Alternating Series Estimation Theorem (ASET) [10].

ASET has three parts. They are all implied by oscillations in partial sums; first too much, then too little, but the distance between the two goes to zero. Thus part 1 is $s_{n+1} < L < s_n$ where L is the limit of the series and s_k's are partial sums; part 2 is the absolute value of the error is less than the absolute value of a_{k+1}, the first omitted term of the series approximating partial; and part 3 is the sign of the tail, $L - s_n$ is the same as this first omitted term. There are many youtube animations that show all three parts.

We'll give a quick proof of part 3; we'll need it later. Consider

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{n} a_k + (a_{n+1} + a_{n+2}) + (a_{n+3} + a_{n+4}) + \dots.$$

If the first omitted term, a_{n+1} is negative then, as $|a_n|$ is a descending sequence, $(a_{n+1} + a_{n+2}) < 0$, note a_{n+2} has to be positive; they're alternating. This pattern is maintained for all such pairings, so the tail is negative, thus the same sign as a_{n+1}. Likewise, if a_{n+1} is positive then a_{n+2} is negative and $(a_{n+1} + a_{n+2}) > 0$ and this pattern holds for subsequent pairs; the tail is positive, the same sign as a_{n+1}.

It follows that if r is a root of $\sin(x)$, then all Maclaurin polynomials can't be 0 at r: head(r) + tail(r) = 0; by way of ASET, tail(r) $\neq 0$; implies head(r) $\neq 0$ and head(r) is the partial. We'll need this implication as our particular interest is in the roots of Maclaurin polynomials.

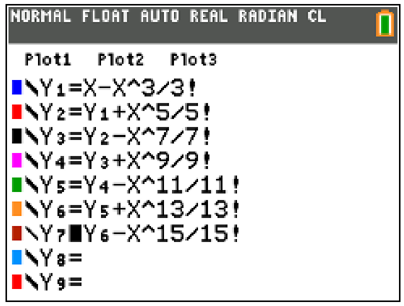

Figure 1: Maclaurin polynomials for $\sin(x)$.

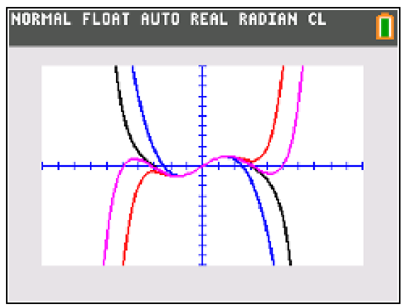

Figure 2: A few Maclaurin polynomials.

Graphs

First, let's get a picture. A TI84-CE calculator can be used to graph Maclaurin polynomials. The first few for our $\sin(x)$ series are given in Figure 1. Figure 2 shows the graph for the first four. The $\sin(x)$ curve is slowly being formed. As the degree of the polynomial grows the number of turning points [3] in the curve increases and the accuracy of the zero estimates of $\sin(x)$ get better. In Figures 3 and 4 we can see that the first non-zero root of Y_6 is close to π and that of Y_7 is closer still: 3.1416138 and 3.1415919.

Per the periodicity of $\sin(x)$, the roots of $\sin(x)$ are of the form $n\pi$ for integer n. The series (1) converges to $\sin(x)$ for all of the reals; an infinite circle of convergence. Thus each additional Maclaurin polynomial crosses the x-axis and gives an additional approximation to the roots (or zeros) of $\sin(x)$. The limit of these polynomial roots are the same as those of $\sin(x)$: $\pm n\pi$.

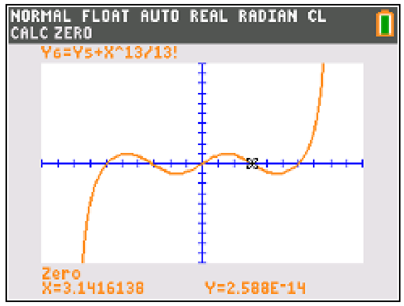

Figure 3: $Y_6(X)$ has a first root of 3.1416138, π with three digit accuracy.

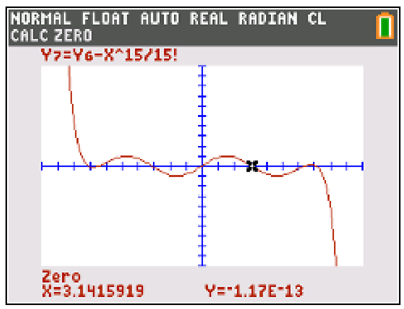

Figure 4: $Y_7(X)$ has a first root of 3.1415919: π with five digit accuracy.

Arithmetic Observations

If $\pi = p/q$ then $k!\pi$ for sufficiently large k is always an integer and equal to a multiple of π. This follows from the arithmetic observations that

$$\frac{p}{q} \cdot \frac{pqk!}{pq}$$

is an integer if k has q^2 factor, i.e. if $k \geq q^2$. But, in turn this means that for sufficiently large k

$$\sin(k!) = 0,$$

if π is assumed to be rational: p/q.

Algebraic Observations

Consider the roots for the first few Maclaurin polynomials for $\sin(k!)$ [6, 9]:

$$T_3(x) = x - \frac{x^3}{3!} \text{ implies } T_3(3!) = 3!(1 - (3!)^2)$$

$$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \text{ implies } T_5(5!) = 5!(1 - \frac{(5!)^2}{3!} + \frac{(5!)^4}{5!})$$

and

$$T_7(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \text{ implies } T_7(7!) = 7!(1 - \frac{(7!)^3}{3!} + \frac{(7!)^5}{5!} - \frac{(7!)^7}{7!}).$$

It is clear that all $T_k(k!)$ are integers. We are now ready to prove π is irrational.

Proof

Theorem 1. *π is irrational.*

Proof. Define the partial series of the Maclaurin expansion of $\sin(x)$ as

$$T_j(x) = \sum_{k=1}^{j} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!}.$$

The sequence of the roots of $T_j(x)$ converges to the roots of $\sin(x)$ as

$$\lim_{j \to \infty} T_j(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} x^{2k-1}}{(2k-1)!} = \sin(x).$$

But if $\pi = p/q$ this implies

$$\lim_{j \to \infty} T_j(j!) = \lim_{j \to \infty} \sin(j!) = 0.$$

Given $\epsilon < 1$ there exists a N such that for all $j > N$

$$T_j(j!)| < \epsilon < 1.$$

But $T_j(j!)$ is an integer polynomial evaluated at an integer. It is not zero because the roots of $T_j(j!)$ are not shared with $\sin(x)$. That's our use of ASET. This forces the existence of a positive integer less than one, a contradiction. □

Remarks

One can come to an understanding of the nature of this proof and of irrational numbers by considering what

$$\lim_{j \to \infty} \hat{T}_j(x) \tag{2}$$

must be. This is a power series with coefficients consisting of sequences that go to infinity. Hard to write down! With an integer x value and a finite j value it must evaluate to an integer. But if x is irrational, say π then

$$\lim_{k \to \infty} A_k \pi - B_k \pi = 0$$

is a possibility, where A_k and B_k are integer sequences going to infinity. The terms $A_k \pi$ and $B_k \pi$ always have infinite decimals and the difference can shrink to 0.

It is likely that (2) can define a function, but it must have a complicated nature. We just need the roots of $T_j(x)$ and $\hat{T}_j(x)$ are the same and as the former converges to roots of $\sin(x)$, so too will the latter.

Conclusion

This proof seems to be easier than the proof by Niven [7]. It does require knowledge of infinite series, a topic later than integration (what Niven's proof uses) in calculus textbooks. But the steps are simpler and not too removed from the level of beginning calculus. It almost seems to be simple algebra in nature. It might make a good application within a section on alternating series in calculus textbooks.

References

[1] Apostol, T. M. (1974). *Mathematical Analysis*, 2nd ed. Reading, Massachusetts: Addison-Wesley.

[2] L. Berggren, J. Borwein, and P. Borwein, *Pi: A Source Book*, 3rd ed., Springer, New York, 2004.

[3] Blitzer, R. (2010). *Algebra and Trignometry*, 3rd ed., Pearson.

[4] P. Eymard and J.-P. Lafon, *The Number* π, American Mathematical Society, Providence, RI, 2004.

[5] T. W. Jones, Discovering and Proving Pis Irrationality, *Monthly*, **53** (2010) 553557.

[6] Larson, R. and Edwards, B.H. (2010). *Calculus*, 9th ed., Belmont, CA: Brooks/Cole.

[7] I. Niven, A simple proof that π is irrational, *Bull. Amer. Math. Soc.* **53** (1947) 509.

[8] Rudin, W. (1976). *Principles of Mathematical Analysis*, 3rd ed. New York: McGraw-Hill.

[9] Thomas, G.B. (1968). *Calculus and Analytic Geometry*, 4th ed., Reading, MA: Addison-Wesley.

[10] Thomas, G.B. and Finney, R.L. (1981). *Elements of Calculus and Analytic Geometry*, Reading, MA: Addison-Wesley.