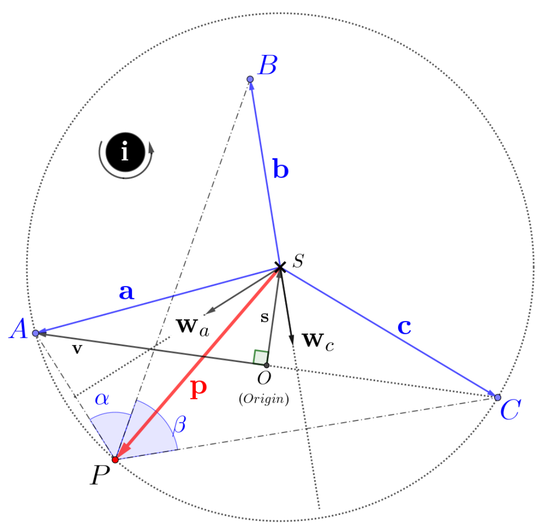
A Solution to the "Snellius-Pothenot" Problem via Rotations and Reflections in Geometric Algebra (GA)

July 10, 2024

James Smith
LinkedIn group "Pre-University Geometric Algebra"
https://www.linkedin.com/groups/8278281/

Abstract

Using GA's capacities for rotating and reflecting vectors, we solve the classic 2-D version of the Snellious-Pothenot surveying problem. The method used here provides two solutions, which can be averaged to better estimate the location of the unknown point P. A link to a GeoGebra worksheet of the solutions is provided so that the reader may test the validity of the method.



The vector $\mathbf{p}$ is the reflection of $\mathbf{a}$ with respect to $\mathbf{w}_a$, where $\mathbf{w}_a = \mathbf{a} + \mathbf{b} + [(\mathbf{a} - \mathbf{b})\,\mathbf{i}]\,/\tan\alpha$. The vector $\mathbf{p}$ is also the reflection of $\mathbf{c}$ with respect to $\mathbf{w}_c$, where $\mathbf{w}_c = \mathbf{c} + \mathbf{b} - [(\mathbf{c} - \mathbf{b})\,\mathbf{i}]\,/\tan\beta$. The vector $\mathbf{s} = \mathbf{v}\mathbf{i}/\tan(\alpha + \beta)$.

1 Statement of the Problem

Fig. 1 Shows the problem statement.

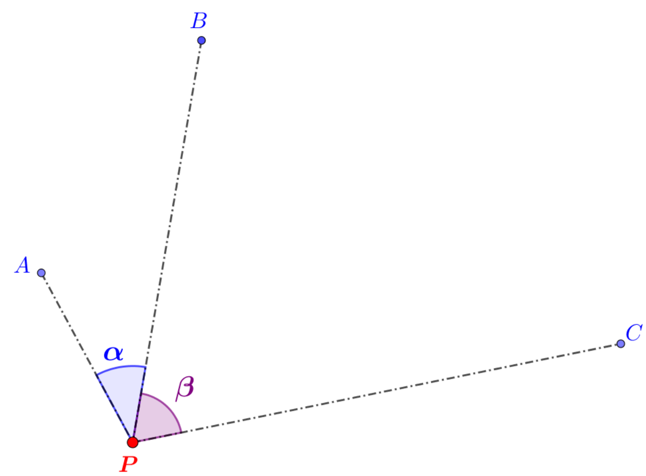


Figure 1: Problem statement: Express the location of point P in terms of the locations of points A, B, C and the angles α y β.

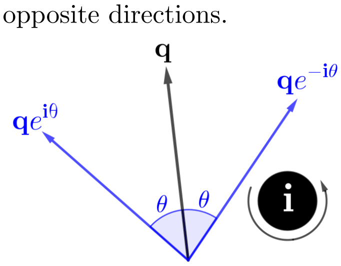
2 Ideas that We Will Use

See also Macdonald [1].

1. For any two vectors $\mathbf{q}$ and $\mathbf{t}$, $\mathbf{q} \wedge \mathbf{t} = [(\mathbf{qi}) \cdot \mathbf{t}]\,\mathbf{i} = -[\mathbf{q} \cdot (\mathbf{ti})]\,\mathbf{i}$.

2. For any two vectors $\mathbf{q}$ and $\mathbf{t}$, $\mathbf{q} \wedge \mathbf{t} = \langle \mathbf{qt} \rangle_2$.

3. The vectors $\mathbf{q}e^{\mathbf{i}\theta}$ and $\mathbf{q}e^{-\mathbf{i}\theta}$ are rotations of $\mathbf{q}$ by the same angle θ, but in opposite directions.



4. The reflection of a vector $\mathbf{q}$ with respect to vector $\mathbf{t}$ can be written as the product $\mathbf{tqt}^{-1}$, which is equal to $[\mathbf{tqt}]\,/\|\mathbf{t}\|^2$ (Fig. 2).

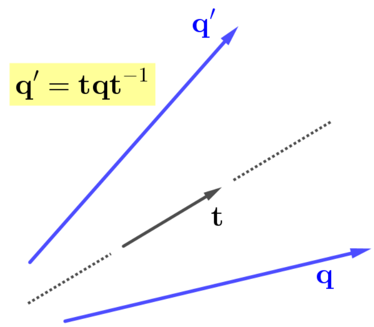


Figure 2: The reflection of a vector $\mathbf{q}$ with respect to vector $\mathbf{t}$ can be written as the product $\mathbf{tqt}^{-1}$, which is equal to $[\mathbf{tqt}]/\|\mathbf{t}\|^2$.

3 Formulation in GA Terms

Fig. 3 Shows the formulation. Note the sign convention for the angles α and β.

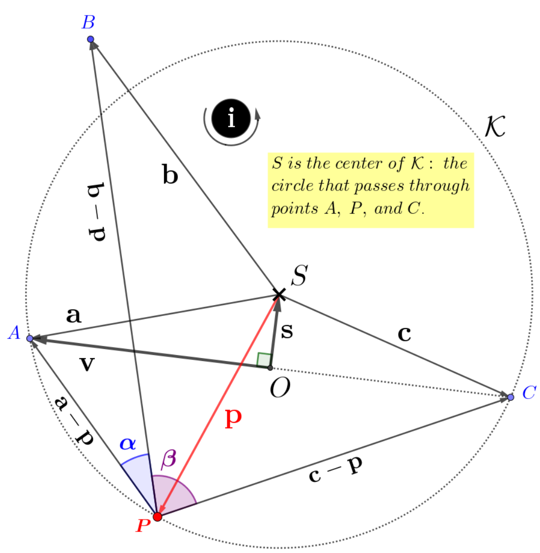


Figure 3: Formulation in terms of GA. As our origin, we use the midpoint of $\overline{AC}$ (point O). The vector $\mathbf{s}$ is from O to the center of $\mathcal{K}$, and the vector $\mathbf{v}$ is from O to point A. Vector $\mathbf{p}$ is from S to point P. **Regarding the algebraic signs of α and β:** Angles are measured in the same direction as the rotation of $\mathbf{i}$. In this diagram, α —the angle of rotation from $\overline{PB}$ to $\overline{PA}$—is in the same sense as $\mathbf{i}$. Therefore, the angle α in this diagram is positive. The angle β is the rotation from $\overline{PC}$ to $\overline{PB}$. This rotation, too, is in the same sense as $\mathbf{i}$. Therefore, the angle β in this diagram is positive.

4 Solution Strategy

We will begin by determining the location of the center of circle $\mathcal{K}$ (Fig. 3).

Then, we will right-multiply the vector $\mathbf{a}-\mathbf{p}$ by $e^{-\mathbf{i}\alpha}$ to make it parallel to $\mathbf{b}-\mathbf{p}$, after which we use the fact that $(\mathbf{b}-\mathbf{p})\wedge\left[(\mathbf{a}-\mathbf{p})\,e^{-\mathbf{i}\alpha}\right]=0$ to obtain an equation for $\mathbf{p}$. We will obtain a separate equation for $\mathbf{p}$ by right-multiplying the vector $\mathbf{c}-\mathbf{p}$ by $e^{\mathbf{i}\beta}$ to make it parallel to $\mathbf{b}-\mathbf{p}$, then using the fact that $(\mathbf{b}-\mathbf{p})\wedge\left[(\mathbf{c}-\mathbf{p})\,e^{\mathbf{i}\beta}\right]=0$.

5 Solution

Readers who wish to test the solutions that are derived here can access the associated interactive GeoGebra worksheet ([2]).

5.1 Determining the Location of the Center of Circle $\mathcal{K}$

The vector $\mathbf{s}$, from the midpoint of $\overline{AC}$ to the center of $\mathcal{K}$, is

$$\mathbf{s}=\frac{\mathbf{v}\mathbf{i}}{\tan\left(\alpha+\beta\right)}. \tag{5.1}$$

Figs. 4 and 5 present two cases that show why this relationship holds.

5.2 Finding $\mathbf{p}$ from the Rotations of the Vectors $\mathbf{a}-\mathbf{p}$ and $\mathbf{c}-\mathbf{p}$.

We will see that each rotation provides a separate solution. These could be averaged to better estimate the location of P.

5.2.1 Finding $\mathbf{p}$ from the Rotation of $\mathbf{a}-\mathbf{p}$.

The vector $(\mathbf{a}-\mathbf{p})\,e^{-\mathbf{i}\alpha}$ is parallel to $\mathbf{b}-\mathbf{p}$. Therefore,

$$(\mathbf{b}-\mathbf{p})\wedge\left[(\mathbf{a}-\mathbf{p})\,e^{-\mathbf{i}\alpha}\right]=0,\text{and}$$
$$\left\langle(\mathbf{b}-\mathbf{p})\left[(\mathbf{a}-\mathbf{p})\,e^{-\mathbf{i}\alpha}\right]\right\rangle_2=0.$$

Expanding the exponential and the geometric product $(\mathbf{b}-\mathbf{p})\,(\mathbf{a}-\mathbf{p})$,

$$\left\langle\left[\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\wedge\mathbf{a}-\mathbf{b}\cdot\mathbf{p}-\mathbf{b}\wedge\mathbf{p}-\mathbf{p}\cdot\mathbf{a}-\mathbf{p}\wedge\mathbf{a}+p^2\right](\cos\alpha-\mathbf{i}\sin\alpha)\right\rangle_2=0.$$

Because the product of any outer product with $\mathbf{i}$ is a scalar, the preceding equation simplifies to

$$[\mathbf{b}\wedge\mathbf{a}-\mathbf{b}\wedge\mathbf{p}-\mathbf{p}\wedge\mathbf{a}]\cos\alpha-\left(\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{p}-\mathbf{p}\cdot\mathbf{a}+p^2\right)\mathbf{i}\sin\alpha=0$$

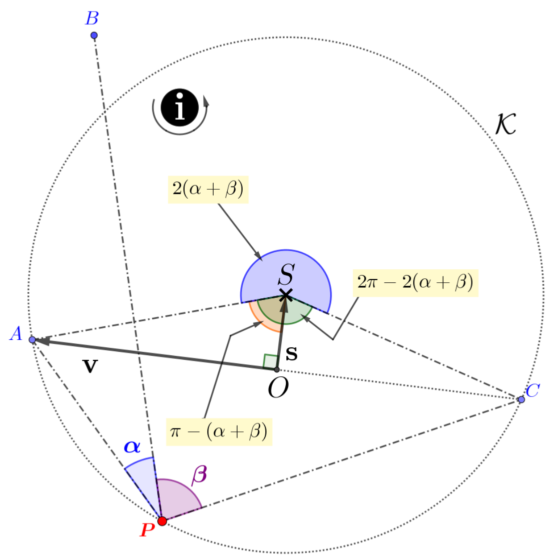


Figure 4: One of the arrangements of points used in deriving an equation for the location of the center of the circle $\mathcal{K}$. Point P is on the arc APC, and B is on the opposite side of chord $\overline{AC}$. To find the length of $\mathbf{s}$, we must first find the central angle that subtends $\mathbf{v}$. To do so, we note that the magnitude of the central angle that subtends the same arc as $\angle CPA$ is $2\,(\alpha+\beta)$. (See Fig. 3 regarding the sign convention for α and β.) Thus, the magnitude of the central angle that subtends $\overline{AC}$ is $2\pi - 2\,(\alpha+\beta)$, and the magnitude of the angle that subtends $\mathbf{v}$ is $\frac{1}{2}\,[2\pi - 2\,(\alpha+\beta)] = \pi - (\alpha+\beta)$. Using the trigonometric identity $\|\tan[\pi - (\theta+\phi)]\| = \|\tan(\theta+\phi)\|$, we find that the length of $\mathbf{s}$ is $\|\mathbf{v}\|/\|\tan(\alpha+\beta)\|$. Vector $\mathbf{s}$ is perpendicular to $\mathbf{v}$ because the bisector of any chord (in this case, $\overline{AC}$) is perpendicular to that chord. The sense of rotation from $\mathbf{v}$ to $\mathbf{s}$ is contrary to the sense of $\mathbf{i}$. For that reason, $\mathbf{s} = -\|\mathbf{v}\|\mathbf{i}/\|\tan(\alpha+\beta)\|$. Because $\alpha+\beta$ is a positive angle between $\pi/2$ and π, $\tan(\alpha+\beta)$ is a negative number. From the foregoing, we can see that the equation $\mathbf{s} = \mathbf{v}\mathbf{i}/\tan(\alpha+\beta)$ captures both the magnitude of $\mathbf{s}$ and the sense of vector $\mathbf{s}$'s rotation with respect to $\mathbf{v}$.

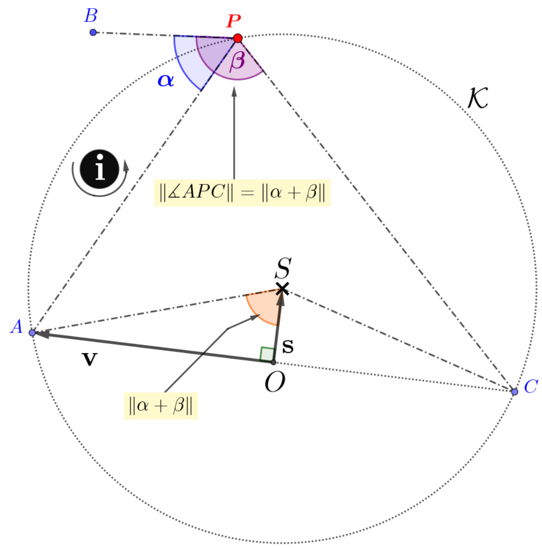


Figure 5: Another arrangement of points used in deriving an equation for the location of the center of circle $\mathcal{K}$. Point P is on the arc APC, and B is on the same side of chord $\overline{AC}$ as point P. (Compare to Fig. 4.) As in the case of Fig. 4, we must find the magnitude of the angle that subtends $\mathbf{v}$. From elementary geometry, that magnitude is the same as the magnitude of $\angle APC$. Because the angles α and β are of opposite signs, the magnitude of that angle is $\|\alpha + \beta\|$. The negative angle (β) is larger than the positive one (α); therefore, $\alpha + \beta$ is negative. In addition, $\|\alpha + \beta\| < \pi/2$ because the arc that is subtended by $\angle APC$ is smaller than π . Thus, just as in Fig. 4, $\tan(\alpha + \beta)$ is negative, leading once again to $\mathbf{s} = \mathbf{v}\mathbf{i}/\tan(\alpha + \beta)$.

Now, we divide by $\cos\alpha$, then use the identities $\mathbf{q}\wedge\mathbf{t} = [(\mathbf{qi})\cdot\mathbf{t}]\,\mathbf{i}$ and $= [(\mathbf{qi})\cdot\mathbf{t}] - [\mathbf{q}\cdot(\mathbf{ti})]$:

$$\begin{aligned}
[(\mathbf{bi})\cdot\mathbf{a}]\,\mathbf{i} - [(\mathbf{bi})\cdot\mathbf{p}]\,\mathbf{i} - [(\mathbf{pi})\cdot\mathbf{a}]\,\mathbf{i} - \left(\mathbf{b}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{a} + p^2\right)\mathbf{i}\tan\alpha = 0\\
[\mathbf{a}\cdot(\mathbf{bi})]\,\mathbf{i} - [\mathbf{p}\cdot(\mathbf{bi})]\,\mathbf{i} + [\mathbf{p}\cdot(\mathbf{ai})]\,\mathbf{i} - \left(\mathbf{a}\cdot\mathbf{b} - \mathbf{p}\cdot\mathbf{a} - \mathbf{p}\cdot\mathbf{b} + p^2\right)\mathbf{i}\tan\alpha = 0.
\end{aligned}$$

Multiplying both sides by $-\mathbf{i}$, then rearranging,

$$\mathbf{p}\cdot\left[\mathbf{a}+\mathbf{b}+\frac{(\mathbf{a}-\mathbf{b})\,\mathbf{i}}{\tan\alpha}\right] = p^2 + \mathbf{a}\cdot\left[\mathbf{b}-\frac{\mathbf{bi}}{\tan\alpha}\right].$$

Now, we define $\mathbf{w}_a = \mathbf{a}+\mathbf{b}+\dfrac{(\mathbf{a}-\mathbf{b})\,\mathbf{i}}{\tan\alpha}$, in order to transform the right-hand side:

$$\mathbf{p}\cdot\mathbf{w}_a = p^2 + \mathbf{a}\cdot\underbrace{\left[\mathbf{a}+\mathbf{b}+\frac{(\mathbf{a}-\mathbf{b})\,\mathbf{i}}{\tan\alpha}\right]}_{\mathbf{w}_a} - \underbrace{\mathbf{a}\cdot\mathbf{a}}_{a^2} - \underbrace{\mathbf{a}\cdot\frac{\mathbf{ai}}{\tan\alpha}}_{=0}$$

Because $\mathbf{a}$ and $\mathbf{p}$ are radii of the same circle $(\mathcal{K})$, $a^2 = p^2$. Therefore,

$$\mathbf{p}\cdot\mathbf{w}_a = \mathbf{a}\cdot\mathbf{w}_a.$$

Next, we recognize that because $\|\mathbf{a}\| = \|\mathbf{p}\|$, and because $\mathbf{p}\neq\mathbf{a}$, $\mathbf{p}$ must be the reflection of $\mathbf{a}$ with respect to $\mathbf{w}_a$. That is,

We don't treat the possibility that $\mathbf{p}=\mathbf{a}$ because for a surveyor in the field, point P would not be "unknown" if it were the same point as A!

$$\begin{aligned}
\mathbf{p} &= [\mathbf{w}_a]\,\mathbf{a}\left[\mathbf{w}_a^{-1}\right]\\
&= \frac{[\mathbf{w}_a]\,\mathbf{a}\,[\mathbf{w}_a]}{\|\mathbf{w}_a\|^2}\\
&= \frac{\mathbf{w}_a\,\{2\mathbf{a}\cdot[\mathbf{w}_a] - [\mathbf{w}_a]\,\mathbf{a}\}}{\|\mathbf{w}_a\|^2}\\
&= 2\left[\frac{\mathbf{w}_a\cdot\mathbf{a}}{\|\mathbf{w}_a\|^2}\right]\mathbf{w}_a - \mathbf{a}. \qquad (5.2)
\end{aligned}$$

An identity: For any two vectors $\mathbf{q}$ and $\mathbf{t}$, $\mathbf{qt} = 2\mathbf{q}\cdot\mathbf{t} - \mathbf{tq}$.

Therefore, the location of P with respect to the midpoint of $\overline{AC}$ is given by the vector $\mathbf{p}^*$ (Fig. 6):

$$\begin{aligned}
\mathbf{p}^* &= \mathbf{s}+\mathbf{p}\\
&= \frac{\mathbf{vi}}{\tan(\alpha+\beta)} + 2\left[\frac{\mathbf{w}_a\cdot\mathbf{a}}{\|\mathbf{w}_a\|^2}\right]\mathbf{w}_a - \mathbf{a}. \qquad (5.3)
\end{aligned}$$

5.2.2 Finding p from the Rotation of c − p.

The vector $(\mathbf{c}-\mathbf{p})\,e^{\mathbf{i}\beta}$ is parallel to $\mathbf{b}-\mathbf{p}$. Therefore,

$$\begin{aligned}
(\mathbf{b}-\mathbf{p})\wedge\left[(\mathbf{c}-\mathbf{p})\,e^{\mathbf{i}\beta}\right] = 0, \text{and}\\
\langle(\mathbf{b}-\mathbf{p})\left[(\mathbf{c}-\mathbf{p})\,e^{\mathbf{i}\beta}\right]\rangle_2 = 0.
\end{aligned}$$

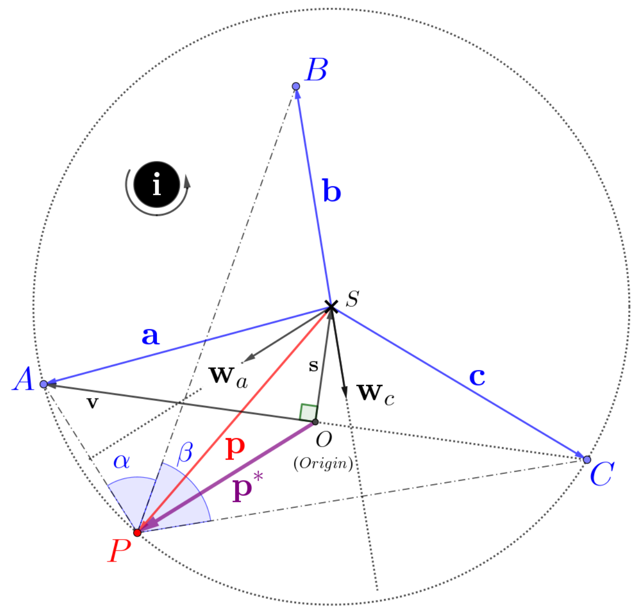


Figure 6: Vector $\mathbf{p}$ is the reflection of $\mathbf{a}$ with respect to $\mathbf{w}_a$, where $\mathbf{w}_a = \mathbf{a} + \mathbf{b} + [(\mathbf{a} - \mathbf{b})\,\mathbf{i}] \,/\, \tan\alpha$. Vector $\mathbf{p}$ is also the reflection of $\mathbf{c}$ with respect to $\mathbf{w}_c$, where $\mathbf{w}_c = \mathbf{c} + \mathbf{b} - [(\mathbf{c} - \mathbf{b})\,\mathbf{i}] \tan\beta$. Therefore, the location of P with respect to the midpoint of $\overline{AC}$ is given by the vector $\mathbf{p}^* = \dfrac{\mathbf{vi}}{\tan(\alpha+\beta)} + 2\left[\dfrac{\mathbf{w}_a \cdot \mathbf{a}}{\|\mathbf{w}_a\|^2}\right] \mathbf{w}_a - \mathbf{a}$, and also by $\mathbf{p}^* = \dfrac{\mathbf{vi}}{\tan(\alpha+\beta)} + 2\left[\dfrac{\mathbf{w}_c \cdot \mathbf{c}}{\|\mathbf{w}_a\|^2}\right] \mathbf{w}_c - \mathbf{c}$.

Expanding the the exponential and the geometric product $(\mathbf{b} - \mathbf{p})(\mathbf{c} - \mathbf{p})$,

$$\left\langle \left[\mathbf{b}\cdot\mathbf{c} + \mathbf{b}\wedge\mathbf{c} - \mathbf{b}\cdot\mathbf{p} - \mathbf{b}\wedge\mathbf{p} - \mathbf{p}\cdot\mathbf{c} - \mathbf{p}\wedge\mathbf{c} + p^2\right](\cos\beta + \mathbf{i}\sin\beta)\right\rangle_2 = 0.$$

Because the product of any outer product with $\mathbf{i}$ is a scalar, the preceding equation simplifies to

$$[\mathbf{b}\wedge\mathbf{c} - \mathbf{b}\wedge\mathbf{p} - \mathbf{p}\wedge\mathbf{c}]\cos\alpha + \left(\mathbf{b}\cdot\mathbf{c} - \mathbf{b}\cdot\mathbf{p} - \mathbf{p}\cdot\mathbf{c} + p^2\right)\mathbf{i}\sin\beta = 0.$$

Dividing by $\cos\beta$, then using the identities $\mathbf{q}\wedge\mathbf{t} = [(\mathbf{qi})\cdot\mathbf{t}]\,\mathbf{i}$ and $= [(\mathbf{qi})\cdot\mathbf{t}] - [\mathbf{q}\cdot(\mathbf{ti})]$,

$$[\mathbf{c}\cdot(\mathbf{bi})]\,\mathbf{i} - [\mathbf{p}\cdot(\mathbf{bi})]\,\mathbf{i} + [\mathbf{p}\cdot(\mathbf{ci})]\,\mathbf{i} + \left(\mathbf{c}\cdot\mathbf{b} - \mathbf{p}\cdot\mathbf{b} - \mathbf{p}\cdot\mathbf{c} + p^2\right)\mathbf{i}\tan\beta = 0.$$

Multiplying both sides by $-\mathbf{i}$, then rearranging,

$$\mathbf{p}\cdot\underbrace{\left[\mathbf{c} + \mathbf{b} - \frac{(\mathbf{c}-\mathbf{b})\,\mathbf{i}}{\tan\beta}\right]}_{\mathbf{w}_c} = p^2 + \mathbf{c}\cdot\left[\mathbf{b} + \frac{\mathbf{bi}}{\tan\beta}\right].$$

Continuing as we did when finding $\mathbf{p}$ from the rotation of $\mathbf{a}-\mathbf{p}$,

$$\mathbf{p}\cdot\mathbf{w}_c = p^2 + \mathbf{c}\cdot\underbrace{\left[\mathbf{c}+\mathbf{b}-\frac{(\mathbf{c}-\mathbf{b})\,\mathbf{i}}{\tan\beta}\right]}_{\mathbf{w}_c} - \underbrace{\mathbf{c}\cdot\mathbf{c}}_{c^2} + \underbrace{\mathbf{c}\cdot\left[\frac{\mathbf{ci}}{\tan\beta}\right]}_{=0}, \text{ and}$$

$$\mathbf{p}\cdot\mathbf{w}_c = \mathbf{c}\cdot\mathbf{w}_c,$$

leading to

$$\begin{aligned}\mathbf{p} &= [\mathbf{w}_c]\,\mathbf{c}\,\left[\mathbf{w}_c^{-1}\right] \\ &= 2\left[\frac{\mathbf{w}_c\cdot\mathbf{c}}{\|\mathbf{w}_c\|^2}\right]\mathbf{w}_c - \mathbf{c}. \qquad (5.4)\end{aligned}$$

Therefore, the derivation that starts from the rotation of $\mathbf{c}-\mathbf{p}$ finds that the location of P with respect to the midpoint of $\overline{AC}$ is given by the same vector $\mathbf{p}^*$ that we found when starting from the rotation of $\mathbf{a}-\mathbf{p}$. However, the vector $\mathbf{p}$ is now expressed in terms of $\mathbf{c}$ (Fig. 6):

$$\begin{aligned}\mathbf{p}^* &= \mathbf{s}+\mathbf{p} \\ &= \frac{\mathbf{vi}}{\tan(\alpha+\beta)} + 2\left[\frac{\mathbf{w}_c\cdot\mathbf{c}}{\|\mathbf{w}_a\|^2}\right]\mathbf{w}_c - \mathbf{c}. \qquad (5.5)\end{aligned}$$

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition), CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] J. A. Smith, “Snellious-Pothenot Solution via Geometric Algebra” ,`www.geogebra.org/m/tmpxpx4z`, 2024.