Hyperbolic Quantum Time Evolution Kernel

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Abstract

We find calculational evidence supporting the hypothesis that the time evolution of a relativistic quantum wave function can be written using an integral kernel formulation that uses the wave function's past values on a relativistic hyperbola.

We assume it to be known [1] that if we define an integral kernel as

$$K(\Delta t, x - x') \;=\; \sqrt{\frac{m}{2\pi i\hbar\Delta t}}e^{\frac{im(x-x')^2}{2\hbar\Delta t}},$$

where $\Delta t > 0$ and $x, x' \in \mathbb{R}$, and define a time evolution of a wave function $\psi : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $(t, x) \mapsto \psi(t, x)$ as

$$\psi(t + \Delta t, x) \;=\; \int_{-\infty}^{\infty} K(\Delta t, x - x')\psi(t, x')dx',$$

then the wave function satisfies the non-relativistic Schrödinger equation

$$i\hbar\partial_t\psi(t, x) \;=\; -\frac{\hbar^2}{2m}\partial_x^2\psi(t, x).$$

In this article we study generalizations of this integral kernel that approximate the solutions to the relativistic Schrödinger equation [2]

$$i\hbar\partial_t\psi(t, x) \;=\; \sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\psi(t, x).$$

Since the precise relativistic Schrödinger equation is difficult to study, here we will focus on the approximations

$$i\hbar\partial_t\psi(t, x) \;=\; \Big(mc^2 \;-\; \frac{\hbar^2}{2m}\partial_x^2 \;-\; \frac{\hbar^4}{8m^3c^2}\partial_x^4\Big)\psi(t, x),$$

$$i\hbar\partial_t\psi(t, x) \;=\; \Big(mc^2 \;-\; \frac{\hbar^2}{2m}\partial_x^2 \;-\; \frac{\hbar^4}{8m^3c^2}\partial_x^4 \;-\; \frac{\hbar^6}{16m^5c^4}\partial_x^6\Big)\psi(t, x)$$

and

$$i\hbar\partial_t\psi(t, x) \;=\; \Big(mc^2 \;-\; \frac{\hbar^2}{2m}\partial_x^2 \;-\; \frac{\hbar^4}{8m^3c^2}\partial_x^4 \;-\; \frac{\hbar^6}{16m^5c^4}\partial_x^6 \;-\; \frac{5\hbar^8}{128m^7c^6}\partial_x^8 \;-\; \frac{7\hbar^{10}}{256m^9c^8}\partial_x^{10}\Big)\psi(t, x)$$

only. Here we call these equations the 4th order, the 6th order and the 10th order approximations of the relativistic Schrödinger equation. (The meaning of the word "order" could also be interpreted differently.) These approximations are based on the Taylor series

$$\sqrt{1+z} \;=\; 1 \;+\; \frac{1}{2}z \;+\; \sum_{k=2}^{\infty} \frac{(-1)^{k+1}\cdot(2k-3)!!}{k!\cdot 2^k} z^k.$$

In this article we will use formulas

$$\int_{-\infty}^{\infty} e^{-ax^2+bx}dx \;=\; e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} xe^{-ax^2+bx}dx \;=\; \frac{b}{2a}e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^2e^{-ax^2+bx}dx \;=\; \Big(\frac{b^2}{4a^2}+\frac{1}{2a}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^3e^{-ax^2+bx}dx \;=\; \Big(\frac{b^3}{8a^3}+\frac{3b}{4a^2}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^4e^{-ax^2+bx}dx \;=\; \Big(\frac{b^4}{16a^4}+\frac{3b^2}{4a^3}+\frac{3}{4a^2}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^5e^{-ax^2+bx}dx \;=\; \Big(\frac{b^5}{32a^5}+\frac{5b^3}{8a^4}+\frac{15b}{8a^3}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^6e^{-ax^2+bx}dx \;=\; \Big(\frac{b^6}{64a^6}+\frac{15b^4}{32a^5}+\frac{45b^2}{16a^4}+\frac{15}{8a^3}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^7e^{-ax^2+bx}dx \;=\; \Big(\frac{b^7}{128a^7}+\frac{21b^5}{64a^6}+\frac{105b^3}{32a^5}+\frac{105b}{16a^4}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^8e^{-ax^2+bx}dx \;=\; \Big(\frac{b^8}{256a^8}+\frac{7b^6}{32a^7}+\frac{105b^4}{32a^6}+\frac{105b^2}{8a^5}+\frac{105}{16a^4}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^9e^{-ax^2+bx}dx \;=\; \Big(\frac{b^9}{512a^9}+\frac{9b^7}{64a^8}+\frac{189b^5}{64a^7}+\frac{315b^3}{16a^6}+\frac{945b}{32a^5}\Big)e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^{10} e^{-ax^2+bx} dx = \Big(\frac{b^{10}}{1024a^{10}} + \frac{45b^8}{512a^9} + \frac{315b^6}{128a^8} + \frac{1575b^4}{64a^7} + \frac{4725b^2}{64a^6} + \frac{945}{32a^5}\Big) e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^{11} e^{-ax^2+bx} dx = \Big(\frac{b^{11}}{2048a^{11}} + \frac{55b^9}{1024a^{10}} + \frac{495b^7}{256a^9} + \frac{3465b^5}{128a^8} + \frac{17325b^3}{128a^7} + \frac{10395b}{64a^6}\Big) e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^{12} e^{-ax^2+bx} dx = \Big(\frac{b^{12}}{4096a^{12}} + \frac{33b^{10}}{1024a^{11}} + \frac{1485b^8}{1024a^{10}} + \frac{3465b^6}{128a^9} + \frac{51975b^4}{256a^8} + \frac{31185b^2}{64a^7} + \frac{10395}{64a^6}\Big) e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^{13} e^{-ax^2+bx} dx = \Big(\frac{b^{13}}{8192a^{13}} + \frac{39b^{11}}{2048a^{12}} + \frac{2145b^9}{2048a^{11}} + \frac{6435b^7}{256a^{10}} + \frac{135135b^5}{512a^9} + \frac{135135b^3}{128a^8} + \frac{135135b}{128a^7}\Big) e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^{14} e^{-ax^2+bx} dx = \Big(\frac{b^{14}}{16384a^{14}} + \frac{91b^{12}}{8192a^{13}} + \frac{3003b^{10}}{4096a^{12}} + \frac{45045b^8}{2048a^{11}} + \frac{315315b^6}{1024a^{10}} + \frac{945945b^4}{512a^9} + \frac{945945b^2}{256a^8} + \frac{135135}{128a^7}\Big) e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}},$$

$$\int_{-\infty}^{\infty} x^{15} e^{-ax^2+bx} dx = \Big(\frac{b^{15}}{32768a^{15}} + \frac{105b^{13}}{16384a^{14}} + \frac{4095b^{11}}{8192a^{13}} + \frac{75075b^9}{4096a^{12}} + \frac{675675b^7}{2048a^{11}} + \frac{2837835b^5}{1024a^{10}} + \frac{4729725b^3}{512a^9} + \frac{2027025b}{256a^8}\Big) e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}$$

and

$$\int\limits_{-\infty}^{\infty} x^{16} e^{-ax^2+bx} dx \;=\; \Big(\frac{b^{16}}{65536a^{16}} + \frac{15b^{14}}{4096a^{15}} + \frac{1365b^{12}}{4096a^{14}} + \frac{15015b^{10}}{1024a^{13}} + \frac{675675b^8}{2048a^{12}} + \frac{945945b^6}{256a^{11}} + \frac{4729725b^4}{256a^{10}} + \frac{2027025b^2}{64a^9} + \frac{2027025}{256a^8}\Big) e^{\frac{b^2}{4a}} \sqrt{\frac{\pi}{a}}.$$

We can assume that the reader has already learned the first formula from somewhere [3]. The second formula can be proven with an antiderivative or with an argument using antisymmetricity. The rest of the formulas can be derived by using the recursion formula

$$\int\limits_{-\infty}^{\infty} x^k e^{-ax^2+bx} dx \;=\; \frac{b}{2a} \int\limits_{-\infty}^{\infty} x^{k-1} e^{-ax^2+bx} dx \;+\; \frac{k-1}{2a} \int\limits_{-\infty}^{\infty} x^{k-2} e^{-ax^2+bx} dx$$

that is true for $k \in \{2, 3, 4, \ldots\}$, and that can be derived by using integration by parts. Strictly speaking, apart from the first formula, these formulas require that $\mathrm{Re}(a) > 0$. Otherwise the integrals do not converge. Despite this, we will use these formulas even in situations where $\mathrm{Re}(a) = 0$ and $\mathrm{Im}(a) \neq 0$, because in the context of quantum path integrals these parameter values with these formulas seem to produce sensical results. We can wonder why it is so, but that is not the topic of this article.

In the following calculations we are going to be using a lot of Taylor series with respect to the quantity $\frac{1}{c^2}$. This hopefully looks reasonable, because the speed of light c is usually considered to be large, so consequently $\frac{1}{c^2}$ is then small, and Taylor series with respect to it have a chance of converging nicely. Although factually the quantity $\frac{1}{c^2}$ is a constant, we can think abstractly that this quantity would undergo a limit process $\frac{1}{c^2} \to 0$ in some equations. A closer look at the calculations raises the question that maybe we should also somehow take into account the smallness of Δt that is present in the series. Actually it turns out that the magnitude of Δt complicates the analysis a lot, so to keep things simple, we are going to ignore the magnitude of Δt in the used logic: In the calculations we are not going to assume that Δt would be especially small.

Let's attempt to generalize the non-relativistic time evolution kernel for the 4th order approximation of the relativistic Schrödinger equation. How could we accomplish this? One way to approach this is to first fix some $t \in \mathbb{R}$, and then assume that the wave function has the plane wave form

$$\psi(t', x') \;=\; e^{-\frac{it'}{\hbar}\left(mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}\right)} e^{\frac{i}{\hbar}px'}$$

for the past time values $t' \leq t$ and all $x' \in \mathbb{R}$ with some constant $p \in \mathbb{R}$. According to the 4th order approximation of the relativistic Schrödinger equation the wave function is supposed to maintain this same plane wave form under the time evolution for the future time values above t. This means that we want the kernel $K(\Delta t, x - x')$ to have the property that the equation

$$e^{-\frac{i(t+\Delta t)}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}e^{\frac{i}{\hbar}px}\Big(1+O\Big(\frac{1}{c^4}\Big)\Big)$$
$$= \int\limits_{-\infty}^{\infty} K(\Delta t, x-x')e^{-\frac{it}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}e^{\frac{i}{\hbar}px'}dx'$$

is true. It is best to allow the presence of some error term, because it is difficult to avoid it. If we wanted to be rigorous, we should probably specify how the error term behaves as a function of Δt, but to keep things simple, we ignore this issue. We just assume that the error term behaves as nicely as needed. If this equation and the simplifying assumptions are true for all $p \in \mathbb{R}$, and if the kernel $K(\Delta t, x - x')$ itself does not depend on the parameter p, then because physical wave packets can be written as linear combinations of these type of plane waves, we can then conclude that the kernel will produce approximations of the solutions to the 4th order approximation with arbitrary physical initial conditions. In other words, the kernel will have the property that if

$$\psi(t+\Delta t, x) \;=\; \int\limits_{-\infty}^{\infty} K(\Delta t, x-x')\psi(t,x')dx',$$

then

$$i\hbar\partial_t\psi(t,x) \;\approx\; \Big(mc^2 \;-\; \frac{\hbar^2}{2m}\partial_x^2 \;-\; \frac{\hbar^4}{8m^3c^2}\partial_x^4\Big)\psi(t,x).$$

An obvious ansatz for the kernel is

$$K(\Delta t, x-x') \;=\; a_0 e^{a_1(x-x')^2+a_2(x-x')^4}$$

with some coefficients $a_0, a_1, a_2 \in \mathbb{C}$. We can anticipate that $a_1 \propto \frac{1}{c^0}$ and $a_2 \propto \frac{1}{c^2}$, and use these relations in the approximations. So the equation that we want to be true is

$$e^{-\frac{i(t+\Delta t)}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}e^{\frac{i}{\hbar}px}\Big(1+O\Big(\frac{1}{c^4}\Big)\Big)$$
$$= \int\limits_{-\infty}^{\infty} a_0 e^{a_1(x-x')^2+a_2(x-x')^4}e^{-\frac{it}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}e^{\frac{i}{\hbar}px'}dx'.$$

Some things cancel, and this equation is equivalent with the equation

$$e^{-\frac{i\Delta t}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}\Big(1+O\Big(\frac{1}{c^4}\Big)\Big) \;=\; a_0\int\limits_{-\infty}^{\infty} e^{a_1\xi^2+a_2\xi^4}e^{\frac{i}{\hbar}p\xi}d\xi. \qquad (1)$$

Here we denoted $\xi = x' - x$. The left side of (1) can be written in the form

$$e^{-\frac{i\Delta t mc^2}{\hbar}}e^{-\frac{i\Delta t p^2}{2\hbar m}}\Big(1 \;+\; \frac{i\Delta t p^4}{8\hbar m^3c^2} \;+\; O\Big(\frac{1}{c^4}\Big)\Big).$$

The right side of (1) can be written in the form

$$\begin{aligned} & a_0\int\limits_{-\infty}^{\infty} e^{a_1\xi^2+\frac{i}{\hbar}p\xi}\Big(1+a_2\xi^4+O\Big(\frac{1}{c^4}\Big)\Big)d\xi \\ &= a_0\sqrt{\frac{\pi}{-a_1}}e^{\frac{p^2}{4\hbar^2 a_1}}\Big(1 \;+\; a_2\Big(\frac{p^4}{16\hbar^4 a_1^4}+\frac{3p^2}{4\hbar^2 a_1^3}+\frac{3}{4a_1^2}\Big) \;+\; O\Big(\frac{1}{c^4}\Big)\Big) \\ &= a_0\sqrt{\frac{\pi}{-a_1}}e^{\frac{p^2}{4\hbar^2 a_1}}\Big(1+\frac{3a_2}{4a_1^2}\Big)\Big(1 \;+\; \frac{3a_2p^2}{4\hbar^2 a_1^3} \;+\; \frac{a_2p^4}{16\hbar^4 a_1^4} \;+\; O\Big(\frac{1}{c^4}\Big)\Big). \end{aligned}$$

There is a factor $e^{-\frac{i\Delta t p^2}{2\hbar m}}$ on the left side of (1), and there is a similar factor $e^{\frac{p^2}{4\hbar^2 a_1}}$ on the right side of (1), so there is no other option but to adjust the coefficient a_1 in a such way that these factors become equal. This means that we have to set $a_1 = \frac{im}{2\hbar\Delta t}$. This looks good, because we just rederived the phase factor in the non-relativistic time evolution kernel. There is a term $\frac{i\Delta t p^4}{8\hbar m^3c^2}$ on the left side of (1), and there is a similar term $\frac{a_2p^4}{16\hbar^4 a_1^4}$ on the right side of (1), so there is no other option but to adjust the coefficient a_2 in a such way that these terms become equal. This means that we have to set $a_2 = \frac{im}{8\hbar\Delta t^3c^2}$. There is a term $\frac{3a_2p^2}{4\hbar^2 a_1^3}$ on the right side of (1), but there is no similar term proportional to p^2 on the left side of (1) at all. This means that there is no other option but to set $a_2 = 0$. So it turned out that the coefficient a_2 would have to satisfy two different constraints related to the terms proportional to p^2 and p^4. Since these constraints cannot be satisfied simultaneously, we have to conclude that our ansatz is not working.

Let's see what happens if we attempt to do the same with the 6th order approximation of the relativistic Schrödinger equation. This time an obvious ansatz is

$$K(\Delta t, x-x') \;=\; a_0e^{a_1(x-x')^2+a_2(x-x')^4+a_3(x-x')^6}$$

where $a_1 \propto \frac{1}{c^0}$, $a_2 \propto \frac{1}{c^2}$ and $a_3 \propto \frac{1}{c^4}$. This time the equation that we want to be true is

$$
\begin{aligned}
&e^{-\frac{i(t+\Delta t)}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}\right)}e^{\frac{i}{\hbar}px}\left(1+O\Big(\frac{1}{c^6}\Big)\right)\\
&=\int\limits_{-\infty}^{\infty} a_0 e^{a_1(x-x')^2+a_2(x-x')^4+a_3(x-x')^6}e^{-\frac{it}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}\right)}e^{\frac{i}{\hbar}px'}dx'.
\end{aligned}
$$

This is equivalent with the equation

$$
\begin{aligned}
&e^{-\frac{i\Delta t}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}\right)}\left(1+O\Big(\frac{1}{c^6}\Big)\right)\\
&= a_0\int\limits_{-\infty}^{\infty} e^{a_1\xi^2+a_2\xi^4+a_3\xi^6}e^{\frac{i}{\hbar}p\xi}d\xi.
\end{aligned}
\tag{2}
$$

The left side of (2) can be written in the form

$$
e^{-\frac{i\Delta t mc^2}{\hbar}}e^{-\frac{i\Delta tp^2}{2\hbar m}}\left(1 + \frac{i\Delta tp^4}{8\hbar m^3c^2} - \frac{i\Delta tp^6}{16\hbar m^5c^4} - \frac{\Delta t^2p^8}{128\hbar^2m^6c^4} + O\Big(\frac{1}{c^6}\Big)\right).
$$

The right side of (2) can be written in the form

$$
\begin{aligned}
&a_0\int\limits_{-\infty}^{\infty} e^{a_1\xi^2+\frac{i}{\hbar}p\xi}\left(1+a_2\xi^4+a_3\xi^6+\frac{1}{2}a_2^2\xi^8+O\Big(\frac{1}{c^6}\Big)\right)d\xi\\
&= a_0\sqrt{\frac{\pi}{-a_1}}e^{\frac{p^2}{4\hbar^2a_1}}\Bigg(1 + a_2\Big(\frac{p^4}{16\hbar^4a_1^4}+\frac{3p^2}{4\hbar^2a_1^3}+\frac{3}{4a_1^2}\Big)\\
&\quad + a_3\Big(-\frac{p^6}{64\hbar^6a_1^6}-\frac{15p^4}{32\hbar^4a_1^5}-\frac{45p^2}{16\hbar^2a_1^4}-\frac{15}{8a_1^3}\Big)\\
&\quad + \frac{1}{2}a_2^2\Big(\frac{p^8}{256\hbar^8a_1^8}+\frac{7p^6}{32\hbar^6a_1^7}+\frac{105p^4}{32\hbar^4a_1^6}+\frac{105p^2}{8\hbar^2a_1^5}+\frac{105}{16a_1^4}\Big) + O\Big(\frac{1}{c^6}\Big)\Bigg)\\
&= a_0\sqrt{\frac{\pi}{-a_1}}e^{\frac{p^2}{4\hbar^2a_1}}\Bigg(1+\frac{3a_2}{4a_1^2}-\frac{15a_3}{8a_1^3}+\frac{105a_2^2}{32a_1^4}\\
&\quad + \Big(\frac{3a_2}{4a_1^3}-\frac{45a_3}{16a_1^4}+\frac{105a_2^2}{16a_1^5}\Big)\frac{p^2}{\hbar^2} + \Big(\frac{a_2}{16a_1^4}-\frac{15a_3}{32a_1^5}+\frac{105a_2^2}{64a_1^6}\Big)\frac{p^4}{\hbar^4}\\
&\quad + \Big(-\frac{a_3}{64a_1^6}+\frac{7a_2^2}{64a_1^7}\Big)\frac{p^6}{\hbar^6} + \frac{a_2^2}{512a_1^8}\frac{p^8}{\hbar^8} + O\Big(\frac{1}{c^6}\Big)\Bigg)\\
&= a_0\sqrt{\frac{\pi}{-a_1}}e^{\frac{p^2}{4\hbar^2a_1}}\Big(1+\frac{3a_2}{4a_1^2}-\frac{15a_3}{8a_1^3}+\frac{105a_2^2}{32a_1^4}\Big)\\
&\quad \Bigg(1 + \Big(\frac{3a_2}{4a_1^3}-\frac{45a_3}{16a_1^4}+\frac{6a_2^2}{a_1^5}\Big)\frac{p^2}{\hbar^2} + \Big(\frac{a_2}{16a_1^4}-\frac{15a_3}{32a_1^5}+\frac{51a_2^2}{32a_1^6}\Big)\frac{p^4}{\hbar^4}\\
&\quad + \Big(-\frac{a_3}{64a_1^6}+\frac{7a_2^2}{64a_1^7}\Big)\frac{p^6}{\hbar^6} + \frac{a_2^2}{512a_1^8}\frac{p^8}{\hbar^8} + O\Big(\frac{1}{c^6}\Big)\Bigg).
\end{aligned}
$$

Again there is a factor $e^{-\frac{i\Delta t p^2}{2\hbar m}}$ on the left side of (2), and a similar factor $e^{\frac{p^2}{4\hbar^2 a_1}}$ on the right side of (2), so there is no other option but to set $a_1 = \frac{im}{2\hbar\Delta t}$. There is a term $-\frac{\Delta t^2 p^8}{128\hbar^2 m^6 c^4}$ on the left side of (2), and there is a similar term $\frac{a_2^2}{512a_1^8}\frac{p^8}{\hbar^8}$ on the right side of (2), so there is no other option but to adjust the coefficient a_2 in a such way that these terms become equal. This means that we have to set $a_2 = \pm\frac{im}{8\hbar\Delta t^3 c^2}$. There is a term $-\frac{i\Delta t p^6}{16\hbar m^5 c^4}$ on the left side of (2), and there is a similar term $\big(-\frac{a_3}{64a_1^6} + \frac{7a_2^2}{64a_1^7}\big)\frac{p^6}{\hbar^6}$ on the right side of (2), so there is no other option but to adjust the coefficient a_3 in a such way that these terms become equal. This means that we have to set $a_3 = \frac{5im}{32\hbar\Delta t^5 c^4}$. There is a term $\frac{i\Delta t p^4}{8\hbar m^3 c^2}$ on the left side of (2), and there is a similar term $\big(\frac{a_2}{16a_1^4} - \frac{15a_3}{32a_1^5} + \frac{51a_2^2}{32a_1^6}\big)\frac{p^4}{\hbar^4}$ on the right side of (2), so there is no other option but to demand that these terms would be equal. However, there are no degrees of freedom left to be adjusted for these terms to become equal. The coefficients a_1, a_2 and a_3 have already been fixed above, and with the fixed values we have a relation

$$\Big(\frac{a_2}{16a_1^4} - \frac{15a_3}{32a_1^5} + \frac{51a_2^2}{32a_1^6}\Big)\frac{1}{\hbar^4} \;=\; \frac{1}{8m^3c^2}\Big(\pm\frac{i\Delta t}{\hbar} - \frac{6}{mc^2}\Big) \;\neq\; \frac{i\Delta t}{8\hbar m^3 c^2}.$$

There is a term $\big(\frac{3a_2}{4a_1^3} - \frac{45a_3}{16a_1^4} + \frac{6a_2^2}{a_1^5}\big)\frac{p^2}{\hbar^2}$ on the right side of (2), but there is no similar term proportional to p^2 on the left side of (2) at all. There is no other option but to demand that the term on the right side vanishes. However, there are no degrees of freedom left to be adjusted for this relation to become true. The coefficients a_1, a_2 and a_3 have already been fixed above, and with the fixed values we have a relation

$$\Big(\frac{3a_2}{4a_1^3} - \frac{45a_3}{16a_1^4} + \frac{6a_2^2}{a_1^5}\Big)\frac{1}{\hbar^2} \;=\; \frac{3\hbar^3}{4m^2\Delta t c^2}\Big(\mp\frac{\Delta t}{\hbar} - \frac{43i}{8mc^2}\Big) \;\neq\; 0.$$

So it turned out that the object (a_2, a_3) would have to satisfy four different constraints related to the terms proportional to p^2, p^4, p^6 and p^8. Since there are only two degrees of freedom in the object (a_2, a_3), the four constraints cannot be satisfied simultaneously. Again we have to conclude that our ansatz is not working.

Since the ansatz didn't work, the details of the calculation are probably not very interesting, but anyway, if somebody is interested in repeating the calculation, he or she will probably find the equation

$$\frac{1}{1 + \frac{3a_2}{4a_1^2} - \frac{15a_3}{8a_1^3} + \frac{105a_2^2}{32a_1^4}} \;=\; 1 - \frac{3a_2}{4a_1^2} + \frac{15a_3}{8a_1^3} - \frac{87a_2^2}{32a_1^4} + O\Big(\frac{1}{c^6}\Big)$$

relevant.

These calculations support the hypothesis that the non-relativistic time evolution kernel most apparently cannot be generalized into a relativistic form with an ansatz that would look like

$$K(\Delta t, x - x') \propto e^{\frac{im(x-x')^2}{2\hbar\Delta t} + a_2(x-x')^4 + a_3(x-x')^6 + \cdots}.$$

The issue with the number of constraints being larger than the number of adjustable parameters was worse with the 6th order approximation than with the 4th order approximation, so it is a reasonable belief that the issue will probably only get even worse with higher order approximations. We should contemplate on what this result means. The phase factor $e^{\frac{im(x-x')^2}{2\hbar\Delta t}}$ in the non-relativistic time evolution kernel can be interpreted to be the quantity $e^{\frac{i\Delta t}{\hbar}L}$, where $L = \frac{1}{2}mv^2$ is the non-relativistic Lagrangian of a free point particle, and where we have substituted $v = \frac{x-x'}{\Delta t}$. If we encounter the question that how could the time evolution kernel be generalized into a relativistic form, one obvious idea is that maybe we could replace the Lagrangian with the relativistic Lagrangian $L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}$. Then there is a problem that the square root expression makes integrals very hard. One idea that may surface is that maybe the new relativistic time evolution kernel with the relativistic Lagrangian does work, but it's just difficult to gain information about its functioning, because we just don't know the right integration tricks. At this point we have a reason to believe that that idea is probably not right: It is a reasonable conclusion that if the relativistic Lagrangian in the time evolution kernel did work, it should imply that we could write a Taylor series with respect to the quantity $\frac{1}{c^2}$ of the time evolution kernel, and we should get an approximation of the time evolution equation that would work as an approximation in the almost non-relativistic cases. As we just learned above, the ansatz that used a series with the coefficients $a_1 \propto \frac{1}{c^0}$, $a_2 \propto \frac{1}{c^2}$, $a_3 \propto \frac{1}{c^4} \ldots$ and so on, did not work. Logically, with some weight, this fact then supports the hypothesis that even the precise relativistic Lagrangian with the square root expression genuinely doesn't work.

Let's think more about what options there exist when trying to write a time evolution of a wave function using an integral kernel. The starting point is that we would like to write the value of $\psi(t+\Delta t, x)$ as a linear combination of $\psi(t', x')$, where hopefully the region $(t', x') \approx (t, x)$ dominates the integral. The most obvious choice for the spacetime points (t', x') is that we set $t' = t$, and then let either $x' \in \mathbb{R}$ or $x - c\Delta t < x' < x + c\Delta t$. Let's consider a new idea that we let (t', x') assume all the values such that $t' \leq t$ and

$$c^2(t + \Delta t - t')^2 - (x - x')^2 = c^2\Delta t^2.$$

These constraints define a hyperbola that has a special significance in Special Relativity. This idea is reasonable, because in the almost non-relativistic cases the use of this hyperbola is roughly equivalent to using a straight line $t' = t$. Suppose we want to parametrize this hyperbola. The most obvious parametrization could be that we define a function $\mathbb{R} \to \mathbb{R}$, $x' \mapsto t'(x')$ by a formula

$$t'(x') \;=\; t + \Delta t \;-\; \sqrt{\Delta t^2 + \frac{1}{c^2}(x - x')^2}.$$

Let's consider more elaborate parametrizations, and make x' into a function $\xi \mapsto x'(\xi)$. (The prime symbols in the notations $x' \mapsto t'(x')$ and $\xi \mapsto x'(\xi)$ do not mean that the functions would be derivatives of something defined earlier, but instead the prime symbols are part of the original notations for these functions. The reason for this notation is that above we have already fixed the meaning of the parameters t, t', x and x', and now the intended use of our new functions is that they produce values for the parameters t' and x'.) Suppose we want the function $\xi \mapsto x'(\xi)$ to have the property, that the spatial distance between the points $(t'(x'(\xi)), x'(\xi))$ and $(t'(x'(\xi{+}\Delta\xi)), x'(\xi{+}\Delta\xi))$ is approximately $\Delta\xi$, when measured in the frame of reference where these points are simultaneous. In other words

$$-\Delta\xi^2 \;\approx\; c^2\big(t'(x'(\xi + \Delta\xi)) - t'(x'(\xi))\big)^2 \;-\; \big(x'(\xi + \Delta x) - x'(\xi)\big)^2.$$

In the limit $\Delta\xi \to 0$ this becomes a differential equation

$$-1 \;=\; \Big(c^2\Big(\frac{dt'(x'(\xi))}{dx'}\Big)^2 \;-\; 1\Big)\Big(\frac{dx'(\xi)}{d\xi}\Big)^2.$$

By using the derivative formula

$$\frac{dt'(x')}{dx'} \;=\; \frac{1}{c^2}\frac{x - x'}{\sqrt{\Delta t^2 + \frac{1}{c^2}(x - x')^2}}$$

we see that the differential equation is

$$\frac{dx'(\xi)}{d\xi} \;=\; \pm\sqrt{1 + \frac{(x - x'(\xi))^2}{c^2\Delta t^2}}.$$

By using the properties of the hyperbolic functions we see that the wanted solution is

$$x'(\xi) \;=\; x \;+\; c\Delta t \sinh\Big(\frac{\xi}{c\Delta t}\Big).$$

Then

$$t'(x'(\xi)) \;=\; t + \Delta t \;-\; \Delta t \cosh\Big(\frac{\xi}{c\Delta t}\Big).$$

Strictly speaking, we do not here have a rigorous justification for choosing this parametrization. We just hope that people who know Special Relativity will agree that there is an intuitive feeling that this probably is a good

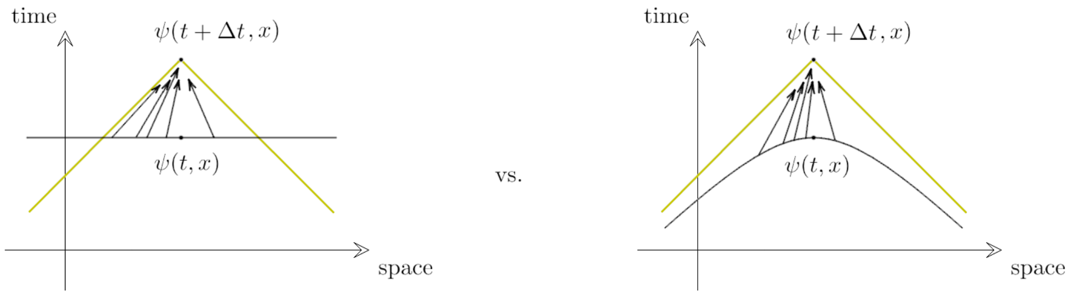

Figure 1: The big question is that when we write $\psi(t+\Delta t, x)$ as a linear combination of the past values $\psi(t', x')$, where the region $(t', x') \approx (t, x)$ hopefully dominates the integral, what spacetime points (t', x') precisely should we use? Those (t', x') that are on the straight line $t' = t$, or those (t', x') that are on the hyperbola $c^2(t+\Delta t - t')^2 - (x-x')^2 = c^2\Delta t^2$?

parametrization. One of the properties of this parametrization is that if the wave function ψ and the other possible factors in an integrand are scalars, then an integral of the values of ψ over the hyperbola will be Lorentz invariant, when written with respect to the parameter ξ.

Before attempting to come up with an unnecessarily generic parametrized ansatz for the integral kernel, we should ask whether it could be possible to guess what it will look like. As we mentioned above, the non-relativistic phase factor $e^{\frac{im(x-x')^2}{2\hbar\Delta t}}$ can be interpreted to be $e^{\frac{i\Delta t}{\hbar}L}$, where $L = \frac{1}{2}mv^2$ is the non-relativistic Lagrangian, and where we have substituted $v = \frac{x-x'}{\Delta t}$. An obvious guess is that maybe we can turn this into a relativistic phase factor by replacing the Lagrangian with the relativistic Lagrangian $L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$, and by replacing Δt with $t + \Delta t - t'(x'(\xi))$. According to our parametrization we have the relations

$$\frac{x - x'(\xi)}{t + \Delta t - t'(x'(\xi))} = -\frac{c\sinh\left(\frac{\xi}{c\Delta t}\right)}{\cosh\left(\frac{\xi}{c\Delta t}\right)},$$

$$\sqrt{1 - \frac{(x-x'(\xi))^2}{c^2(t+\Delta t - t'(x'(\xi)))^2}} = \frac{1}{\cosh\left(\frac{\xi}{c\Delta t}\right)}$$

and

$$\big(t + \Delta t - t'(x'(\xi))\big)\sqrt{1 - \frac{(x-x'(\xi))^2}{c^2(t+\Delta t - t'(x'(\xi)))^2}} = \Delta t.$$

So it turned out that $e^{\frac{i(t+\Delta t-t'(x'(\xi)))}{\hbar}L} = e^{-\frac{i\Delta tmc^2}{\hbar}}$, and there is no dependence on ξ or $x'-x$ in this phase factor at all. Many people will probably think that the fact that the dependece on ξ or $x'-x$ vanished completely must be a sign of something having gone wrong. For example, one might think that the relativistic phase factor would have to be of such kind that we could derive the non-relativistic phase factor $e^{\frac{im(x-x')^2}{2\hbar\Delta t}}$ out of it as a approximation. Then one might also think that the non-relativistic phase factor cannot be derived out of a phase factor where there is no non-trivial dependence on ξ or $x'-x$ at all. The truth turns out to be surprising, because actually the non-relativistic phase factor can be derived from a formulation that uses the relativistic phase factor $e^{-\frac{i\Delta tmc^2}{\hbar}}$. The way it works is that the the non-relativistic phase factor with the dependence on $x'-x$ arises from an expression related to the curvature of the hyperbola and the phase factor $e^{-\frac{it'mc^2}{\hbar}}$ in the past values of the wave function $\psi(t',x')$. With this information it now makes sense to try an ansatz that the relativistic time evolution kernel will be of the form $K(\Delta t)$ and not $K(\Delta t,\xi)$. This maybe means that $K(\Delta t)$ is not really an integration kernel now, or that it is a trivial constant kernel.

Let's try to approximate a solution to the 4th order approximation of the relativistic Schrödinger equation by using a function $\Delta t \mapsto K(\Delta t)$ and the parametrized hyperbola. So we want the equation

$$e^{-\frac{i(t+\Delta t)}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}e^{\frac{i}{\hbar}px}\Big(1+O\Big(\frac{1}{c^4}\Big)\Big)$$
$$= \int_{-\infty}^{\infty} K(\Delta t)e^{-\frac{i}{\hbar}t'(x'(\xi))\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}e^{\frac{i}{\hbar}px'(\xi)}d\xi$$

to be true. In this situation it makes sense to use the approximations

$$x'(\xi) \;=\; x \;+\; \xi \;+\; \frac{\xi^3}{6c^2\Delta t^2} \;+\; O\Big(\frac{1}{c^4}\Big)$$

and

$$t'(x'(\xi)) \;=\; t \;-\; \frac{\xi^2}{2c^2\Delta t} \;-\; \frac{\xi^4}{24c^4\Delta t^3} \;+\; O\Big(\frac{1}{c^6}\Big).$$

Some things cancel, and the equation that we want to be true is approximatively

$$e^{-\frac{i\Delta t}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}\Big(1+O\Big(\frac{1}{c^4}\Big)\Big)$$
$$= K(\Delta t)\int_{-\infty}^{\infty} e^{\left(\frac{i\xi^2}{2\hbar c^2\Delta t}+\frac{i\xi^4}{24\hbar c^4\Delta t^3}+O\left(\frac{1}{c^6}\right)\right)\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}\right)}e^{\frac{i}{\hbar}p\left(\xi+\frac{\xi^3}{6c^2\Delta t^2}+O\left(\frac{1}{c^4}\right)\right)}d\xi. \tag{3}$$

The left side of (3) can be written in the form

$$e^{-\frac{i\Delta t m c^2}{\hbar}} e^{-\frac{i\Delta t p^2}{2\hbar m}} \Big(1 \,+\, \frac{i\Delta t p^4}{8\hbar m^3 c^2} \,+\, O\Big(\frac{1}{c^4}\Big)\Big).$$

The right side of (3) can be written in the form

$$K(\Delta t) \int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\left(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2}+O\left(\frac{1}{c^4}\right)} e^{\frac{i}{\hbar}p\left(\xi+\frac{\xi^3}{6\Delta t^2c^2}+O\left(\frac{1}{c^4}\right)\right)} d\xi$$

$$= K(\Delta t) \int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\frac{ip\xi}{\hbar}} e^{\left(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2}+O\left(\frac{1}{c^4}\right)} d\xi$$

$$= K(\Delta t) \int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\frac{ip\xi}{\hbar}} \Big(1 \,+\, \Big(\frac{ip^2\xi^2}{4\hbar m\Delta t} + \frac{ip\xi^3}{6\hbar\Delta t^2} + \frac{im\xi^4}{24\hbar\Delta t^3}\Big)\frac{1}{c^2} \,+\, O\Big(\frac{1}{c^4}\Big)\Big) d\xi$$

$$= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}} e^{-\frac{i\Delta t p^2}{2\hbar m}} \Big(1 \,+\, \frac{ip^2}{4\hbar m\Delta t c^2}\Big(\frac{\Delta t^2 p^2}{m^2} + \frac{i\hbar\Delta t}{m}\Big)$$
$$+\, \frac{ip}{6\hbar\Delta t^2 c^2}\Big(-\frac{\Delta t^3 p^3}{m^3} - \frac{3i\hbar\Delta t^2 p}{m^2}\Big)$$
$$+\, \frac{im}{24\hbar\Delta t^3 c^2}\Big(\frac{\Delta t^4 p^4}{m^4} + \frac{6i\hbar\Delta t^3 p^2}{m^3} - \frac{3\hbar^2\Delta t^2}{m^2}\Big) \,+\, O\Big(\frac{1}{c^4}\Big)\Big)$$

$$= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}} e^{-\frac{i\Delta t p^2}{2\hbar m}} \Big(1 \,+\, \Big(-\frac{p^2}{4m^2c^2} + \frac{i\Delta t p^4}{4\hbar m^3 c^2}\Big)$$
$$+\, \Big(\frac{p^2}{2m^2c^2} - \frac{i\Delta t p^4}{6\hbar m^3 c^2}\Big) \,+\, \Big(-\frac{i\hbar}{8m\Delta t c^2} - \frac{p^2}{4m^2c^2} + \frac{i\Delta t p^4}{24\hbar m^3 c^2}\Big) \,+\, O\Big(\frac{1}{c^4}\Big)\Big)$$

$$= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}} e^{-\frac{i\Delta t p^2}{2\hbar m}} \Big(1 \,-\, \frac{i\hbar}{8m\Delta t c^2} \,+\, \Big(-\frac{1}{4} + \frac{1}{2} - \frac{1}{4}\Big)\frac{p^2}{m^2c^2}$$
$$+\, \Big(\frac{1}{4} - \frac{1}{6} + \frac{1}{24}\Big)\frac{i\Delta t p^4}{\hbar m^3 c^2} \,+\, O\Big(\frac{1}{c^4}\Big)\Big)$$

$$= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}} e^{-\frac{i\Delta t p^2}{2\hbar m}} \Big(1 \,-\, \frac{i\hbar}{8m\Delta t c^2} \,+\, \frac{i\Delta t p^4}{8\hbar m^3 c^2} \,+\, O\Big(\frac{1}{c^4}\Big)\Big)$$

$$= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}} e^{-\frac{i\Delta t p^2}{2\hbar m}} \Big(1 \,-\, \frac{i\hbar}{8m\Delta t c^2}\Big)\Big(1 \,+\, \frac{i\Delta t p^4}{8\hbar m^3 c^2} \,+\, O\Big(\frac{1}{c^4}\Big)\Big).$$

The factor $e^{-\frac{i\Delta t p^2}{2\hbar m}}$ appears similarly on the left and the right sides of (3), so no parameter needs to be adjusted for these to become equal. Also the term $\frac{i\Delta t p^4}{8\hbar m^3 c^2}$ appears similarly on the left and the right sides of (3). No similar term proportional to p^2 is present on either side. Terms proportional to p^2 are present in the intermediate steps, but they vanish due to cancellation. All we have to do is to set

$$K(\Delta t) \,=\, \frac{\sqrt{\frac{m}{2\pi i\hbar\Delta t}} e^{-\frac{i\Delta t m c^2}{\hbar}}}{1 - \frac{i\hbar}{8m\Delta t c^2}},$$

and then Equation (3) is true.

This was an interesting result, so next it would make sense to check whether a similar calculation could be repeated with the 6th order approximation of the relativistic Schrödinger equation. This means that now we want the equation

$$e^{-\frac{i(t+\Delta t)}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}\right)}e^{\frac{i}{\hbar}px}\left(1+O\left(\frac{1}{c^6}\right)\right)$$
$$=\int_{-\infty}^{\infty}K(\Delta t)e^{-\frac{i}{\hbar}t'(x'(\xi))\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}\right)}e^{\frac{i}{\hbar}px'(\xi)}d\xi$$

to be true. In this situation it makes sense to use the approximations

$$x'(\xi)\;=\;x\;+\;\xi\;+\;\frac{\xi^3}{6c^2\Delta t^2}\;+\;\frac{\xi^5}{120c^4\Delta t^4}\;+\;O\left(\frac{1}{c^6}\right)$$

and

$$t'(x'(\xi))\;=\;t\;-\;\frac{\xi^2}{2c^2\Delta t}\;-\;\frac{\xi^4}{24c^4\Delta t^3}\;-\;\frac{\xi^6}{720c^6\Delta t^5}\;+\;O\left(\frac{1}{c^8}\right).$$

Some things cancel, and the equation that we want to be true is approximatively

$$e^{-\frac{i\Delta t}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}\right)}\left(1+O\left(\frac{1}{c^6}\right)\right)$$
$$=\;K(\Delta t)\int_{-\infty}^{\infty}e^{\left(\frac{i\xi^2}{2\hbar c^2\Delta t}+\frac{i\xi^4}{24\hbar c^4\Delta t^3}+\frac{i\xi^6}{720\hbar c^6\Delta t^5}+O\left(\frac{1}{c^8}\right)\right)\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}\right)}$$
$$e^{\frac{i}{\hbar}p\left(\xi+\frac{\xi^3}{6c^2\Delta t^2}+\frac{\xi^5}{120c^4\Delta t^4}+O\left(\frac{1}{c^6}\right)\right)}d\xi. \tag{4}$$

The left side of (4) can be written in the form

$$e^{-\frac{i\Delta tmc^2}{\hbar}}e^{-\frac{i\Delta tp^2}{2\hbar m}}\left(1\;+\;\frac{i\Delta tp^4}{8\hbar m^3c^2}\;-\;\frac{i\Delta tp^6}{16\hbar m^5c^4}\;-\;\frac{\Delta t^2p^8}{128\hbar^2m^6c^4}\;+\;O\left(\frac{1}{c^6}\right)\right).$$

The right side of (4) can be written in the form

$$K(\Delta t)\int\limits_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\left(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2}+\left(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t}+\frac{ip^2\xi^4}{48\hbar m\Delta t^3}+\frac{im\xi^6}{720\hbar\Delta t^5}\right)\frac{1}{c^4}+O\left(\frac{1}{c^6}\right)}$$

$$e^{\frac{i}{\hbar}p\left(\xi+\frac{\xi^3}{6c^2\Delta t^2}+\frac{\xi^5}{120c^4\Delta t^4}+O\left(\frac{1}{c^6}\right)\right)}d\xi$$

$$= K(\Delta t)\int\limits_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\frac{ip\xi}{\hbar}} e^{\left(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2}}$$

$$e^{\left(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t}+\frac{ip^2\xi^4}{48\hbar m\Delta t^3}+\frac{ip\xi^5}{120\hbar\Delta t^4}+\frac{im\xi^6}{720\hbar\Delta t^5}\right)\frac{1}{c^4}} e^{O\left(\frac{1}{c^6}\right)}d\xi$$

$$= K(\Delta t)\int\limits_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\frac{ip\xi}{\hbar}}\Bigg(1 + \left(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2}$$

$$+ \frac{1}{2}\left(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\right)^2\frac{1}{c^4} + O\left(\frac{1}{c^6}\right)\Bigg)$$

$$\left(1 + \left(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t}+\frac{ip^2\xi^4}{48\hbar m\Delta t^3}+\frac{ip\xi^5}{120\hbar\Delta t^4}+\frac{im\xi^6}{720\hbar\Delta t^5}\right)\frac{1}{c^4}+O\left(\frac{1}{c^8}\right)\right)$$

$$\left(1+O\left(\frac{1}{c^6}\right)\right)d\xi$$

$$= K(\Delta t)\int\limits_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\frac{ip\xi}{\hbar}}\Bigg(1 + \left(\frac{ip^2}{4\hbar m\Delta t c^2}-\frac{ip^4}{16\hbar m^3\Delta t c^4}\right)\xi^2$$

$$+ \frac{ip}{6\hbar\Delta t^2c^2}\xi^3 + \left(\frac{im}{24\hbar\Delta t^3c^2}+\frac{ip^2}{48\hbar m\Delta t^3c^4}-\frac{p^4}{32\hbar^2m^2\Delta t^2c^4}\right)\xi^4$$

$$+ \left(\frac{ip}{120\hbar\Delta t^4c^4}-\frac{p^3}{24\hbar^2 m\Delta t^3c^4}\right)\xi^5 + \left(\frac{im}{720\hbar\Delta t^5c^4}-\frac{7p^2}{288\hbar^2\Delta t^4c^4}\right)\xi^6$$

$$- \frac{mp}{144\hbar^2\Delta t^5c^4}\xi^7 - \frac{m^2}{1152\hbar^2\Delta t^6c^4}\xi^8 + O\left(\frac{1}{c^6}\right)\Bigg)d\xi$$

$$
\begin{aligned}
= \; & K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}e^{-\frac{i\Delta t p^2}{2\hbar m}}\Big(1 \\
& + \Big(\frac{ip^2}{4\hbar m\Delta t c^2} - \frac{ip^4}{16\hbar m^3\Delta t c^4}\Big)\Big(\frac{\Delta t^2 p^2}{m^2} + \frac{i\hbar\Delta t}{m}\Big) \\
& + \frac{ip}{6\hbar\Delta t^2 c^2}\Big(-\frac{\Delta t^3 p^3}{m^3} - \frac{3i\hbar\Delta t^2 p}{m^2}\Big) \\
& + \Big(\frac{im}{24\hbar\Delta t^3 c^2} + \frac{ip^2}{48\hbar m\Delta t^3 c^4} - \frac{p^4}{32\hbar^2 m^2\Delta t^2 c^4}\Big)\Big(\frac{\Delta t^4 p^4}{m^4} + \frac{6i\hbar\Delta t^3 p^2}{m^3} - \frac{3\hbar^2\Delta t^2}{m^2}\Big) \\
& + \Big(\frac{ip}{120\hbar\Delta t^4 c^4} - \frac{p^3}{24\hbar^2 m\Delta t^3 c^4}\Big)\Big(-\frac{\Delta t^5 p^5}{m^5} - \frac{10i\hbar\Delta t^4 p^3}{m^4} + \frac{15\hbar^2\Delta t^3 p}{m^3}\Big) \\
& + \Big(\frac{im}{720\hbar\Delta t^5 c^4} - \frac{7p^2}{288\hbar^2\Delta t^4 c^4}\Big)\Big(\frac{\Delta t^6 p^6}{m^6} + \frac{15i\hbar\Delta t^5 p^4}{m^5} - \frac{45\hbar^2\Delta t^4 p^2}{m^4} - \frac{15i\hbar^3\Delta t^3}{m^3}\Big) \\
& - \frac{mp}{144\hbar^2\Delta t^5 c^4}\Big(-\frac{\Delta t^7 p^7}{m^7} - \frac{21i\hbar\Delta t^6 p^5}{m^6} + \frac{105\hbar^2\Delta t^5 p^3}{m^5} + \frac{105i\hbar^3\Delta t^4 p}{m^4}\Big) \\
& - \frac{m^2}{1152\hbar^2\Delta t^6 c^4}\Big(\frac{\Delta t^8 p^8}{m^8} + \frac{28i\hbar\Delta t^7 p^6}{m^7} - \frac{210\hbar^2\Delta t^6 p^4}{m^6} - \frac{420i\hbar^3\Delta t^5 p^2}{m^5} + \frac{105\hbar^4\Delta t^4}{m^4}\Big) \\
& + O\Big(\frac{1}{c^6}\Big)\Big) \\
= \; & K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}e^{-\frac{i\Delta t p^2}{2\hbar m}}\Big(1 \\
& + \Big(-\frac{p^2}{4m^2c^2} + \frac{i\Delta t p^4}{4\hbar m^3 c^2} + \frac{p^4}{16m^4c^4} - \frac{i\Delta t p^6}{16\hbar m^5 c^4}\Big) \\
& + \Big(\frac{p^2}{2m^2c^2} - \frac{i\Delta t p^4}{6\hbar m^3 c^2}\Big) \\
& + \Big(-\frac{i\hbar}{8m\Delta t c^2} - \frac{p^2}{4m^2c^2} - \frac{i\hbar p^2}{16m^3\Delta t c^4} + \frac{i\Delta t p^4}{24\hbar m^3 c^2} - \frac{p^4}{32m^4c^4} - \frac{i\Delta t p^6}{6\hbar m^5 c^4} - \frac{\Delta t^2 p^8}{32\hbar^2 m^6 c^4}\Big) \\
& + \Big(\frac{i\hbar p^2}{8m^3\Delta t c^4} - \frac{13p^4}{24m^4c^4} + \frac{49i\Delta t p^6}{120\hbar m^5 c^4} + \frac{\Delta t^2 p^8}{24\hbar^2 m^6 c^4}\Big) \\
& + \Big(\frac{\hbar^2}{48m^2\Delta t^2 c^4} + \frac{29i\hbar p^2}{96m^3\Delta t c^4} + \frac{103p^4}{96m^4c^4} - \frac{523i\Delta t p^6}{1440\hbar m^5 c^4} - \frac{7\Delta t^2 p^8}{288\hbar^2 m^6 c^4}\Big) \\
& + \Big(-\frac{35i\hbar p^2}{48m^3\Delta t c^4} - \frac{35p^4}{48m^4c^4} + \frac{7i\Delta t p^6}{48\hbar m^5 c^4} + \frac{\Delta t^2 p^8}{144\hbar^2 m^6 c^4}\Big) \\
& + \Big(-\frac{35\hbar^2}{384m^2\Delta t^2 c^4} + \frac{35i\hbar p^2}{96m^3\Delta t c^4} + \frac{35p^4}{192m^4c^4} - \frac{7i\Delta t p^6}{288\hbar m^5 c^4} - \frac{\Delta t^2 p^8}{1152\hbar^2 m^6 c^4}\Big) \\
& + O\Big(\frac{1}{c^6}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}e^{-\frac{i\Delta tp^2}{2\hbar m}}\Big(1 \;-\; \frac{i\hbar}{8m\Delta tc^2} \;+\; \Big(\frac{1}{48}-\frac{35}{384}\Big)\frac{\hbar^2}{m^2\Delta t^2c^4}\\
&\quad+\;\Big(-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\Big)\frac{p^2}{m^2c^2} \;+\; \Big(-\frac{1}{16}+\frac{1}{8}+\frac{29}{96}-\frac{35}{48}+\frac{35}{96}\Big)\frac{i\hbar p^2}{m^3\Delta tc^4}\\
&\quad+\;\Big(\frac{1}{4}-\frac{1}{6}+\frac{1}{24}\Big)\frac{i\Delta tp^4}{\hbar m^3c^2} \;+\; \Big(\frac{1}{16}-\frac{1}{32}-\frac{13}{24}+\frac{103}{96}-\frac{35}{48}+\frac{35}{192}\Big)\frac{p^4}{m^4c^4}\\
&\quad+\;\Big(-\frac{1}{16}-\frac{1}{6}+\frac{49}{120}-\frac{523}{1440}+\frac{7}{48}-\frac{7}{288}\Big)\frac{i\Delta tp^6}{\hbar m^5c^4}\\
&\quad+\;\Big(-\frac{1}{32}+\frac{1}{24}-\frac{7}{288}+\frac{1}{144}-\frac{1}{1152}\Big)\frac{\Delta t^2p^8}{\hbar^2m^6c^4} \;+\; O\Big(\frac{1}{c^6}\Big)\Big)\\
&= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}e^{-\frac{i\Delta tp^2}{2\hbar m}}\Big(1 \;-\; \frac{i\hbar}{8m\Delta tc^2} \;-\; \frac{9\hbar^2}{128m^2\Delta t^2c^4}\\
&\quad+\;\Big(\frac{i\Delta t}{8\hbar m^3c^2}+\frac{1}{64m^4c^4}\Big)p^4 \;-\; \frac{i\Delta tp^6}{16\hbar m^5c^4} \;-\; \frac{\Delta t^2p^8}{128\hbar^2m^6c^4} \;+\; O\Big(\frac{1}{c^6}\Big)\Big)\\
&= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}e^{-\frac{i\Delta tp^2}{2\hbar m}}\Big(1-\frac{i\hbar}{8m\Delta tc^2}-\frac{9\hbar^2}{128m^2\Delta t^2c^4}\Big)\\
&\quad\Big(1 \;+\; \frac{i\Delta tp^4}{8\hbar m^3c^2} \;-\; \frac{i\Delta tp^6}{16\hbar m^5c^4} \;-\; \frac{\Delta t^2p^8}{128\hbar^2m^6c^4} \;+\; O\Big(\frac{1}{c^6}\Big)\Big).
\end{aligned}
$$

The factor $e^{-\frac{i\Delta tp^2}{2\hbar m}}$ and the terms $\frac{i\Delta tp^4}{8\hbar m^3c^2}$, $-\frac{i\Delta tp^6}{16\hbar m^5c^4}$ and $-\frac{\Delta t^2p^8}{128\hbar^2m^6c^4}$ appear similarly on both the left and the right sides of (4). The terms proportional to $\frac{p^2}{c^2}$ vanish in the same way as earlier above. Also terms proportional to $\frac{p^2}{c^4}$ are present in the intermediate steps, and they vanish similarly. In the last step we use the equation

$$
\frac{1}{1-\frac{i\hbar}{8m\Delta tc^2}-\frac{9\hbar^2}{128m^2\Delta t^2c^4}} \;=\; 1 \;+\; \frac{i\hbar}{8m\Delta tc^2} \;+\; \frac{7\hbar^2}{128m^2\Delta t^2c^4} \;+\; O\Big(\frac{1}{c^6}\Big),
$$

and there the term proportional to p^4 gets its coefficient right because

$$
\Big(1 \;+\; \frac{i\hbar}{8m\Delta tc^2} \;+\; O\Big(\frac{1}{c^4}\Big)\Big)\Big(\frac{i\Delta t}{8\hbar m^3c^2} \;+\; \frac{1}{64m^4c^4}\Big)p^4 \;=\; \frac{i\Delta tp^4}{8\hbar m^3c^2} \;+\; O\Big(\frac{1}{c^6}\Big).
$$

All we have to do is to set

$$
K(\Delta t) \;=\; \frac{\sqrt{\frac{m}{2\pi i\hbar\Delta t}}e^{-\frac{i\Delta tmc^2}{\hbar}}}{1-\frac{i\hbar}{8m\Delta tc^2}-\frac{9\hbar^2}{128m^2\Delta t^2c^4}},
$$

and then Equation (4) is true.

It is extremely unlikely that Equation (4) could be made true with the right choice of $K(\Delta t)$ like this via a mere coincidence, so at this point we know that the use of the hyperbola is probably related to some real result.

Nevertheless, it is still reasonable to ask whether a similar calculation could be repeated with the 10th order approximation of the relativistic Schrödinger equation. At least we might want to see how the function $K(\Delta t)$ turns out in that case. This means that next we want the equation

$$e^{-\frac{i(t+\Delta t)}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}-\frac{5p^8}{128m^7c^6}+\frac{7p^{10}}{256m^9c^8}\right)}e^{\frac{i}{\hbar}px}\left(1+O\left(\frac{1}{c^{10}}\right)\right)$$
$$= \int_{-\infty}^{\infty} K(\Delta t)e^{-\frac{i}{\hbar}t'(x'(\xi))\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}-\frac{5p^8}{128m^7c^6}+\frac{7p^{10}}{256m^9c^8}\right)}e^{\frac{i}{\hbar}px'(\xi)}d\xi$$

to be true. In this situation it makes sense to use the approximations

$$x'(\xi) = x + \xi + \frac{\xi^3}{6c^2\Delta t^2} + \frac{\xi^5}{120c^4\Delta t^4} + \frac{\xi^7}{5040c^6\Delta t^6} + \frac{\xi^9}{362880c^8\Delta t^8} + O\left(\frac{1}{c^{10}}\right)$$

and

$$t'(x'(\xi)) = t - \frac{\xi^2}{2c^2\Delta t} - \frac{\xi^4}{24c^4\Delta t^3} - \frac{\xi^6}{720c^6\Delta t^5} - \frac{\xi^8}{40320c^8\Delta t^7} - \frac{\xi^{10}}{3628800c^{10}\Delta t^9} + O\left(\frac{1}{c^{12}}\right).$$

Some things cancel, and the equation that we want to be true is approximatively

$$e^{-\frac{i\Delta t}{\hbar}\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}-\frac{5p^8}{128m^7c^6}+\frac{7p^{10}}{256m^9c^8}\right)}\left(1+O\left(\frac{1}{c^{10}}\right)\right)$$
$$= K(\Delta t)\int_{-\infty}^{\infty} e^{\left(\frac{i\xi^2}{2\hbar c^2\Delta t}+\frac{i\xi^4}{24\hbar c^4\Delta t^3}+\frac{i\xi^6}{720\hbar c^6\Delta t^5}+\frac{i\xi^8}{40320\hbar c^8\Delta t^7}+\frac{i\xi^{10}}{3628800\hbar c^{10}\Delta t^9}+O\left(\frac{1}{c^{12}}\right)\right)\left(mc^2+\frac{p^2}{2m}-\frac{p^4}{8m^3c^2}+\frac{p^6}{16m^5c^4}-\frac{5p^8}{128m^7c^6}+\frac{7p^{10}}{256m^9c^8}\right)}$$
$$e^{\frac{i}{\hbar}p\left(\xi+\frac{\xi^3}{6c^2\Delta t^2}+\frac{\xi^5}{120c^4\Delta t^4}+\frac{\xi^7}{5040c^6\Delta t^6}+\frac{\xi^9}{362880c^8\Delta t^8}+O\left(\frac{1}{c^{10}}\right)\right)}d\xi. \tag{5}$$

The left side of (5) can be written in the form

$$e^{-\frac{i\Delta tmc^2}{\hbar}}e^{-\frac{i\Delta tp^2}{2\hbar m}}\Big(1 + \frac{i\Delta tp^4}{8\hbar m^3c^2} - \frac{i\Delta tp^6}{16\hbar m^5c^4}$$
$$+ \Big(-\frac{\Delta t^2}{128\hbar^2m^6c^4} + \frac{5i\Delta t}{128\hbar m^7c^6}\Big)p^8 + \Big(\frac{\Delta t^2}{128\hbar^2m^8c^6} - \frac{7i\Delta t}{256\hbar m^9c^8}\Big)p^{10}$$
$$+ \Big(-\frac{i\Delta t^3}{3072\hbar^3m^9c^6} - \frac{7\Delta t^2}{1024\hbar^2m^{10}c^8}\Big)p^{12} + \frac{i\Delta t^3p^{14}}{2048\hbar^3m^{11}c^8}$$
$$+ \frac{\Delta t^4p^{16}}{98304\hbar^4m^{12}c^8} + O\Big(\frac{1}{c^{10}}\Big)\Big).$$

The right side of (5) can be written in the form

$$
\begin{aligned}
& K(\Delta t) \int\limits_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t} + \left(\frac{ip^2\xi^2}{4\hbar m\Delta t} + \frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2} + \left(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t} + \frac{ip^2\xi^4}{48\hbar m\Delta t^3} + \frac{im\xi^6}{720\hbar\Delta t^5}\right)\frac{1}{c^4}} \\
& \qquad e^{\left(\frac{ip^6\xi^2}{32\hbar m^5\Delta t} - \frac{ip^4\xi^4}{192\hbar m^3\Delta t^3} + \frac{ip^2\xi^6}{1440\hbar m\Delta t^5} + \frac{im\xi^8}{40320\hbar\Delta t^7}\right)\frac{1}{c^6}} \\
& \qquad e^{\left(-\frac{5ip^8\xi^2}{256\hbar m^7\Delta t} + \frac{ip^6\xi^4}{384\hbar m^5\Delta t^3} - \frac{ip^4\xi^6}{5760\hbar m^3\Delta t^5} + \frac{ip^2\xi^8}{80640\hbar m\Delta t^7} + \frac{im\xi^{10}}{3628800\hbar\Delta t^9}\right)\frac{1}{c^8} + O\left(\frac{1}{c^{10}}\right)} \\
& \qquad e^{\frac{i}{\hbar}p\left(\xi + \frac{\xi^3}{6c^2\Delta t^2} + \frac{\xi^5}{120c^4\Delta t^4} + \frac{\xi^7}{5040c^6\Delta t^6} + \frac{\xi^9}{362880c^8\Delta t^8} + O\left(\frac{1}{c^{10}}\right)\right)} d\xi \\
= \; & K(\Delta t) \int\limits_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t} + \frac{ip\xi}{\hbar} + \left(\frac{ip^2\xi^2}{4\hbar m\Delta t} + \frac{ip\xi^3}{6\hbar\Delta t^2} + \frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2}} \\
& \qquad e^{\left(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t} + \frac{ip^2\xi^4}{48\hbar m\Delta t^3} + \frac{ip\xi^5}{120\hbar\Delta t^4} + \frac{im\xi^6}{720\hbar\Delta t^5}\right)\frac{1}{c^4}} \\
& \qquad e^{\left(\frac{ip^6\xi^2}{32\hbar m^5\Delta t} - \frac{ip^4\xi^4}{192\hbar m^3\Delta t^3} + \frac{ip^2\xi^6}{1440\hbar m\Delta t^5} + \frac{ip\xi^7}{5040\hbar\Delta t^6} + \frac{im\xi^8}{40320\hbar\Delta t^7}\right)\frac{1}{c^6}} \\
& \qquad e^{\left(-\frac{5ip^8\xi^2}{256\hbar m^7\Delta t} + \frac{ip^6\xi^4}{384\hbar m^5\Delta t^3} - \frac{ip^4\xi^6}{5760\hbar m^3\Delta t^5} + \frac{ip^2\xi^8}{80640\hbar m\Delta t^7} + \frac{ip\xi^9}{362880\hbar\Delta t^8} + \frac{im\xi^{10}}{3628800\hbar\Delta t^9}\right)\frac{1}{c^8}} \\
& \qquad e^{O\left(\frac{1}{c^{10}}\right)} d\xi \\
= \; & K(\Delta t) \int\limits_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t} + \frac{ip\xi}{\hbar}} \Bigg(1 \; + \; \left(\frac{ip^2\xi^2}{4\hbar m\Delta t} + \frac{ip\xi^3}{6\hbar\Delta t^2} + \frac{im\xi^4}{24\hbar\Delta t^3}\right)\frac{1}{c^2} \\
& \quad + \; \frac{1}{2}\left(\frac{ip^2\xi^2}{4\hbar m\Delta t} + \frac{ip\xi^3}{6\hbar\Delta t^2} + \frac{im\xi^4}{24\hbar\Delta t^3}\right)^2 \frac{1}{c^4} \; + \; \frac{1}{6}\left(\frac{ip^2\xi^2}{4\hbar m\Delta t} + \frac{ip\xi^3}{6\hbar\Delta t^2} + \frac{im\xi^4}{24\hbar\Delta t^3}\right)^3 \frac{1}{c^6} \\
& \quad + \; \frac{1}{24}\left(\frac{ip^2\xi^2}{4\hbar m\Delta t} + \frac{ip\xi^3}{6\hbar\Delta t^2} + \frac{im\xi^4}{24\hbar\Delta t^3}\right)^4 \frac{1}{c^8} \; + \; O\left(\frac{1}{c^{10}}\right)\Bigg) \\
& \Bigg(1 \; + \; \left(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t} + \frac{ip^2\xi^4}{48\hbar m\Delta t^3} + \frac{ip\xi^5}{120\hbar\Delta t^4} + \frac{im\xi^6}{720\hbar\Delta t^5}\right)\frac{1}{c^4} \\
& \quad + \; \frac{1}{2}\left(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t} + \frac{ip^2\xi^4}{48\hbar m\Delta t^3} + \frac{ip\xi^5}{120\hbar\Delta t^4} + \frac{im\xi^6}{720\hbar\Delta t^5}\right)^2 \frac{1}{c^8} \; + \; O\left(\frac{1}{c^{12}}\right)\Bigg) \\
& \Bigg(1 \; + \; \left(\frac{ip^6\xi^2}{32\hbar m^5\Delta t} - \frac{ip^4\xi^4}{192\hbar m^3\Delta t^3} + \frac{ip^2\xi^6}{1440\hbar m\Delta t^5} + \frac{ip\xi^7}{5040\hbar\Delta t^6} + \frac{im\xi^8}{40320\hbar\Delta t^7}\right)\frac{1}{c^6} \\
& \quad + \; O\left(\frac{1}{c^{12}}\right)\Bigg) \\
& \Bigg(1 \; + \; \Bigg(-\frac{5ip^8\xi^2}{256\hbar m^7\Delta t} + \frac{ip^6\xi^4}{384\hbar m^5\Delta t^3} - \frac{ip^4\xi^6}{5760\hbar m^3\Delta t^5} + \frac{ip^2\xi^8}{80640\hbar m\Delta t^7} \\
& \qquad + \frac{ip\xi^9}{362880\hbar\Delta t^8} + \frac{im\xi^{10}}{3628800\hbar\Delta t^9}\Bigg)\frac{1}{c^8} \; + \; O\left(\frac{1}{c^{16}}\right)\Bigg) \\
& \left(1 \; + \; O\left(\frac{1}{c^{10}}\right)\right) d\xi
\end{aligned}
$$

$$
\begin{aligned}
= \; & K(\Delta t) \int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\frac{ip\xi}{\hbar}}\Big(1 \;+\; \Big(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\Big)\frac{1}{c^2} \\
& + \Big(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t}+\Big(\frac{ip^2}{48\hbar m\Delta t^3}-\frac{p^4}{32\hbar^2 m^2\Delta t^2}\Big)\xi^4+\Big(\frac{ip}{120\hbar\Delta t^4}-\frac{p^3}{24\hbar^2 m\Delta t^3}\Big)\xi^5 \\
& \quad + \Big(\frac{im}{720\hbar\Delta t^5}-\frac{7p^2}{288\hbar^2\Delta t^4}\Big)\xi^6-\frac{mp\xi^7}{144\hbar^2\Delta t^5}-\frac{m^2\xi^8}{1152\hbar^2\Delta t^6}\Big)\frac{1}{c^4} \\
& + \Big(\frac{ip^6\xi^2}{32\hbar m^5\Delta t} \;+\; \Big(-\frac{ip^4}{192\hbar m^3\Delta t^3}+\frac{p^6}{64\hbar^2 m^4\Delta t^2}\Big)\xi^4 \;+\; \frac{p^5\xi^5}{96\hbar^2 m^3\Delta t^3} \\
& \quad + \Big(\frac{ip^2}{1440\hbar m\Delta t^5}-\frac{p^4}{384\hbar^2 m^2\Delta t^4}-\frac{ip^6}{384\hbar^3 m^3\Delta t^3}\Big)\xi^6 \\
& \quad + \Big(\frac{ip}{5040\hbar\Delta t^6}-\frac{p^3}{180\hbar^2 m\Delta t^5}-\frac{ip^5}{192\hbar^3 m^2\Delta t^4}\Big)\xi^7 \\
& \quad + \Big(\frac{im}{40320\hbar\Delta t^7}-\frac{p^2}{384\hbar^2\Delta t^6}-\frac{11ip^4}{2304\hbar^3 m\Delta t^5}\Big)\xi^8 \\
& \quad + \Big(-\frac{mp}{1728\hbar^2\Delta t^7}-\frac{13ip^3}{5184\hbar^3\Delta t^6}\Big)\xi^9 \;+\; \Big(-\frac{m^2}{17280\hbar^2\Delta t^8}-\frac{11imp^2}{13824\hbar^3\Delta t^7}\Big)\xi^{10} \\
& \quad - \frac{im^2p\xi^{11}}{6912\hbar^3\Delta t^8} \;-\; \frac{im^3\xi^{12}}{82944\hbar^3\Delta t^9}\Big)\frac{1}{c^6} \\
& + \Big(-\frac{5ip^8\xi^2}{256\hbar m^7\Delta t} \;+\; \Big(\frac{ip^6}{384\hbar m^5\Delta t^3}-\frac{5p^8}{512\hbar^2 m^6\Delta t^2}\Big)\xi^4 \;-\; \frac{p^7\xi^5}{192\hbar^2 m^5\Delta t^3} \\
& \quad + \Big(-\frac{ip^4}{5760\hbar m^3\Delta t^5}+\frac{p^6}{768\hbar^2 m^4\Delta t^4}+\frac{ip^8}{512\hbar^3 m^5\Delta t^3}\Big)\xi^6 \\
& \quad + \Big(\frac{p^5}{720\hbar^2 m^3\Delta t^5}+\frac{ip^7}{384\hbar^3 m^4\Delta t^4}\Big)\xi^7 \\
& \quad + \Big(\frac{ip^2}{80640\hbar m\Delta t^7}-\frac{p^4}{11520\hbar^2 m^2\Delta t^6}+\frac{ip^6}{1152\hbar^3 m^3\Delta t^5}+\frac{p^8}{6144\hbar^4 m^4\Delta t^4}\Big)\xi^8 \\
& \quad + \Big(\frac{ip}{362880\hbar\Delta t^8}-\frac{41p^3}{120960\hbar^2 m\Delta t^7}-\frac{ip^5}{1440\hbar^3 m^2\Delta t^6}+\frac{p^7}{2304\hbar^4 m^3\Delta t^5}\Big)\xi^9 \\
& \quad + \Big(\frac{im}{3628800\hbar\Delta t^9}-\frac{319p^2}{2419200\hbar^2\Delta t^8}-\frac{233ip^4}{276480\hbar^3 m\Delta t^7}+\frac{5p^6}{9216\hbar^4 m^2\Delta t^6}\Big)\xi^{10} \\
& \quad + \Big(-\frac{29mp}{1209600\hbar^2\Delta t^9}-\frac{7ip^3}{17280\hbar^3\Delta t^8}+\frac{17p^5}{41472\hbar^4 m\Delta t^7}\Big)\xi^{11} \\
& \quad + \Big(-\frac{29m^2}{14515200\hbar^2\Delta t^{10}}-\frac{91imp^2}{829440\hbar^3\Delta t^9}+\frac{203p^4}{995328\hbar^4\Delta t^8}\Big)\xi^{12} \\
& \quad + \Big(-\frac{7im^2p}{414720\hbar^3\Delta t^{10}}+\frac{17mp^3}{248832\hbar^4\Delta t^9}\Big)\xi^{13} \;+\; \Big(-\frac{im^3}{829440\hbar^3\Delta t^{11}}+\frac{5m^2p^2}{331776\hbar^4\Delta t^{10}}\Big)\xi^{14} \\
& \quad + \frac{m^3p\xi^{15}}{497664\hbar^4\Delta t^{11}} \;+\; \frac{m^4\xi^{16}}{7962624\hbar^4\Delta t^{12}}\Big)\frac{1}{c^8} \;+\; O\Big(\frac{1}{c^{10}}\Big)\Big)d\xi
\end{aligned}
$$

$$
\begin{aligned}
= \; & K(\Delta t) \int_{-\infty}^{\infty} e^{\frac{im\xi^2}{2\hbar\Delta t}+\frac{ip\xi}{\hbar}} \Big(1 \\
& + \Big(\frac{ip^2}{4\hbar m\Delta tc^2} - \frac{ip^4}{16\hbar m^3\Delta tc^4} + \frac{ip^6}{32\hbar m^5\Delta tc^6} - \frac{5ip^8}{256\hbar m^7\Delta tc^8}\Big)\xi^2 \\
& + \frac{ip\xi^3}{6\hbar\Delta t^2c^2} \\
& + \Big(\frac{im}{24\hbar\Delta t^3c^2} + \frac{ip^2}{48\hbar m\Delta t^3c^4} - \frac{p^4}{32\hbar^2m^2\Delta t^2c^4} - \frac{ip^4}{192\hbar m^3\Delta t^3c^6} + \frac{p^6}{64\hbar^2m^4\Delta t^2c^6} \\
& \quad + \frac{ip^6}{384\hbar m^5\Delta t^3c^8} - \frac{5p^8}{512\hbar^2m^6\Delta t^2c^8}\Big)\xi^4 \\
& + \Big(\frac{ip}{120\hbar\Delta t^4c^4} - \frac{p^3}{24\hbar^2m\Delta t^3c^4} + \frac{p^5}{96\hbar^2m^3\Delta t^3c^6} - \frac{p^7}{192\hbar^2m^5\Delta t^3c^8}\Big)\xi^5 \\
& + \Big(\frac{im}{720\hbar\Delta t^5c^4} - \frac{7p^2}{288\hbar^2\Delta t^4c^4} + \frac{ip^2}{1440\hbar m\Delta t^5c^6} - \frac{p^4}{384\hbar^2m^2\Delta t^4c^6} - \frac{ip^4}{5760\hbar m^3\Delta t^5c^8} \\
& \quad - \frac{ip^6}{384\hbar^3m^3\Delta t^3c^6} + \frac{p^6}{768\hbar^2m^4\Delta t^4c^8} + \frac{ip^8}{512\hbar^3m^5\Delta t^3c^8}\Big)\xi^6 \\
& + \Big(-\frac{mp}{144\hbar^2\Delta t^5c^4} + \frac{ip}{5040\hbar\Delta t^6c^6} - \frac{p^3}{180\hbar^2m\Delta t^5c^6} - \frac{ip^5}{192\hbar^3m^2\Delta t^4c^6} + \frac{p^5}{720\hbar^2m^3\Delta t^5c^8} + \frac{ip^7}{384\hbar^3m^4\Delta t^4c^8}\Big)\xi^7 \\
& + \Big(-\frac{m^2}{1152\hbar^2\Delta t^6c^4} + \frac{im}{40320\hbar\Delta t^7c^6} - \frac{p^2}{384\hbar^2\Delta t^6c^6} + \frac{ip^2}{80640\hbar m\Delta t^7c^8} - \frac{11ip^4}{2304\hbar^3m\Delta t^5c^6} \\
& \quad - \frac{p^4}{11520\hbar^2m^2\Delta t^6c^8} + \frac{ip^6}{1152\hbar^3m^3\Delta t^5c^8} + \frac{p^8}{6144\hbar^4m^4\Delta t^4c^8}\Big)\xi^8 \\
& + \Big(-\frac{mp}{1728\hbar^2\Delta t^7c^6} + \frac{ip}{362880\hbar\Delta t^8c^8} - \frac{13ip^3}{5184\hbar^3\Delta t^6c^6} - \frac{41p^3}{120960\hbar^2m\Delta t^7c^8} - \frac{ip^5}{1440\hbar^3m^2\Delta t^6c^8} + \frac{p^7}{2304\hbar^4m^3\Delta t^5c^8}\Big)\xi^9 \\
& + \Big(-\frac{m^2}{17280\hbar^2\Delta t^8c^6} + \frac{im}{3628800\hbar\Delta t^9c^8} - \frac{11imp^2}{13824\hbar^3\Delta t^7c^6} - \frac{319p^2}{2419200\hbar^2\Delta t^8c^8} \\
& \quad - \frac{233ip^4}{276480\hbar^3m\Delta t^7c^8} + \frac{5p^6}{9216\hbar^4m^2\Delta t^6c^8}\Big)\xi^{10} \\
& + \Big(-\frac{im^2p}{6912\hbar^3\Delta t^8c^6} - \frac{29mp}{1209600\hbar^2\Delta t^9c^8} - \frac{7ip^3}{17280\hbar^3\Delta t^8c^8} + \frac{17p^5}{41472\hbar^4m\Delta t^7c^8}\Big)\xi^{11} \\
& + \Big(-\frac{im^3}{82944\hbar^3\Delta t^9c^6} - \frac{29m^2}{14515200\hbar^2\Delta t^{10}c^8} - \frac{91imp^2}{829440\hbar^3\Delta t^9c^8} + \frac{203p^4}{995328\hbar^4\Delta t^8c^8}\Big)\xi^{12} \\
& + \Big(-\frac{7im^2p}{414720\hbar^3\Delta t^{10}c^8} + \frac{17mp^3}{248832\hbar^4\Delta t^9c^8}\Big)\xi^{13} \\
& + \Big(-\frac{im^3}{829440\hbar^3\Delta t^{11}c^8} + \frac{5m^2p^2}{331776\hbar^4\Delta t^{10}c^8}\Big)\xi^{14} \\
& + \frac{m^3p\xi^{15}}{497664\hbar^4\Delta t^{11}c^8} \\
& + \frac{m^4\xi^{16}}{7962624\hbar^4\Delta t^{12}c^8} \; + O\Big(\frac{1}{c^{10}}\Big)\Big)d\xi
\end{aligned}
$$

$$
\begin{aligned}
= \;& K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\,e^{-\frac{i\Delta t p^2}{2\hbar m}}\Big(1 \\
&+ \Big(\frac{ip^2}{4\hbar m\Delta t c^2} - \frac{ip^4}{16\hbar m^3\Delta t c^4} + \frac{ip^6}{32\hbar m^5\Delta t c^6} - \frac{5ip^8}{256\hbar m^7\Delta t c^8}\Big)\Big(\frac{\Delta t^2 p^2}{m^2} + \frac{i\hbar\Delta t}{m}\Big) \\
&+ \frac{ip}{6\hbar\Delta t^2 c^2}\Big(-\frac{\Delta t^3 p^3}{m^3} - \frac{3i\hbar\Delta t^2 p}{m^2}\Big) \\
&+ \Big(\frac{im}{24\hbar\Delta t^3 c^2} + \frac{ip^2}{48\hbar m\Delta t^3 c^4} - \frac{p^4}{32\hbar^2 m^2\Delta t^2 c^4} - \frac{ip^4}{192\hbar m^3\Delta t^3 c^6} + \frac{p^6}{64\hbar^2 m^4\Delta t^2 c^6} + \frac{ip^6}{384\hbar m^5\Delta t^3 c^8} - \frac{5p^8}{512\hbar^2 m^6\Delta t^2 c^8}\Big) \\
&\quad \Big(\frac{\Delta t^4 p^4}{m^4} + \frac{6i\hbar\Delta t^3 p^2}{m^3} - \frac{3\hbar^2\Delta t^2}{m^2}\Big) \\
&+ \Big(\frac{ip}{120\hbar\Delta t^4 c^4} - \frac{p^3}{24\hbar^2 m\Delta t^3 c^4} + \frac{p^5}{96\hbar^2 m^3\Delta t^3 c^6} - \frac{p^7}{192\hbar^2 m^5\Delta t^3 c^8}\Big)\Big(-\frac{\Delta t^5 p^5}{m^5} - \frac{10i\hbar\Delta t^4 p^3}{m^4} + \frac{15\hbar^2\Delta t^3 p}{m^3}\Big) \\
&+ \Big(\frac{im}{720\hbar\Delta t^5 c^4} - \frac{7p^2}{288\hbar^2\Delta t^4 c^4} + \frac{ip^2}{1440\hbar m\Delta t^5 c^6} - \frac{p^4}{384\hbar^2 m^2\Delta t^4 c^6} - \frac{ip^4}{5760\hbar m^3\Delta t^5 c^8} - \frac{ip^6}{384\hbar^3 m^3\Delta t^3 c^6} + \frac{p^6}{768\hbar^2 m^4\Delta t^4 c^8} \\
&\quad + \frac{ip^8}{512\hbar^3 m^5\Delta t^3 c^8}\Big)\Big(\frac{\Delta t^6 p^6}{m^6} + \frac{15i\hbar\Delta t^5 p^4}{m^5} - \frac{45\hbar^2\Delta t^4 p^2}{m^4} - \frac{15i\hbar^3\Delta t^3}{m^3}\Big) \\
&+ \Big(-\frac{mp}{144\hbar^2\Delta t^5 c^4} + \frac{ip}{5040\hbar\Delta t^6 c^6} - \frac{p^3}{180\hbar^2 m\Delta t^5 c^6} - \frac{ip^5}{192\hbar^3 m^2\Delta t^4 c^6} + \frac{p^5}{720\hbar^2 m^3\Delta t^5 c^8} + \frac{ip^7}{384\hbar^3 m^4\Delta t^4 c^8}\Big)\Big(-\frac{\Delta t^7 p^7}{m^7} \\
&\quad - \frac{21i\hbar\Delta t^6 p^5}{m^6} + \frac{105\hbar^2\Delta t^5 p^3}{m^5} + \frac{105i\hbar^3\Delta t^4 p}{m^4}\Big) \\
&+ \Big(-\frac{m^2}{1152\hbar^2\Delta t^6 c^4} + \frac{im}{40320\hbar\Delta t^7 c^6} - \frac{p^2}{384\hbar^2\Delta t^6 c^6} + \frac{ip^2}{80640\hbar m\Delta t^7 c^8} - \frac{11ip^4}{2304\hbar^3 m\Delta t^5 c^6} - \frac{p^4}{11520\hbar^2 m^2\Delta t^6 c^8} \\
&\quad + \frac{ip^6}{1152\hbar^3 m^3\Delta t^5 c^8} + \frac{p^8}{6144\hbar^4 m^4\Delta t^4 c^8}\Big)\Big(\frac{\Delta t^8 p^8}{m^8} + \frac{28i\hbar\Delta t^7 p^6}{m^7} - \frac{210\hbar^2\Delta t^6 p^4}{m^6} - \frac{420i\hbar^3\Delta t^5 p^2}{m^5} + \frac{105\hbar^4\Delta t^4}{m^4}\Big) \\
&+ \Big(-\frac{mp}{1728\hbar^2\Delta t^7 c^6} + \frac{ip}{362880\hbar\Delta t^8 c^8} - \frac{13ip^3}{5184\hbar^3\Delta t^6 c^6} - \frac{41p^3}{120960\hbar^2 m\Delta t^7 c^8} - \frac{ip^5}{1440\hbar^3 m^2\Delta t^6 c^8} + \frac{p^7}{2304\hbar^4 m^3\Delta t^5 c^8}\Big) \\
&\quad \Big(-\frac{\Delta t^9 p^9}{m^9} - \frac{36i\hbar\Delta t^8 p^7}{m^8} + \frac{378\hbar^2\Delta t^7 p^5}{m^7} + \frac{1260i\hbar^3\Delta t^6 p^3}{m^6} - \frac{945\hbar^4\Delta t^5 p}{m^5}\Big) \\
&+ \Big(-\frac{m^2}{17280\hbar^2\Delta t^8 c^6} + \frac{im}{3628800\hbar\Delta t^9 c^8} - \frac{11imp^2}{13824\hbar^3\Delta t^7 c^6} - \frac{319p^2}{2419200\hbar^2\Delta t^8 c^8} - \frac{233ip^4}{276480\hbar^3 m\Delta t^7 c^8} + \frac{5p^6}{9216\hbar^4 m^2\Delta t^6 c^8}\Big) \\
&\quad \Big(\frac{\Delta t^{10} p^{10}}{m^{10}} + \frac{45i\hbar\Delta t^9 p^8}{m^9} - \frac{630\hbar^2\Delta t^8 p^6}{m^8} - \frac{3150i\hbar^3\Delta t^7 p^4}{m^7} + \frac{4725\hbar^4\Delta t^6 p^2}{m^6} + \frac{945i\hbar^5\Delta t^5}{m^5}\Big) \\
&+ \Big(-\frac{im^2 p}{6912\hbar^3\Delta t^8 c^6} - \frac{29mp}{1209600\hbar^2\Delta t^9 c^8} - \frac{7ip^3}{17280\hbar^3\Delta t^8 c^8} + \frac{17p^5}{41472\hbar^4 m\Delta t^7 c^8}\Big)\Big(-\frac{\Delta t^{11} p^{11}}{m^{11}} - \frac{55i\hbar\Delta t^{10} p^9}{m^{10}} + \frac{990\hbar^2\Delta t^9 p^7}{m^9} \\
&\quad + \frac{6930i\hbar^3\Delta t^8 p^5}{m^8} - \frac{17325\hbar^4\Delta t^7 p^3}{m^7} - \frac{10395i\hbar^5\Delta t^6 p}{m^6}\Big) \\
&+ \Big(-\frac{im^3}{82944\hbar^3\Delta t^9 c^6} - \frac{29m^2}{14515200\hbar^2\Delta t^{10} c^8} - \frac{91imp^2}{829440\hbar^3\Delta t^9 c^8} + \frac{203p^4}{995328\hbar^4\Delta t^8 c^8}\Big)\Big(\frac{\Delta t^{12} p^{12}}{m^{12}} + \frac{66i\hbar\Delta t^{11} p^{10}}{m^{11}} - \frac{1485\hbar^2\Delta t^{10} p^8}{m^{10}} \\
&\quad - \frac{13860i\hbar^3\Delta t^9 p^6}{m^9} + \frac{51975\hbar^4\Delta t^8 p^4}{m^8} + \frac{62370i\hbar^5\Delta t^7 p^2}{m^7} - \frac{10395\hbar^6\Delta t^6}{m^6}\Big) \\
&+ \Big(-\frac{7im^2 p}{414720\hbar^3\Delta t^{10} c^8} + \frac{17mp^3}{248832\hbar^4\Delta t^9 c^8}\Big)\Big(-\frac{\Delta t^{13} p^{13}}{m^{13}} - \frac{78i\hbar\Delta t^{12} p^{11}}{m^{12}} + \frac{2145\hbar^2\Delta t^{11} p^9}{m^{11}} + \frac{25740i\hbar^3\Delta t^{10} p^7}{m^{10}} - \frac{135135\hbar^4\Delta t^9 p^5}{m^9} \\
&\quad - \frac{270270i\hbar^5\Delta t^8 p^3}{m^8} + \frac{135135\hbar^6\Delta t^7 p}{m^7}\Big) \\
&+ \Big(-\frac{im^3}{829440\hbar^3\Delta t^{11} c^8} + \frac{5m^2 p^2}{331776\hbar^4\Delta t^{10} c^8}\Big)\Big(\frac{\Delta t^{14} p^{14}}{m^{14}} + \frac{91i\hbar\Delta t^{13} p^{12}}{m^{13}} - \frac{3003\hbar^2\Delta t^{12} p^{10}}{m^{12}} - \frac{45045i\hbar^3\Delta t^{11} p^8}{m^{11}} + \frac{315315\hbar^4\Delta t^{10} p^6}{m^{10}} \\
&\quad + \frac{945945i\hbar^5\Delta t^9 p^4}{m^9} - \frac{945945\hbar^6\Delta t^8 p^2}{m^8} - \frac{135135i\hbar^7\Delta t^7}{m^7}\Big) \\
&+ \frac{m^3 p}{497664\hbar^4\Delta t^{11} c^8}\Big(-\frac{\Delta t^{15} p^{15}}{m^{15}} - \frac{105i\hbar\Delta t^{14} p^{13}}{m^{14}} + \frac{4095\hbar^2\Delta t^{13} p^{11}}{m^{13}} + \frac{75075i\hbar^3\Delta t^{12} p^9}{m^{12}} - \frac{675675\hbar^4\Delta t^{11} p^7}{m^{11}} - \frac{2837835i\hbar^5\Delta t^{10} p^5}{m^{10}} \\
&\quad + \frac{4729725\hbar^6\Delta t^9 p^3}{m^9} + \frac{2027025i\hbar^7\Delta t^8 p}{m^8}\Big) \\
&+ \frac{m^4}{7962624\hbar^4\Delta t^{12} c^8}\Big(\frac{\Delta t^{16} p^{16}}{m^{16}} + \frac{120i\hbar\Delta t^{15} p^{14}}{m^{15}} - \frac{5460\hbar^2\Delta t^{14} p^{12}}{m^{14}} - \frac{120120i\hbar^3\Delta t^{13} p^{10}}{m^{13}} + \frac{1351350\hbar^4\Delta t^{12} p^8}{m^{12}} + \frac{7567560i\hbar^5\Delta t^{11} p^6}{m^{11}} \\
&\quad - \frac{18918900\hbar^6\Delta t^{10} p^4}{m^{10}} - \frac{16216200i\hbar^7\Delta t^9 p^2}{m^9} + \frac{2027025\hbar^8\Delta t^8}{m^8}\Big) \\
&+ O\Big(\frac{1}{c^{10}}\Big)\Big)
\end{aligned}
$$

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\begin{aligned}
= \;& K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\,e^{-\frac{i\Delta t p^2}{2\hbar m}}\Big(1 \\
&+ \Big(-\frac{p^2}{4m^2c^2}+\frac{i\Delta t p^4}{4\hbar m^3c^2}+\frac{p^4}{16m^4c^4}-\frac{i\Delta t p^6}{16\hbar m^5c^4}-\frac{p^6}{32m^6c^6}+\frac{i\Delta t p^8}{32\hbar m^7c^6}+\frac{5p^8}{256m^8c^8}-\frac{5i\Delta t p^{10}}{256\hbar m^9c^8}\Big) \\
&+ \Big(\frac{p^2}{2m^2c^2}-\frac{i\Delta t p^4}{6\hbar m^3c^2}\Big) \\
&+ \Big(-\frac{i\hbar}{8m\Delta t c^2}-\frac{p^2}{4m^2c^2}-\frac{i\hbar p^2}{16m^3\Delta t c^4}+\frac{i\Delta t p^4}{24\hbar m^3c^2}-\frac{p^4}{32m^4c^4}+\frac{i\hbar p^4}{64m^5\Delta t c^6}-\frac{i\Delta t p^6}{6\hbar m^5c^4}-\frac{p^6}{64m^6c^6}-\frac{i\hbar p^6}{128m^7\Delta t c^8}-\frac{\Delta t^2p^8}{32\hbar^2m^6c^4} \\
&\quad +\frac{17i\Delta t p^8}{192\hbar m^7c^6}+\frac{7p^8}{512m^8c^8}+\frac{\Delta t^2p^{10}}{64\hbar^2m^8c^6}-\frac{43i\Delta t p^{10}}{768\hbar m^9c^8}-\frac{5\Delta t^2p^{12}}{512\hbar^2m^{10}c^8}\Big) \\
&+ \Big(\frac{i\hbar p^2}{8m^3\Delta t c^4}-\frac{13p^4}{24m^4c^4}+\frac{49i\Delta t p^6}{120\hbar m^5c^4}+\frac{5p^6}{32m^6c^6}+\frac{\Delta t^2p^8}{24\hbar^2m^6c^4}-\frac{5i\Delta t p^8}{48\hbar m^7c^6}-\frac{5p^8}{64m^8c^8}-\frac{\Delta t^2p^{10}}{96\hbar^2m^8c^6}+\frac{5i\Delta t p^{10}}{96\hbar m^9c^8}+\frac{\Delta t^2p^{12}}{192\hbar^2m^{10}c^8}\Big) \\
&+ \Big(\frac{\hbar^2}{48m^2\Delta t^2c^4}+\frac{29i\hbar p^2}{96m^3\Delta t c^4}+\frac{\hbar^2p^2}{96m^4\Delta t^2c^6}+\frac{103p^4}{96m^4c^4}+\frac{i\hbar p^4}{128m^5\Delta t c^6}-\frac{\hbar^2p^4}{384m^6\Delta t^2c^8}-\frac{523i\Delta t p^6}{1440\hbar m^5c^4}+\frac{13p^6}{192m^6c^6}-\frac{3i\hbar p^6}{256m^7\Delta t c^8} \\
&\quad -\frac{7\Delta t^2p^8}{288\hbar^2m^6c^4}+\frac{227i\Delta t p^8}{2880\hbar m^7c^6}-\frac{41p^8}{1536m^8c^8}+\frac{7\Delta t^2p^{10}}{192\hbar^2m^8c^6}-\frac{1579i\Delta t p^{10}}{23040\hbar m^9c^8}-\frac{i\Delta t^3p^{12}}{384\hbar^3m^9c^6}-\frac{43\Delta t^2p^{12}}{1536\hbar^2m^{10}c^8}+\frac{i\Delta t^3p^{14}}{512\hbar^3m^{11}c^8}\Big) \\
&+ \Big(-\frac{35i\hbar p^2}{48m^3\Delta t c^4}-\frac{\hbar^2p^2}{48m^4\Delta t^2c^6}-\frac{35p^4}{48m^4c^4}-\frac{9i\hbar p^4}{16m^5\Delta t c^6}+\frac{7i\Delta t p^6}{48\hbar m^5c^4}-\frac{31p^6}{960m^6c^6}+\frac{7i\hbar p^6}{48m^7\Delta t c^8}+\frac{\Delta t^2p^8}{144\hbar^2m^6c^4}-\frac{8677i\Delta t p^8}{20160\hbar m^7c^6} \\
&\quad -\frac{49p^8}{384m^8c^8}-\frac{299\Delta t^2p^{10}}{2880\hbar^2m^8c^6}+\frac{469i\Delta t p^{10}}{1920\hbar m^9c^8}+\frac{i\Delta t^3p^{12}}{192\hbar^3m^9c^6}+\frac{307\Delta t^2p^{12}}{5760\hbar^2m^{10}c^8}-\frac{i\Delta t^3p^{14}}{384\hbar^3m^{11}c^8}\Big) \\
&+ \Big(-\frac{35\hbar^2}{384m^2\Delta t^2c^4}+\frac{i\hbar^3}{384m^3\Delta t^3c^6}+\frac{35i\hbar p^2}{96m^3\Delta t c^4}-\frac{101\hbar^2p^2}{384m^4\Delta t^2c^6}+\frac{i\hbar^3p^2}{768m^5\Delta t^3c^8}+\frac{35p^4}{192m^4c^4}+\frac{451i\hbar p^4}{768m^5\Delta t c^6}-\frac{\hbar^2p^4}{256m^6\Delta t^2c^8} \\
&\quad -\frac{7i\Delta t p^6}{288\hbar m^5c^4}-\frac{2101p^6}{1440m^6c^6}+\frac{i\hbar p^6}{8m^7\Delta t c^8}-\frac{\Delta t^2p^8}{1152\hbar^2m^6c^4}+\frac{18743i\Delta t p^8}{20160\hbar m^7c^6}+\frac{36823p^8}{92160m^8c^8}+\frac{151\Delta t^2p^{10}}{1152\hbar^2m^8c^6}-\frac{907i\Delta t p^{10}}{3584\hbar m^9c^8} \\
&\quad -\frac{11i\Delta t^3p^{12}}{2304\hbar^3m^9c^6}-\frac{2699\Delta t^2p^{12}}{46080\hbar^2m^{10}c^8}+\frac{25i\Delta t^3p^{14}}{4608\hbar^3m^{11}c^8}+\frac{\Delta t^4p^{16}}{6144\hbar^4m^{12}c^8}\Big) \\
&+ \Big(\frac{35\hbar^2p^2}{64m^4\Delta t^2c^6}-\frac{i\hbar^3p^2}{384m^5\Delta t^3c^8}+\frac{105i\hbar p^4}{64m^5\Delta t c^6}+\frac{365\hbar^2p^4}{1152m^6\Delta t^2c^8}+\frac{847p^6}{288m^6c^6}+\frac{221i\hbar p^6}{960m^7\Delta t c^8}-\frac{89i\Delta t p^8}{96\hbar m^7c^6}+\frac{27161p^8}{80640m^8c^8} \\
&\quad -\frac{155\Delta t^2p^{10}}{1728\hbar^2m^8c^6}+\frac{107621i\Delta t p^{10}}{362880\hbar m^9c^8}+\frac{13i\Delta t^3p^{12}}{5184\hbar^3m^9c^6}+\frac{8431\Delta t^2p^{12}}{60480\hbar^2m^{10}c^8}-\frac{43i\Delta t^3p^{14}}{2880\hbar^3m^{11}c^8}-\frac{\Delta t^4p^{16}}{2304\hbar^4m^{12}c^8}\Big) \\
&+ \Big(-\frac{7i\hbar^3}{128m^3\Delta t^3c^6}-\frac{\hbar^4}{3840m^4\Delta t^4c^8}+\frac{245\hbar^2p^2}{512m^4\Delta t^2c^6}-\frac{947i\hbar^3p^2}{7680m^5\Delta t^3c^8}-\frac{5495i\hbar p^4}{1536m^5\Delta t c^6}+\frac{3211\hbar^2p^4}{18432m^6\Delta t^2c^8}-\frac{1897p^6}{768m^6c^6} \\
&\quad -\frac{281461i\hbar p^6}{92160m^7\Delta t c^8}+\frac{383i\Delta t p^8}{768\hbar m^7c^6}-\frac{163p^8}{20160m^8c^8}+\frac{2471\Delta t^2p^{10}}{69120\hbar^2m^8c^6}-\frac{17185901i\Delta t p^{10}}{14515200\hbar m^9c^8}-\frac{11i\Delta t^3p^{12}}{13824\hbar^3m^9c^6}-\frac{2941801\Delta t^2p^{12}}{9676800\hbar^2m^{10}c^8} \\
&\quad +\frac{6517i\Delta t^3p^{14}}{276480\hbar^3m^{11}c^8}+\frac{5\Delta t^4p^{16}}{9216\hbar^4m^{12}c^8}\Big) \\
&+ \Big(-\frac{385\hbar^2p^2}{256m^4\Delta t^2c^6}+\frac{319i\hbar^3p^2}{1280m^5\Delta t^3c^8}+\frac{1925i\hbar p^4}{768m^5\Delta t c^6}-\frac{2915\hbar^2p^4}{768m^6\Delta t^2c^8}+\frac{385p^6}{384m^6c^6}+\frac{6633i\hbar p^6}{2560m^7\Delta t c^8}-\frac{55i\Delta t p^8}{384\hbar m^7c^6}-\frac{696443p^8}{161280m^8c^8} \\
&\quad -\frac{55\Delta t^2p^{10}}{6912\hbar^2m^8c^6}+\frac{147631i\Delta t p^{10}}{60480\hbar m^9c^8}+\frac{i\Delta t^3p^{12}}{6912\hbar^3m^9c^6}+\frac{231977\Delta t^2p^{12}}{604800\hbar^2m^{10}c^8}-\frac{4591i\Delta t^3p^{14}}{207360\hbar^3m^{11}c^8}-\frac{17\Delta t^4p^{16}}{41472\hbar^4m^{12}c^8}\Big) \\
&+ \Big(\frac{385i\hbar^3}{3072m^3\Delta t^3c^6}+\frac{319\hbar^4}{15360m^4\Delta t^4c^8}+\frac{385\hbar^2p^2}{512m^4\Delta t^2c^6}+\frac{31207i\hbar^3p^2}{30720m^5\Delta t^3c^8}-\frac{1925i\hbar p^4}{3072m^5\Delta t c^6}+\frac{170269\hbar^2p^4}{36864m^6\Delta t^2c^8}-\frac{385p^6}{2304m^6c^6} \\
&\quad +\frac{81169i\hbar p^6}{11520m^7\Delta t c^8}+\frac{55i\Delta t p^8}{3072\hbar m^7c^6}+\frac{3906331p^8}{430080m^8c^8}+\frac{11\Delta t^2p^{10}}{13824\hbar^2m^8c^6}-\frac{25778951i\Delta t p^{10}}{9676800\hbar m^9c^8}-\frac{i\Delta t^3p^{12}}{82944\hbar^3m^9c^6}-\frac{17164571\Delta t^2p^{12}}{58060800\hbar^2m^{10}c^8} \\
&\quad +\frac{5537i\Delta t^3p^{14}}{414720\hbar^3m^{11}c^8}+\frac{203\Delta t^4p^{16}}{995328\hbar^4m^{12}c^8}\Big) \\
&+ \Big(-\frac{7007i\hbar^3p^2}{3072m^5\Delta t^3c^8}+\frac{43043\hbar^2p^4}{9216m^6\Delta t^2c^8}-\frac{149149i\hbar p^6}{9216m^7\Delta t c^8}-\frac{9009p^8}{1024m^8c^8}+\frac{5291i\Delta t p^{10}}{3072\hbar m^9c^8}+\frac{60229\Delta t^2p^{12}}{414720\hbar^2m^{10}c^8}-\frac{2203i\Delta t^3p^{14}}{414720\hbar^3m^{11}c^8} \\
&\quad -\frac{17\Delta t^4p^{16}}{248832\hbar^4m^{12}c^8}\Big) \\
&+ \Big(-\frac{1001\hbar^4}{6144m^4\Delta t^4c^8}-\frac{11011i\hbar^3p^2}{12288m^5\Delta t^3c^8}-\frac{161161\hbar^2p^4}{12288m^6\Delta t^2c^8}+\frac{511511i\hbar p^6}{36864m^7\Delta t c^8}+\frac{173173p^8}{36864m^8c^8}-\frac{373373i\Delta t p^{10}}{552960\hbar m^9c^8}-\frac{74893\Delta t^2p^{12}}{1658880\hbar^2m^{10}c^8} \\
&\quad +\frac{2273i\Delta t^3p^{14}}{1658880\hbar^3m^{11}c^8}+\frac{5\Delta t^4p^{16}}{331776\hbar^4m^{12}c^8}\Big) \\
&+ \Big(\frac{25025i\hbar^3p^2}{6144m^5\Delta t^3c^8}+\frac{175175\hbar^2p^4}{18432m^6\Delta t^2c^8}-\frac{35035i\hbar p^6}{6144m^7\Delta t c^8}-\frac{25025p^8}{18432m^8c^8}+\frac{25025i\Delta t p^{10}}{165888\hbar m^9c^8}+\frac{455\Delta t^2p^{12}}{55296\hbar^2m^{10}c^8}-\frac{35i\Delta t^3p^{14}}{165888\hbar^3m^{11}c^8} \\
&\quad -\frac{\Delta t^4p^{16}}{497664\hbar^4m^{12}c^8}\Big) \\
&+ \Big(\frac{25025\hbar^4}{98304m^4\Delta t^4c^8}-\frac{25025i\hbar^3p^2}{12288m^5\Delta t^3c^8}-\frac{175175\hbar^2p^4}{73728m^6\Delta t^2c^8}+\frac{35035i\hbar p^6}{36864m^7\Delta t c^8}+\frac{25025p^8}{147456m^8c^8}-\frac{5005i\Delta t p^{10}}{331776\hbar m^9c^8}-\frac{455\Delta t^2p^{12}}{663552\hbar^2m^{10}c^8} \\
&\quad +\frac{5i\Delta t^3p^{14}}{331776\hbar^3m^{11}c^8}+\frac{\Delta t^4p^{16}}{7962624\hbar^4m^{12}c^8}\Big) \\
&+ O\Big(\frac{1}{c^{10}}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \ & K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\,e^{-\frac{i\Delta t p^2}{2\hbar m}}\Big(1 \ - \ \frac{i\hbar}{8m\Delta tc^2} \ + \ \Big(\frac{1}{48}-\frac{35}{384}\Big)\frac{\hbar^2}{m^2\Delta t^2c^4} \ + \ \Big(\frac{1}{384}-\frac{7}{128}+\frac{385}{3072}\Big)\frac{i\hbar^3}{m^3\Delta t^3c^6} \\
& + \ \Big(-\frac{1}{3840}+\frac{319}{15360}-\frac{1001}{6144}+\frac{25025}{98304}\Big)\frac{\hbar^4}{m^4\Delta t^4c^8} \ + \ \Big(-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\Big)\frac{p^2}{m^2c^2} \ + \ \Big(-\frac{1}{16}+\frac{1}{8}+\frac{29}{96}-\frac{35}{48}+\frac{35}{96}\Big)\frac{i\hbar p^2}{m^3\Delta tc^4} \\
& + \ \Big(\frac{1}{96}-\frac{1}{48}-\frac{101}{384}+\frac{35}{64}+\frac{245}{512}-\frac{385}{256}+\frac{385}{512}\Big)\frac{\hbar^2p^2}{m^4\Delta t^2c^6} \\
& + \ \Big(\frac{1}{768}-\frac{1}{384}-\frac{947}{7680}+\frac{319}{1280}+\frac{31207}{30720}-\frac{7007}{3072}-\frac{11011}{12288}+\frac{25025}{6144}-\frac{25025}{12288}\Big)\frac{i\hbar^3p^2}{m^5\Delta t^3c^8} \\
& + \ \Big(\frac{1}{4}-\frac{1}{6}+\frac{1}{24}\Big)\frac{i\Delta tp^4}{\hbar m^3c^2} \ + \ \Big(\frac{1}{16}-\frac{1}{32}-\frac{13}{24}+\frac{103}{96}-\frac{35}{48}+\frac{35}{192}\Big)\frac{p^4}{m^4c^4} \\
& + \ \Big(\frac{1}{64}+\frac{1}{128}-\frac{9}{16}+\frac{451}{768}+\frac{105}{64}-\frac{5495}{1536}+\frac{1925}{768}-\frac{1925}{3072}\Big)\frac{i\hbar p^4}{m^5\Delta tc^6} \\
& + \ \Big(-\frac{1}{384}-\frac{1}{256}+\frac{365}{1152}+\frac{3211}{18432}-\frac{2915}{768}+\frac{170269}{36864}+\frac{43043}{9216}-\frac{161161}{12288}+\frac{175175}{18432}-\frac{175175}{73728}\Big)\frac{\hbar^2p^4}{m^6\Delta t^2c^8} \\
& + \ \Big(-\frac{1}{16}-\frac{1}{6}+\frac{49}{120}-\frac{523}{1440}+\frac{7}{48}-\frac{7}{288}\Big)\frac{i\Delta tp^6}{\hbar m^5c^4} \\
& + \ \Big(-\frac{1}{32}-\frac{1}{64}+\frac{5}{32}+\frac{13}{192}-\frac{31}{960}-\frac{2101}{1440}+\frac{847}{288}-\frac{1897}{768}+\frac{385}{384}-\frac{385}{2304}\Big)\frac{p^6}{m^6c^6} \\
& + \ \Big(-\frac{1}{128}-\frac{3}{256}+\frac{7}{48}+\frac{1}{8}+\frac{221}{960}-\frac{281461}{92160}+\frac{6633}{2560}+\frac{81169}{11520}-\frac{149149}{9216}+\frac{511511}{36864}-\frac{35035}{6144}+\frac{35035}{36864}\Big)\frac{i\hbar p^6}{m^7\Delta tc^8} \\
& + \ \Big(-\frac{1}{32}+\frac{1}{24}-\frac{7}{288}+\frac{1}{144}-\frac{1}{1152}\Big)\frac{\Delta t^2p^8}{\hbar^2m^6c^4} \\
& + \ \Big(\frac{1}{32}+\frac{17}{192}-\frac{5}{48}+\frac{227}{2880}-\frac{8677}{20160}+\frac{18743}{20160}-\frac{89}{96}+\frac{383}{768}-\frac{55}{384}+\frac{55}{3072}\Big)\frac{i\Delta tp^8}{\hbar m^7c^6} \\
& + \ \Big(\frac{5}{256}+\frac{7}{512}-\frac{5}{64}-\frac{41}{1536}-\frac{49}{384}+\frac{36823}{92160}+\frac{27161}{80640}-\frac{163}{20160}-\frac{696443}{161280}+\frac{3906331}{430080}-\frac{9009}{1024}+\frac{173173}{36864}-\frac{25025}{18432}+\frac{25025}{147456}\Big)\frac{p^8}{m^8c^8} \\
& + \ \Big(\frac{1}{64}-\frac{1}{96}+\frac{7}{192}-\frac{299}{2880}+\frac{151}{1152}-\frac{155}{1728}+\frac{2471}{69120}-\frac{55}{6912}+\frac{11}{13824}\Big)\frac{\Delta t^2p^{10}}{\hbar^2m^8c^6} \\
& + \ \Big(-\frac{5}{256}-\frac{43}{768}+\frac{5}{96}-\frac{1579}{23040}+\frac{469}{1920}-\frac{907}{3584}+\frac{107621}{362880}-\frac{17185901}{14515200}+\frac{147631}{60480}-\frac{25778951}{9676800}+\frac{5291}{3072}-\frac{373373}{552960}+\frac{25025}{165888} \\
& \quad -\frac{5005}{331776}\Big)\frac{i\Delta tp^{10}}{\hbar m^9c^8} \ + \ \Big(-\frac{1}{384}+\frac{1}{192}-\frac{11}{2304}+\frac{13}{5184}-\frac{11}{13824}+\frac{1}{6912}-\frac{1}{82944}\Big)\frac{i\Delta t^3p^{12}}{\hbar^3m^9c^6} \\
& + \ \Big(-\frac{5}{512}+\frac{1}{192}-\frac{43}{1536}+\frac{307}{5760}-\frac{2699}{46080}+\frac{8431}{60480}-\frac{2941801}{9676800}+\frac{231977}{604800}-\frac{17164571}{58060800}+\frac{60229}{414720}-\frac{74893}{1658880}+\frac{455}{55296} \\
& \quad -\frac{455}{663552}\Big)\frac{\Delta t^2p^{12}}{\hbar^2m^{10}c^8} \\
& + \ \Big(\frac{1}{512}-\frac{1}{384}+\frac{25}{4608}-\frac{43}{2880}+\frac{6517}{276480}-\frac{4591}{207360}+\frac{5537}{414720}-\frac{2203}{414720}+\frac{2273}{1658880}-\frac{35}{165888}+\frac{5}{331776}\Big)\frac{i\Delta t^3p^{14}}{\hbar^3m^{11}c^8} \\
& + \ \Big(\frac{1}{6144}-\frac{1}{2304}+\frac{5}{9216}-\frac{17}{41472}+\frac{203}{995328}-\frac{17}{248832}+\frac{5}{331776}-\frac{1}{497664}+\frac{1}{7962624}\Big)\frac{\Delta t^4p^{16}}{\hbar^4m^{12}c^8} \ + \ O\Big(\frac{1}{c^{10}}\Big)\Big)
\end{aligned}
$$

$$
\begin{aligned}
= \ & K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}\,e^{-\frac{i\Delta t p^2}{2\hbar m}}\Big(1 \ - \ \frac{i\hbar}{8m\Delta tc^2} \ - \ \frac{9\hbar^2}{128m^2\Delta t^2c^4} \ + \ \frac{75i\hbar^3}{1024m^3\Delta t^3c^6} \\
& + \ \frac{3675\hbar^4}{32768m^4\Delta t^4c^8} \ + \ \Big(\frac{i\Delta t}{8\hbar m^3c^2}+\frac{1}{64m^4c^4}-\frac{9i\hbar}{1024m^5\Delta tc^6}-\frac{75\hbar^2}{8192m^6\Delta t^2c^8}\Big)p^4 \\
& + \ \Big(-\frac{i\Delta t}{16\hbar m^5c^4}-\frac{1}{128m^6c^6}+\frac{9i\hbar}{2048m^7\Delta tc^8}\Big)p^6 \\
& + \Big(-\frac{\Delta t^2}{128\hbar^2m^6c^4}+\frac{41i\Delta t}{1024\hbar m^7c^6}+\frac{89}{16384m^8c^8}\Big)p^8 \\
& + \Big(\frac{\Delta t^2}{128\hbar^2m^8c^6}-\frac{29i\Delta t}{1024\hbar m^9c^8}\Big)p^{10} \ + \ \Big(-\frac{i\Delta t^3}{3072\hbar^3m^9c^6}-\frac{169\Delta t^2}{24576\hbar^2m^{10}c^8}\Big)p^{12} \\
& + \frac{i\Delta t^3p^{14}}{2048\hbar^3m^{11}c^8} \ + \ \frac{\Delta t^4p^{16}}{98304\hbar^4m^{12}c^8} \ + \ O\Big(\frac{1}{c^{10}}\Big)\Big)
\end{aligned}
$$

$$= K(\Delta t)\sqrt{\frac{2\pi i\hbar\Delta t}{m}}e^{-\frac{i\Delta tp^2}{2\hbar m}}\Big(1 - \frac{i\hbar}{8m\Delta tc^2} - \frac{9\hbar^2}{128m^2\Delta t^2c^4} + \frac{75i\hbar^3}{1024m^3\Delta t^3c^6}$$
$$+ \frac{3675\hbar^4}{32768m^4\Delta t^4c^8}\Big)\Big(1 + \frac{i\Delta tp^4}{8\hbar m^3c^2} - \frac{i\Delta tp^6}{16\hbar m^5c^4}$$
$$+ \Big(-\frac{\Delta t^2}{128\hbar^2m^6c^4} + \frac{5i\Delta t}{128\hbar m^7c^6}\Big)p^8 + \Big(\frac{\Delta t^2}{128\hbar^2m^8c^6} - \frac{7i\Delta t}{256\hbar m^9c^8}\Big)p^{10}$$
$$+ \Big(-\frac{i\Delta t^3}{3072\hbar^3m^9c^6} - \frac{7\Delta t^2}{1024\hbar^2m^{10}c^8}\Big)p^{12} + \frac{i\Delta t^3p^{14}}{2048\hbar^3m^{11}c^8} + \frac{\Delta t^4p^{16}}{98304\hbar^4m^{12}c^8}$$
$$+ O\Big(\frac{1}{c^{10}}\Big)\Big).$$

The coefficients of the terms proportional to p^4, p^6, p^8, p^{10}, p^{12}, p^{14} and p^{16} are all the same on both the left and the right sides of Equation (5). The puzzle pieces of the calculation fit together perfectly. In the last step we use the equation

$$\frac{1}{1 - \frac{i\hbar}{8m\Delta tc^2} - \frac{9\hbar^2}{128m^2\Delta t^2c^4} + \frac{75i\hbar^3}{1024m^3\Delta t^3c^6} + \frac{3675\hbar^4}{32768m^4\Delta t^4c^8}}$$
$$= 1 + \frac{i\hbar}{8m\Delta tc^2} + \frac{7\hbar^2}{128m^2\Delta t^2c^4} - \frac{59i\hbar^3}{1024m^3\Delta t^3c^6} - \frac{3265\hbar^4}{32768m^4\Delta t^4c^8} + O\Big(\frac{1}{c^{10}}\Big),$$

and there the term proportional to p^4 gets its coefficient right because

$$\Big(1 + \frac{i\hbar}{8m\Delta tc^2} + \frac{7\hbar^2}{128m^2\Delta t^2c^4} - \frac{59i\hbar^3}{1024m^3\Delta t^3c^6} + O\Big(\frac{1}{c^8}\Big)\Big)$$
$$\Big(\frac{i\Delta t}{8\hbar m^3c^2} + \frac{1}{64m^4c^4} - \frac{9i\hbar}{1024m^5\Delta tc^6} - \frac{75\hbar^2}{8192m^6\Delta t^2c^8}\Big) = \frac{i\Delta t}{8\hbar m^3c^2} + O\Big(\frac{1}{c^{10}}\Big),$$

the term proportional to p^6 gets it coefficient right because

$$\Big(1 + \frac{i\hbar}{8m\Delta tc^2} + \frac{7\hbar^2}{128m^2\Delta t^2c^4} + O\Big(\frac{1}{c^6}\Big)\Big)$$
$$\Big(-\frac{i\Delta t}{16\hbar m^5c^4} - \frac{1}{128m^6c^6} + \frac{9i\hbar}{2048m^7\Delta tc^8}\Big) = -\frac{i\Delta t}{16\hbar m^5c^4} + O\Big(\frac{1}{c^{10}}\Big),$$

the term proportional to p^8 gets it coefficient right because

$$\Big(1 + \frac{i\hbar}{8m\Delta tc^2} + \frac{7\hbar^2}{128m^2\Delta t^2c^4} + O\Big(\frac{1}{c^6}\Big)\Big)$$
$$\Big(-\frac{\Delta t^2}{128\hbar^2m^6c^4} + \frac{41i\Delta t}{1024\hbar m^7c^6} + \frac{89}{16384m^8c^8}\Big)$$
$$= -\frac{\Delta t^2}{128\hbar^2m^6c^4} + \frac{5i\Delta t}{128\hbar m^7c^6} + O\Big(\frac{1}{c^{10}}\Big),$$

the term proportional to p^{10} gets it coefficient right because

$$\Big(1+\frac{i\hbar}{8m\Delta tc^2}+O\Big(\frac{1}{c^4}\Big)\Big)\Big(\frac{\Delta t^2}{128\hbar^2m^8c^6}-\frac{29i\Delta t}{1024\hbar m^9c^8}\Big) = \frac{\Delta t^2}{128\hbar^2m^8c^6} - \frac{7i\Delta t}{256\hbar m^9c^8} + O\Big(\frac{1}{c^{10}}\Big)$$

and the term proportional to p^{12} gets it coefficient right because

$$\Big(1+\frac{i\hbar}{8m\Delta tc^2}+O\Big(\frac{1}{c^4}\Big)\Big)\Big(-\frac{i\Delta t^3}{3072\hbar^3m^9c^6}-\frac{169\Delta t^2}{24576\hbar^2m^{10}c^8}\Big) = -\frac{i\Delta t^3}{3072\hbar^3m^9c^6} - \frac{7\Delta t^2}{1024\hbar^2m^{10}c^8} + O\Big(\frac{1}{c^{10}}\Big).$$

All we have to do is to set

$$K(\Delta t) = \frac{\sqrt{\frac{m}{2\pi i\hbar\Delta t}}e^{-\frac{i\Delta tmc^2}{\hbar}}}{1-\frac{i\hbar}{8m\Delta tc^2}-\frac{9\hbar^2}{128m^2\Delta t^2c^4}+\frac{75i\hbar^3}{1024m^3\Delta t^3c^6}+\frac{3675\hbar^4}{32768m^4\Delta t^4c^8}},$$

and then Equation (5) is true.

If somebody is interested in repeating the calculation, he or she will probably find it smart to check the formulas

$$\Big(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\Big)^2 = -\frac{p^4\xi^4}{16\hbar^2m^2\Delta t^2}-\frac{p^3\xi^5}{12\hbar^2m\Delta t^3}-\frac{7p^2\xi^6}{144\hbar^2\Delta t^4}-\frac{mp\xi^7}{72\hbar^2\Delta t^5}-\frac{m^2\xi^8}{576\hbar^2\Delta t^6},$$

$$\Big(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\Big)^3 = -\frac{ip^6\xi^6}{64\hbar^3m^3\Delta t^3}-\frac{ip^5\xi^7}{32\hbar^3m^2\Delta t^4}-\frac{11ip^4\xi^8}{384\hbar^3m\Delta t^5}-\frac{13ip^3\xi^9}{864\hbar^3\Delta t^6}-\frac{11imp^2\xi^{10}}{2304\hbar^3\Delta t^7} -\frac{im^2p\xi^{11}}{1152\hbar^3\Delta t^8}-\frac{im^3\xi^{12}}{13824\hbar^3\Delta t^9},$$

$$\Big(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\Big)^4 = \frac{p^8\xi^8}{256\hbar^4m^4\Delta t^4}+\frac{p^7\xi^9}{96\hbar^4m^3\Delta t^5}+\frac{5p^6\xi^{10}}{384\hbar^4m^2\Delta t^6}+\frac{17p^5\xi^{11}}{1728\hbar^4m\Delta t^7}+\frac{203p^4\xi^{12}}{41472\hbar^4\Delta t^8} +\frac{17mp^3\xi^{13}}{10368\hbar^4\Delta t^9}+\frac{5m^2p^2\xi^{14}}{13824\hbar^4\Delta t^{10}}+\frac{m^3p\xi^{15}}{20736\hbar^4\Delta t^{11}}+\frac{m^4\xi^{16}}{331776\hbar^4\Delta t^{12}},$$

$$\Big(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t}+\frac{ip^2\xi^4}{48\hbar m\Delta t^3}+\frac{ip\xi^5}{120\hbar\Delta t^4}+\frac{im\xi^6}{720\hbar\Delta t^5}\Big)^2 = -\frac{p^8\xi^4}{256\hbar^2m^6\Delta t^2}+\frac{p^6\xi^6}{384\hbar^2m^4\Delta t^4}+\frac{p^5\xi^7}{960\hbar^2m^3\Delta t^5}-\frac{p^4\xi^8}{3840\hbar^2m^2\Delta t^6} -\frac{p^3\xi^9}{2880\hbar^2m\Delta t^7}-\frac{11p^2\xi^{10}}{86400\hbar^2\Delta t^8}-\frac{mp\xi^{11}}{43200\hbar^2\Delta t^9}-\frac{m^2\xi^{12}}{518400\hbar^2\Delta t^{10}},$$

$$\Big(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\Big)\Big(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t}+\frac{ip^2\xi^4}{48\hbar m\Delta t^3}+\frac{ip\xi^5}{120\hbar\Delta t^4}+\frac{im\xi^6}{720\hbar\Delta t^5}\Big)$$
$$= \frac{p^6\xi^4}{64\hbar^2m^4\Delta t^2}+\frac{p^5\xi^5}{96\hbar^2m^3\Delta t^3}-\frac{p^4\xi^6}{384\hbar^2m^2\Delta t^4}-\frac{p^3\xi^7}{180\hbar^2m\Delta t^5}-\frac{p^2\xi^8}{384\hbar^2\Delta t^6}$$
$$-\frac{mp\xi^9}{1728\hbar^2\Delta t^7}-\frac{m^2\xi^{10}}{17280\hbar^2\Delta t^8},$$
$$\Big(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\Big)^2\Big(-\frac{ip^4\xi^2}{16\hbar m^3\Delta t}+\frac{ip^2\xi^4}{48\hbar m\Delta t^3}+\frac{ip\xi^5}{120\hbar\Delta t^4}+\frac{im\xi^6}{720\hbar\Delta t^5}\Big)$$
$$= \frac{ip^8\xi^6}{256\hbar^3m^5\Delta t^3}+\frac{ip^7\xi^7}{192\hbar^3m^4\Delta t^4}+\frac{ip^6\xi^8}{576\hbar^3m^3\Delta t^5}-\frac{ip^5\xi^9}{720\hbar^3m^2\Delta t^6}-\frac{233ip^4\xi^{10}}{138240\hbar^3m\Delta t^7}$$
$$-\frac{7ip^3\xi^{11}}{8640\hbar^3\Delta t^8}-\frac{91imp^2\xi^{12}}{414720\hbar^3\Delta t^9}-\frac{7im^2p\xi^{13}}{207360\hbar^3\Delta t^{10}}-\frac{im^3\xi^{14}}{414720\hbar^3\Delta t^{11}},$$
$$\Big(\frac{ip^2\xi^2}{4\hbar m\Delta t}+\frac{ip\xi^3}{6\hbar\Delta t^2}+\frac{im\xi^4}{24\hbar\Delta t^3}\Big)$$
$$\Big(\frac{ip^6\xi^2}{32\hbar m^5\Delta t}-\frac{ip^4\xi^4}{192\hbar m^3\Delta t^3}+\frac{ip^2\xi^6}{1440\hbar m\Delta t^5}+\frac{ip\xi^7}{5040\hbar\Delta t^6}+\frac{im\xi^8}{40320\hbar\Delta t^7}\Big)$$
$$= -\frac{p^8\xi^4}{128\hbar^2m^6\Delta t^2}-\frac{p^7\xi^5}{192\hbar^2m^5\Delta t^3}+\frac{p^5\xi^7}{1152\hbar^2m^3\Delta t^5}+\frac{p^4\xi^8}{23040\hbar^2m^2\Delta t^6}$$
$$-\frac{p^3\xi^9}{6048\hbar^2m\Delta t^7}-\frac{11p^2\xi^{10}}{161280\hbar^2\Delta t^8}-\frac{mp\xi^{11}}{80640\hbar^2\Delta t^9}-\frac{m^2\xi^{12}}{967680\hbar^2\Delta t^{10}},$$
$$\Big(-\frac{i\hbar}{8m\Delta tc^2}-\frac{9\hbar^2}{128m^2\Delta t^2c^4}+\frac{75i\hbar^3}{1024m^3\Delta t^3c^6}+\frac{3675\hbar^4}{32768m^4\Delta t^4c^8}\Big)^2$$
$$= -\frac{\hbar^2}{64m^2\Delta t^2c^4}+\frac{9i\hbar^3}{512m^3\Delta t^3c^6}+\frac{381\hbar^4}{16384m^4\Delta t^4c^8}+O\Big(\frac{1}{c^{10}}\Big),$$
$$\Big(-\frac{i\hbar}{8m\Delta tc^2}-\frac{9\hbar^2}{128m^2\Delta t^2c^4}+\frac{75i\hbar^3}{1024m^3\Delta t^3c^6}+\frac{3675\hbar^4}{32768m^4\Delta t^4c^8}\Big)^3$$
$$= \frac{i\hbar^3}{512m^3\Delta t^3c^6}+\frac{45\hbar^4}{4096m^4\Delta t^4c^8}+O\Big(\frac{1}{c^{10}}\Big)$$

and

$$\Big(-\frac{i\hbar}{8m\Delta tc^2}-\frac{9\hbar^2}{128m^2\Delta t^2c^4}+\frac{75i\hbar^3}{1024m^3\Delta t^3c^6}+\frac{3675\hbar^4}{32768m^4\Delta t^4c^8}\Big)^4$$
$$= \frac{\hbar^4}{4096m^4\Delta t^4c^8}+O\Big(\frac{1}{c^{10}}\Big)$$

as separate exercises.

We should contemplate on what this discovery means. When one fact is that we are using the quantity Δt, because we are interested in the differential operator ∂_t, and another fact is that we are using some kind of approximations, one might think that it must be so that the approximations become very accurate in the limit $\Delta t\to 0$. Let's take a closer look at whether this is the case. Suppose we want to use an approximation

$$\int_{-\infty}^{\infty} e^{a_1x^2+a_2x^4+a_3x^6}dx \approx \int_{-\infty}^{\infty} e^{a_1x^2}\Big(1 + (a_2x^4+a_3x^6) + \frac{1}{2}(a_2x^4+a_3x^6)^2\Big)dx.$$

When does this make sense, and when does it not? To answer the question we do the change of variable $x = \frac{1}{\sqrt{|a_1|}}u$, and write the integral in the form

$$\int_{-\infty}^{\infty} e^{\frac{a_1}{|a_1|}u^2+\frac{a_2}{|a_1|^2}u^4+\frac{a_3}{|a_1|^3}u^6}\frac{1}{\sqrt{|a_1|}}du.$$

From here we see that if the relations $\frac{|a_2|}{|a_1|^2} \lesssim 1$ and $\frac{|a_3|}{|a_1|^3} \lesssim 1$ are true, then the approximation makes sense, and if these relations are not true, then the approximation does not make sense. In our study above we have been using coefficients that satisfy the relations

$$a_1 \propto \frac{1}{\Delta t},$$

$$a_2 \propto \frac{1}{\Delta t^3 c^2}, \qquad a_2 \propto \frac{1}{\Delta t^3 c^4}, \quad \ldots$$

$$a_3 \propto \frac{1}{\Delta t^5 c^4}, \qquad a_3 \propto \frac{1}{\Delta t^5 c^6}, \quad \ldots$$

and so on. So the relevant ratios have been

$$\frac{|a_2|}{|a_1|^2} \propto \frac{1}{\Delta t c^2}, \qquad \frac{|a_2|}{|a_1|^2} \propto \frac{1}{\Delta t c^4}, \quad \ldots$$

$$\frac{|a_3|}{|a_1|^3} \propto \frac{1}{\Delta t^2 c^4}, \qquad \frac{|a_3|}{|a_1|^3} \propto \frac{1}{\Delta t^2 c^6}, \quad \ldots$$

and so on. If we ignore Δt, and only pay attention to c, then from that point of view these ratios seem to be small, because the speed of light c is usually considered to be large. However, if we assume that Δt is small, and take this into account, then we see that these ratios are actually not small. The ratios diverge in the limit $\Delta t \to 0$. This means that the approximations we have been using in the above study have been kind of approximations that are only valid with sufficiently large Δt. The approximations will stop working in the limit $\Delta t \to 0$. This is a peculiar result, and eventually it is not obvious what it means. An obvious attempt to conclude something is that there will probably be some difficulty in turning the integral formulation of the time evolution equation into a differential equation.

Nevertheless, it is extremely unlikely that Equation (5) could be made true with the right choice of $K(\Delta t)$ via a mere coincidence, so it is a safe conclusion that our discovery is probably related to some real result. The problem is that it's just not obvious what that result is. In other words, at this point it is a reasonable conjecture that there exists a function $\Delta t \mapsto K(\Delta t)$ that works so that when the time evolution of a wave function is defined according to the formula

$$\psi(t+\Delta t, x) = K(\Delta t)\int\limits_{-\infty}^{\infty} \psi\Big(t+\Delta t - \Delta t\cosh\Big(\frac{\xi}{c\Delta t}\Big),\; x + c\Delta t\sinh\Big(\frac{\xi}{c\Delta t}\Big)\Big)d\xi,$$

then a relation that looks like

$$i\hbar\partial_t\psi(t,x) \;\text{“} = \text{ or } \approx \text{”}\; \sqrt{(mc^2)^2 - c^2\hbar^2\partial_x^2}\psi(t,x)$$

will be true. A precise description of the function $\Delta t \mapsto K(\Delta t)$ is not yet known, but apparently it has a representation that looks like

$$K(\Delta t) = \frac{\sqrt{\frac{m}{2\pi i\hbar\Delta t}}e^{-\frac{i\Delta t mc^2}{\hbar}}}{1 - \frac{i\hbar}{8m\Delta tc^2} - \frac{9\hbar^2}{128m^2\Delta t^2c^4} + \frac{75i\hbar^3}{1024m^3\Delta t^3c^6} + \frac{3675\hbar^4}{32768m^4\Delta t^4c^8} - \cdots}.$$

When only a finite number of terms of the series are being used, this representation is more useful with large Δt, and less useful with small Δt.

One concern is that the Taylor series of the quantity $\sqrt{(mc^2)^2 + c^2p^2}$ with respect to p converges for $|p| < mc$, and diverges for $|p| > mc$. A simple conclusion from this fact is that if we are interested in studying the relativistic Schrödinger equation, we maybe shouldn't be using the Taylor series of the square root. However, this doesn't necessarily mean that the above calculations with the Taylor series would be nonsense. Sometimes it can happen that first we prove some result in a limited domain, where some series converges, and then the result can be extended outside that domain with an argument using analytic continuation. Since at this point we do not yet know what the precise result of our conjecture is, consequently it is difficult to speculate about how precisely the analytic continuation should be used.

Earlier I uploaded an article *Extreme Oscillation Phenomenon of Relativistic Propagator* [4] to viXra, and there I discussed the paradoxical issue that the solutions of the relativistic Schrödinger equation seem to exhibit small amplitude leaking from outside the past light cone. Our new conjecture discovered here maybe affects the earlier discussion, so we should contemplate on the relations between these phenomena. Let's assume that there is “=” sign in our conjecture. What would this imply? Then we would have two different integral formulations of the solutions to the relativistic Schrödinger equation. One formulation uses the straight line $t' = t$, and the other formulation uses the hyperbola $c^2(t+\Delta t-t')^2-(x-x')^2=c^2\Delta t^2$, and these formulations would be equivalent. If we use the straight line, it looks like that $\psi(t+\Delta t,x)$ depends on the past values of ψ outside the past light cone, which looks bad. The values of ψ outside the past light cone are on the lines $\{t\}\times\,]-\infty,x-c\Delta t]$ and $\{t\}\times[x+c\Delta t,\infty[$. If we use the hyperbola, it looks like that $\psi(t+\Delta t,x)$ is fully determined by what is inside the past light cone, which looks good. Then there is a paradox: Is $\psi(t+\Delta t,x)$ affected by the values of ψ outside the past light cone or not? One possible solution to this paradox is that if we assume that the wave function satisfies the relativistic Schrödinger equation on all past spacetime points $t'\leq t$, then it could be, although it is not obvious whether this is true, that the values of ψ on the lines $\{t\}\times\,]-\infty,x-c\Delta t]$ and $\{t\}\times[x+c\Delta t,\infty[$ will be having been determined by the values of ψ inside the past light cone. So when we write $\psi(t+\Delta t,x)$ in terms of the values of ψ on the lines $\{t\}\times\,]-\infty,x-c\Delta t]$ and $\{t\}\times[x+c\Delta t,\infty[$, it looks bad and suspicious, but it would still be equivalent to $\psi(t+\Delta t,x)$ having been written in terms of the values of ψ inside the past light cone. Maybe the solution to the paradoxes of the relativistic Schrödinger equation is no more complicated than this?

References

[1] https://en.wikipedia.org/wiki/Propagator

[2] *Relativistic Quantum Mechanics*, J. D. Bjorken & S. D. Drell, 1964, McGraw-Hill, Inc.

[3] https://en.wikipedia.org/wiki/Gaussian_integral

[4] *Extreme Oscillation Phenomenon of Relativistic Propagator*, J. S. Puuronen, 2024, viXra:2402.0053