Conditions for convergence of the sequence $\frac{1}{n^u|\sin n|^v}$

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Abstract

It is known that if the sequence $\frac{1}{n^u|\sin n|^v}$ converges then $\mu(\pi) \leq 1 + \frac{u}{v}$, but the convergence of this sequence has not been solved. In this study, the conditions for convergence of $\frac{1}{n^u|\sin n|^v}$ were clarified by focusing on n such that the value of $|\sin n|$ becomes explosively small. As a result, it was confirmed that $\mu(\pi) < 1 + \frac{u}{v}$ is a sufficient condition for convergence of $\frac{1}{n^u|\sin n|^v}$. This is the same result as in the previous study, but because the method of proof is different, we succeeded in identifying a range of values for $\lim_{n\to\infty} \frac{1}{n^u|\sin n|^v}$ when $\mu(\pi) = 1 + \frac{u}{v}$.

1 Introduction

The irrational measure $\mu(\alpha)$ of a real number α is defined as follows

$$\mu(\alpha) = \inf_{\mu_0 \in \mathbb{R}^+} \left[\text{There exists constant } C > 0 \text{ such that} \left|\alpha - \frac{p}{q}\right| \geq \frac{C}{q^{\mu_0}} \text{ for all } \frac{p}{q} \in \mathbb{Q} \text{ distinct from } \alpha \right].$$

The bound of $\mu(\pi)$ has been updated over the years and according to Zeilberger and Zudilin [5] it was shown to be $\mu(\pi) < 7.103205334137 \cdots$.

The Flint Hills series is definded as follows

$$\sum_{n=1}^{\infty} \frac{1}{n^3|\sin n|^2}.$$

Alekseyev [1] reported that if the sequence $\frac{1}{n^3|\sin n|^2}$ converges then $\mu(\pi) \leq 2.5$, but the convergence of this sequence is unresolved. Lacey [3] studied this sequence of numbers by focusing on n such that the value of $|\sin n|$ is smaller than all $|\sin k|$ $(k < n)$. Such an n is given by the following algorithm

$$f(|\sin 1|) = 1$$

$$\text{For } n \geq 2, f(|\sin n|) = \begin{cases} 1 \ (|\sin n| < \min\{|\sin 1|, |\sin 2|, \cdots, |\sin(n-1)|\}) \\ 0 \ (\text{otherwise}) \end{cases}.$$

The sequence $\alpha(k)$ is defined as follows

$$\alpha(k) = [\ k \text{ such that } f(|\sin k|) = 1\ ].$$

Specifically, $\alpha(k) = 1, 3, 22, 333, 355, \cdots$ and so on indefinitely. The aim of this study is to investigate the behaviour of $|\sin \alpha(k)|$ and to identify the conditions under which the generalised sequence $\frac{1}{n^u |\sin n|^v}$ $(u, v \in \mathbb{R}^+)$ converges.

2 Convergence of the sequence $\frac{1}{n^u |\sin n|^v}$

Theorem 1. If $\mu(\pi) < 1 + \frac{u}{v}$, then the sequence $\frac{1}{n^u |\sin n|^v}$ converges.

To prove Theorem 1 now let us prove Theorem 2.

Theorem 2. If the sequence $\frac{1}{(\alpha(k))^u |\sin \alpha(k)|^v}$ converges, then the sequence $\frac{1}{n^u |\sin n|^v}$ converges.

Proof. For n such that $f(|\sin n|) = 0$, the value of $\frac{1}{|\sin n|^v}$ is not huge, so $\frac{1}{n^u |\sin n|^v}$ converges to 0. Thus, if the sequence $\frac{1}{(\alpha(k))^u |\sin \alpha(k)|^v}$ converges, then the sequence $\frac{1}{n^u |\sin n|^v}$ converges.

□

From Theorem 2, it is sufficient to prove the convergence of $\frac{1}{(\alpha(k))^u |\sin \alpha(k)|^v}$ to prove Theorem 1. We use

$$\lim_{x \to \infty} \frac{\sin x}{x} = 1 \tag{1}$$

for the proof of Theorem 2. There exists $l(k) \in \mathbb{Z}^+$ such that $|l(k)\pi - \alpha(k)| < \epsilon$ for any $\epsilon > 0$ was shown by Olds et al. [4], and their result is Lemma 3.

Lemma 3. Let a be an irrational number. The straight line $y = ax$ has on its sides an infinite number of grid points closer than an arbitrarily given distance $\epsilon > 0$.

From this we can eliminate the from it follows that $|\sin \alpha(k)|$ from $\frac{1}{(\alpha(k))^u |\sin \alpha(k)|^v}$.

$$\lim_{|l(k)\pi-\alpha(k)|\to 0} |\sin|l(k)\pi-\alpha(k)|| = \lim_{|l(k)\pi-\alpha(k)|\to 0} |l(k)\pi-\alpha(k)| \tag{2}$$

Theorem 1. If $\mu(\pi) < 1 + \frac{u}{v}$, then the sequence $\frac{1}{n^u|\sin n|^v}$ converges.

Proof. On the left-hand side of (2) it follows

$$|\sin|l(k)\pi-\alpha(k)|| = |\sin\alpha(k)|\,. \tag{3}$$

If we take k such that $|l(k)\pi-\alpha(k)| \to 0$, the value is sufficiently large so that $k \to \infty$. Therefore, together with (3) , the following equation holds.

$$\lim_{k\to\infty} \frac{1}{(\alpha(k))^u|\sin\alpha(k)|^v} = \lim_{k\to\infty} \frac{1}{(\alpha(k))^u|l(k)\pi-\alpha(k)|^v} = \lim_{k\to\infty}\left(\frac{1}{(\alpha(k))^u(l(k))^v}\cdot\frac{1}{\left|\pi-\frac{\alpha(k)}{l(k)}\right|^v}\right)$$

From the definition of an irrational measure, (4) holds and (5) can be derived from (4) .

$$\frac{1}{\left|\pi-\frac{\alpha(k)}{l(k)}\right|} \le \frac{(l(k))^{\mu(\pi)}}{C} \tag{4}$$

$$0 \le \frac{1}{(\alpha(k))^u(l(k))^v}\cdot\frac{1}{\left|\pi-\frac{\alpha(k)}{l(k)}\right|^v} \le \frac{(l(k))^{v(\mu(\pi)-1)}}{C^v(\alpha(k))^u} \le \frac{(l(k)\pi)^{v(\mu(\pi)-1)}}{C^v(\alpha(k))^u} \tag{5}$$

Since the values of $l(k)\pi$ and $\alpha(k)$ are asymptotic, the proof is completed using squeeze theorem when $\mu(\pi) < 1 + \frac{u}{v}$.

$$\lim_{k\to\infty}\left(\frac{1}{(\alpha(k))^u(l(k))^v}\cdot\frac{1}{\left|\pi-\frac{\alpha(k)}{l(k)}\right|^v}\right) = 0$$

□

If $\mu(\pi) = 1 + \frac{u}{v}$ in the proof of Theorem 1, then

$$0 \le \lim_{k\to\infty}\left(\frac{1}{(\alpha(k))^u(l(k))^v}\cdot\frac{1}{\left|\pi-\frac{\alpha(k)}{l(k)}\right|^v}\right) \le \frac{1}{C^v}\,.$$

Therefore, the following predictions can be made.

Conjecture 4. If $\mu(\pi) \le 1 + \frac{u}{v}$, then the sequence $\frac{1}{n^u|\sin n|^v}$ converges.

If this conjecture is correct, together with Theorem 5 given by Meiburg [2], the sequence $\frac{1}{n^u|\sin n|^v}$ to converge if and only if $\mu(\pi) \le 1 + \frac{u}{v}$.

Theorem 5. If the sequence $\frac{1}{n^u|\sin n|^v}$ converges, then $\mu(\pi) \le 1 + \frac{u}{v}$.

3 Conclusion

We succeed in showing the results presented by Alekseyev [1] in a different way. The value of $\lim_{n\to\infty} \frac{1}{n^u|\sin n|^v}$ for $\mu(\pi) = 1 + \frac{u}{v}$, which had not yet been solved, was identified in the range of its values. Convergence is still unknown, but if it is shown to converge, it will provide new knowledge in the lower bound of $\mu(\pi)$.

References

[1] Max A. Alekseyev. On convergence of the Flint Hills series. arXiv:1104.5100, 2011.

[2] Alex Meiburg. Bounds on Irrationality Measures and the Flint-Hills Series. arXiv:2208.13356, 2022.

[3] Laurence Francis Lacey. Statistical investigation of the Flint Hills series. Researchgate. DOI : 10.13140/RG.2.2.34340.96647, 2022.

[4] C. D. Olds, Anneli Lax, and Giuliana P. Davidoff. The Geometry of Numbers. Translated by H. Kato and K. Takada, *Kyoouritusyuppan*, 13-15, 2021.

[5] Doron Zeilberger and Wadim Zudilin. The irraitionality measure of π is at most $7.103205334137\ldots$. *Moscow Journal of Combinatorics and Number Theory*, 9(4) : 407-419, 2020.