Simple Calculations of Spin Angular Momentum

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In quantum mechanics, students learn that angular momentum has two parts: intrinsic (or spin), and wave (or orbital) contributions. This separation is analogous to the separation of momentum into two parts when analyzing waves: intrinsic momentum associated with motion of the inertial medium, and wave momentum associated with propagation of energy by the wave. However, spin angular momentum can seem mysterious to students because, unlike the moment of momentum, it is independent of any coordinate origin. This difficulty can be overcome by teaching students the coordinate-independent definition of angular momentum density: the vector field whose curl is equal to twice the intrinsic momentum density. This definition of intrinsic angular momentum density, or spin density, is applicable in both classical and quantum physics. This paper gives specific examples illustrating how spin density describes the angular momentum of rigidly rotating objects. The relationships between spin density, velocity, and angular velocity are similar to the relationships between vector potential, magnetic field, and electric current in magnetostatics. Appreciation of the coordinate-independent description of angular momentum will remove one obstacle to students' understanding of quantum mechanics.

Keywords: angular momentum, Dirac equation, elastic solid, intrinsic momentum, quantum mechanics, spin, wave mechanics

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1. INTRODUCTION

Students are routinely taught that angular momentum ($\mathbf{L}$) is calculated as the "moment of momentum" $\mathbf{L} = \mathbf{r} \times \mathbf{P}$, where $\mathbf{r}$ is a position vector in some coordinate system and $\mathbf{P} = M\mathbf{u}$ is the momentum of an object with mass M and velocity $\mathbf{u}$. For finite-sized objects, the total angular momentum is the integral of an angular momentum density $\text{L} = \mathbf{r} \times \mathbf{p}$, where $\mathbf{p} = \rho\mathbf{u}(\mathbf{r})$ is the momentum density at position $\mathbf{r}$ and ρ is the mass density. This construction interprets angular momentum as a property of the coordinate system, making it unsuitable to be regarded as a fundamental physical quantity.

In contrast, the intrinsic spin angular momentum of elementary particles as defined in relativistic quantum mechanics is independent of coordinates. Belinfante [2] and Rosenfeld [9] independently demonstrated that the symmetric stress-energy tensor of general relativity requires the existence of a quantum mechanical intrinsic momentum density:

$$\mathbf{p} = (1/2)\nabla \times \mathbf{s}\,, \tag{1}$$

where $\mathbf{s}$ is the density of spin angular momentum. This relationship between spin and momentum densities also has applications in classical physics. [1, 3, 6–8] In particular, it was shown that using this relationship to define spin density in an inertial medium such as an elastic solid, the total spin angular momentum is equal to the total moment of momentum computed from $\mathbf{r} \times \mathbf{p}$: [5]

$$\begin{aligned}\int \mathbf{r} \times \frac{1}{2}(\nabla \times \mathbf{s}) d^3r &= \frac{1}{2}\int \left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \mathbf{r}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{r}\right) d^3r \\ = \frac{1}{2}\int &\left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \partial_i(r_i\mathbf{s}) + \mathbf{s}(\nabla\cdot\mathbf{r}) - \mathbf{s}\cdot\nabla\mathbf{r}\right) d^3r \\ = \int \mathbf{s}\, d^3r\,. \end{aligned} \tag{2}$$

The magnitude of spin density is assumed to fall to zero sufficiently rapidly at large distances for the two total derivatives above ($\nabla(\mathbf{r}\cdot\mathbf{s})$ and $\partial_i(r_i\mathbf{s})$) not to contribute to the integral. An exception to this assumption will be discussed in Sec. 2.3.

Equation 2 demonstrates that spin angular momentum is just a coordinate-independent expression for ordinary angular momentum.

The rotational kinetic energy density (k) is given by:

$$k = \frac{1}{2}\mathbf{w}\cdot\mathbf{s}\,, \tag{3}$$

where $\mathbf{w} = (1/2)\nabla \times \mathbf{u}$ is the angular velocity.

The total kinetic energy is:[5]

$$K = \int \frac{1}{2}\mathbf{w}\cdot\mathbf{s}\, d^3r = \int \frac{1}{2}\rho u^2 d^3r \tag{4}$$

This follows from the vector identity:

$$\nabla\cdot(\mathbf{u}\times\mathbf{s}) = \mathbf{s}\cdot(\nabla\times\mathbf{u}) - \mathbf{u}\cdot(\nabla\times\mathbf{s}) \tag{5}$$

since the volume integral of the divergence term can be converted to a surface integral that is assumed to vanish.

According to Eq. 4, spin density ($\mathbf{s}$) is the momentum conjugate to angular velocity for any Lagrangian whose dependence on velocity is only in the kinetic energy term:

$$\frac{\delta}{\delta w_i}\int \frac{1}{2} w_j s_j\, dV = \frac{1}{2}\int \left(\frac{\delta w_j}{\delta w_i} s_j + w_j \frac{\delta s_j}{\delta w_i}\right) dV = \frac{1}{2}s_i + \frac{1}{2}s_i = s_i\,, \tag{6}$$

where integration by parts was used twice to evaluate the second term in the integral.

It was also found that the intrinsic and wave contributions to angular momentum for shear waves in an elastic solid are equivalent to those of relativistic quantum mechanics. [4] Therefore an understanding of classical spin angular momentum is relevant for understanding quantum mechanics as well.

Since $\mathbf{p} = (1/2)\nabla \times \mathbf{s}$, we can apply Stokes' theorem to obtain:

$$\oint \mathbf{s}\cdot d\boldsymbol{\ell} = 2\iint \mathbf{p}\cdot\hat{\mathbf{n}} dS \tag{7}$$

This relationship can be helpful for determining spin density from a known momentum density profile.

Table I shows a clear correspondence between incompressible motion and magnetostatics with magnetic vector potential $\mathbf{A}$, magnetic field $\mathbf{B}$, electric current $\mathbf{J}$, and magnetostatic energy density ε.

TABLE I: Comparison of Magnetostatics and Incompressible Motion

Magnetostatics	**Incompressible Motion**
$\nabla \times \mathbf{A} = \mathbf{B}$	$\nabla \times \mathbf{s} = 2\rho\mathbf{u}$
$\nabla \times \mathbf{B} = \mu_0\mathbf{J}$	$\nabla \times \mathbf{u} = 2\mathbf{w}$
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{u} = 0$
$\varepsilon = \frac{B^2}{2\mu_0}$	$k = \frac{1}{2}\rho u^2$

Therefore an understanding of spin density could also help students understand magnetostatics. An interesting distinction between the two physical phenomena is that while total angular momentum is an important physical quantity, the volume integral of magnetic potential is not known to be a useful concept.

In this paper, we calculate spin angular momentum for three simple physical examples with azimuthal symmetry: (1) a cylinder rotating about its axis, (2) a hollow cylinder rotating about its axis, and (3) a cylinder translating along its axis. Symmetry with respect to rotation about the axis simplifies the mathematical descriptions so that relationships between physical quantities can be easily understood.

2. EXAMPLES

2.1. Rotating Cylinder

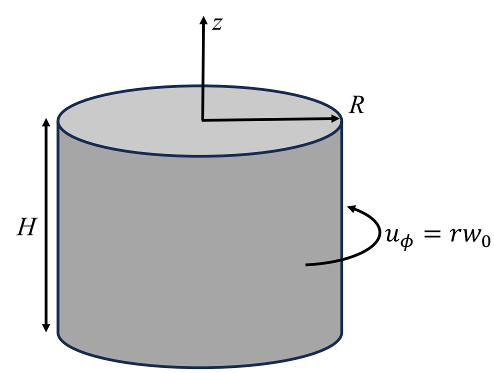

FIG. 1: A Rotating Cylinder.

We will use spin density to describe a cylinder aligned with the z-axis and rotating rigidly with angular velocity w_0 (Fig. 1). The momentum density and angular velocity are given by:

$$p_\phi = \left\{ \begin{array}{cl} \rho r w_0 & \text{for } r \leq R \text{ and } 0 \leq z \leq H \\ 0 & \text{for } r > R \text{ or } z < 0 \text{ or } z > H \end{array} \right\} \tag{8a}$$

$$w_z = \left\{ \begin{array}{cl} w_0 & \text{for } r < R \text{ and } 0 < z < H \\ 0 & \text{for } r > R \text{ or } z < 0 \text{ or } z > H \end{array} \right\} \tag{8b}$$

For $0 \leq z \leq H$ and $r < R$, the differential equation for $\mathbf{s}(\mathbf{r})$ is:

$$\frac{1}{4\rho}\nabla \times (\nabla \times \mathbf{s}) = w_0\hat{\mathbf{z}} \tag{9}$$

Since the right-hand side is the in z-direction, we look for a solution with $s_r = s_\phi = 0$. Azimuthal symmetry implies that the spin density satisfies the equation:

$$-\frac{1}{4\rho r}\partial_r(r\partial_r s_z) = w_0\,. \tag{10}$$

The general solution is:

$$s_z = -\rho w_0 r^2 + c_1 \ln r + c_2 \tag{11}$$

where c_1 and c_2 are arbitrary constants. Requiring a finite value at $r = 0$ implies that $c_1 = 0$, and requiring $s_z \to 0$ for $r \to R$ requires $c_2 = \rho w_0 R^2$. Therefore the solution inside the cylinder is:

$$s_z = \rho w_0 (R^2 - r^2)\,. \tag{12}$$

Outside the cylinder, the equation for s_z is:

$$-\frac{1}{4\rho r}\partial_r(r\partial_r s_z) = 0\,. \tag{13}$$

The solution to this equation is $s_z = c_1 \ln r + c_2$. Requiring $s_z = 0$ at $r \to \infty$ requires $c_1 = c_2 = 0$. The complete solution is therefore:

$$s_z = \left\{ \begin{array}{cl} \rho w_0 (R^2 - r^2) & \text{for } r < R \text{ and } 0 < z < H \\ 0 & \text{for } r > R \text{ or } z < 0 \text{ or } z > H \end{array} \right\} \tag{14}$$

Define the step function $\chi(x)$:

$$\chi(x) = \left\{ \begin{array}{ll} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{array} \right\} \tag{15}$$

Then:

$$s_z = \rho w_0 (R^2 - r^2)\,(1 - \chi(r - R))\,(\chi(z) - \chi(z - H)) \tag{16}$$

The calculated velocity is:

$$u_\phi = -\frac{1}{2\rho}\partial_r s_z = r w_0\,(1 - \chi(r - R))\,(\chi(z) - \chi(z - H)) \tag{17}$$

The calculated components of angular velocity are:

$$w_z = \tfrac{1}{2r}\partial_r(r u_\phi) = w_0 \left(1 - \tfrac{1}{2} R\delta(r - R)\right)(\chi(z) - \chi(z - H)) \tag{18a}$$

$$w_r = -\tfrac{1}{2}\partial_z(u_\phi) = -\tfrac{1}{2} r w_0\,(1 - \chi(r - R))\,(\delta(z) - \delta(z - H)) \tag{18b}$$

Note that this satisfies $\nabla \cdot \mathbf{w} = 0$ everywhere.

Once we established that the spin density is entirely in the z-direction inside the cylinder, we could have applied Stokes' theorem to a rectangular loop with one side along the z-axis and the opposite side at radius r:

$$H((s_z(0) - s_z(r)) = H\left(\rho r^2 w_0\right) \tag{19}$$

Solving for $s_z(r)$ yields:

$$s_z(r) = s_z(0) - \rho r^2 w_0 \tag{20}$$

Since $s_z(R) = 0$, the value on the axis is $s_z(0) = \rho R^2 w_0$ and the value of s_z inside the cylinder is therefore:

$$s_z(r) = \rho w_0 \left(R^2 - r^2\right) \tag{21}$$

Applying step functions to delineate the boundaries yields Eq. 16.

The total angular momentum is:

$$S_z = \int s_z\, d^3r = \rho w_0 2\pi H \int_0^R (R^2 - r^2) r\, dr = \rho w_0 2\pi H \left(\frac{R^4}{4}\right) \tag{22}$$

Identifying the mass as $M = \rho\pi R^2 H$, this is:

$$S_z = \frac{MR^2}{2} w_0 \tag{23}$$

This is the usual expression for angular momentum of a cylinder with moment of inertia $I = MR^2/2$.

Since the angular velocity is constant within the cylinder, the kinetic energy is easily calculated:

$$K = \int \frac{1}{2}\mathbf{w}\cdot\mathbf{s}\, d^3r = \frac{1}{2} w_0 S_z = \frac{1}{2} I w_0^2 \tag{24}$$

2.2. Rotating Hollow Cylinder

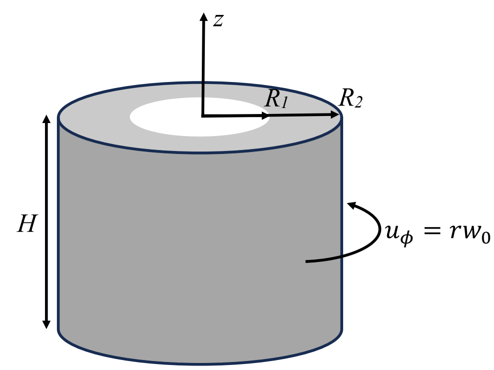

FIG. 2: A Rotating Hollow Cylinder

A rotating hollow (or annular) cylinder, as shown in Fig. 2, can be regarded as the difference between a larger cylinder with radius R_2 and a smaller cylinder with radius R_1 sharing the same rotation axis.

From Eq. 16 this yields:

$$s_z = \rho w_0 \left(\chi(z) - \chi(z-H)\right) \left\{ (R_2^2 - r^2)\left(1 - \chi(r - R_2)\right) - (R_1^2 - r^2)\left(1 - \chi(r - R_1)\right) \right\} \tag{25}$$

Rearranging:

$$s_z = \rho w_0 \left(\chi(z) - \chi(z-H)\right) \left\{ (R_2^2 - R_1^2) + (R_1^2 - r^2)\chi(r - R_1) - (R_2^2 - r^2)\chi(r - R_2) \right\} \tag{26}$$

This means that spin density is constant at $\rho w_0(R_2^2 - R_1^2)$ for $r < R_1$, then becomes $\rho w_0(R_2^2 - r^2)$ for $R_1 \le r \le R_2$, then drops to zero for $r > R_2$. Although it is somewhat counterintuitive to have nonzero spin density in the empty space near the center of the annulus, this profile does yield the correct total angular momentum.

$$\begin{aligned} \int s_z\, d^3r &= \rho w_0 2\pi H \left\{ \int_0^{R_1} (R_2^2 - R_1^2) r\, dr + \int_{R_1}^{R_2} (R_2^2 - r^2) r\, dr \right\} \\ &= \rho w_0 2\pi H \left\{ (R_2^2 - R_1^2)\tfrac{R_1^2}{2} + R_2^2 \left(\tfrac{R_2^2 - R_1^2}{2}\right) - \tfrac{R_2^4 - R_1^4}{4} \right\} \\ &= \rho w_0 2\pi H \left(\tfrac{R_2^4 - R_1^4}{4}\right) \end{aligned} \tag{27}$$

Factoring out the mass $M = \rho\pi(R_2^2 - R_1^2)H$ yields:

$$S_z = \frac{M(R_2^2 + R_1^2)}{2} w_0 \tag{28}$$

This is the usual expression for angular momentum of a hollow cylinder with moment of inertia $I = M(R_2^2 + R_1^2)/2$.
Like the solid cylinder, the angular velocity is constant within the hollow cylinder, so the total kinetic energy is again simply $K = (1/2)Iw_0^2$.

2.3. Translating Cylinder

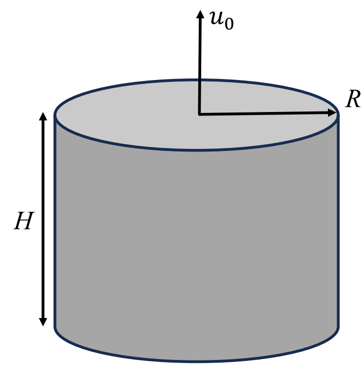

FIG. 3: A Translating Cylinder.

Our final example is a cylinder translating along the direction of its axis as in Fig. 3. In this case $\nabla \cdot \mathbf{p} \neq 0$ at the top and bottom of the cylinder, but we still assume that $\mathbf{p} = (1/2)\nabla \times \mathbf{s}$. The momentum density profile is:

$$p_z = \left\{ \begin{array}{cl} \rho u_0 & \text{for } r \leq R \text{ and } 0 < z < H \\ 0 & \text{for } r > R \text{ or } z < 0 \text{ or } z > H \end{array} \right\} \tag{29}$$

This has an angular velocity profile of:

$$w_\phi(r) = -\frac{1}{2\rho}\partial_r p_z(r,z) = \frac{1}{2}u_0 \delta(r-R)\left(\chi(z) - \chi(z-H)\right) . \tag{30}$$

We can use Stokes' Theorem to find the z-component of spin density. For $r < R$ we have:

$$\oint_0^{2\pi} s_\phi r \, d\phi = \oiint p_z r d\phi dr \tag{31}$$

which yields:

$$s_\phi = \rho u_0 r \left(\chi(z) - \chi(z-H)\right) . \tag{32}$$

For $r > R$ the area integral is constant but the line integral increases with radius, so:

$$s_\phi = \frac{\rho u_0 R^2}{r}\left(\chi(z) - \chi(z-H)\right) . \tag{33}$$

Combining the different regions:

$$s_\phi = \rho u_0 \left(r(1 - \chi(r-R)) + \frac{\rho u_0 R^2}{r}\chi(r-R) \right) \left(\chi(z) - \chi(z-H)\right) . \tag{34}$$

In this case the spin density falls to zero at infinity only as $1/r$, so the volume integral of spin density could depend on where the integration boundary is chosen. If the boundary is chosen to have azimuthal symmetry around the axis of the cylinder, then the spin density integrates to zero:

$$\mathbf{S} = \int_V s_\phi(r,z)\hat{\boldsymbol{\phi}} \, r d\phi \, dr \, dz = 0 \, . \tag{35}$$

In this case there are always equal and opposite contributions from points separated by 180-degree rotation.

However, if the integration boundary is not symmetrical with respect to the cylinder's axis, then the cancellation is incomplete and a net integrated angular momentum would result. Consider a boundary with radius $R_B + x_0 \cos\phi$ where $R_B >> R$ and $x_0 << R_B$. This approximates a displacement from the cylinder axis by $x_0\hat{\mathbf{x}}$. The integral of spin density is:

$$\begin{aligned}\mathbf{S} &= \int_{z=0}^{H}\int_{\phi=0}^{2\pi}\int_{r=R}^{R_B+x_0\cos\phi} \tfrac{\rho u_0 R^2}{r}\left(\hat{\mathbf{y}}\cos\phi - \hat{\mathbf{x}}\sin\phi\right) dz r d\phi dr \\ &= H\int_{\phi=0}^{2\pi}\rho u_0 R^2\left(\hat{\mathbf{y}}\cos\phi - \hat{\mathbf{x}}\sin\phi\right)\left(R_B + x_0\cos\phi - R\right) d\phi \\ &= H\pi\rho u_0 R^2\hat{\mathbf{y}}x_0 = M u_0 x_0\hat{\mathbf{y}}\end{aligned} \tag{36}$$

This is the same result we would have gotten by integrating $(\mathbf{r} - x_0\hat{\mathbf{x}}) \times \mathbf{p}$. In this case, shifting the integrated volume of spin density has the same effect as an opposite shift of the origin for calculating moment of momentum.

This example illustrates the effect of spin density contributions at large distances from the motion (or from integration boundaries). For a rigid solid object, there is no problem limiting integration to the solid region. More generally, the total calculated spin angular momentum varies with the choice of integration boundary, but the density of spin angular momentum is always well-defined and independent of coordinates.

The kinetic energy for the translating cylinder is calculated to be:

$$K = \int_V \frac{1}{2} w_\phi s_\phi d^3r = 2\pi H\int_{r=0}^{R}\frac{1}{4}u_0\delta(r-R)\rho u_0 r^2\, dr = 2\pi H\left(\frac{1}{4}\rho u_0^2 R^2\right) = \frac{1}{2}M u_0^2 \tag{37}$$

3. CONCLUSIONS

These examples demonstrate the role of spin density in describing rotational motion. Unlike the moment of momentum density, spin density is independent of coordinates and can therefore be regarded as a fundamental physical quantity. Since there is a close analogy between the variables of incompressible motion and the variables of magnetostatics, an understanding of spin density will help students to understand the relationships between magnetostatic variables. And since the relationship between spin and momentum densities is the same for both classical and quantum physics, an understanding of spin density will make quantum mechanics somewhat less mysterious for students.

DATA AVAILABILITY STATEMENT

No new data were generated or analyzed in this study.

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