PASCAL TRIANGLE:
A COMBINATORIAL APPROACH TO POWER SUMS

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Abstract. We explore three combinatorial sequences derived from Pascal's triangle: Binomial Coefficients, the Narayana Numbers and a variant of the Binomial Coefficients. The goal is to express particular cases of the sum of powers of the first n natural numbers using combinatorial sequences.

$$1^p + 2^p + 3^p + 4^p + \ldots + n^p, \ \textit{where} \ \ p, \ n \in \mathbb{N}$$

The methodology we employ is based on the differences between terms. We multiply each term by n to equal the next exponent and then add each term. Finally, we identify patterns in the sequences at the intermediate or final stage.

1 Introduction.

Pascal's triangle is a representation of the binomial coefficients in the form of a triangle, named after the French mathematician Blaise Pascal(1623-1662).

The binomial coefficient is given by the formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad 0 \leqslant k \leqslant n$$

Pascal's Triangle	Numerical values
$\binom{0}{0}$	1
$\binom{1}{0} \binom{1}{1}$	1 1
$\binom{2}{0} \binom{2}{1} \binom{2}{2}$	1 2 1
$\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$	1 3 3 1
$\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$	1 4 6 4 1

1.1 Binomial coefficients in the Diagonals

The binomial coefficients that form the diagonals in the triangle are the first family of sequences that we can derive from it, where we denote the binomial coefficients as C(n, k).

Table-C(n, k)

C(n, 0)	C(n, 1)	C(n, 2)	C(n, 3)	C(n, 4)	C(n, 5)	C(n, 6)	C(n, 7)	C(n, 8)	C(n, 9)
1	1	1	1	1	1	1	1	1	1
1	2	3	4	5	6	7	8	9	10
1	3	6	10	15	21	28	36	45	55
1	4	10	20	35	56	84	120	165	220
1	5	15	35	70	126	210	330	495	715
1	6	21	56	126	252	462	792	1287	2002
1	7	28	84	210	462	924	1716	3003	5005
1	8	36	120	330	792	1716	3432	6435	11440
1	9	45	165	495	1287	3003	6435	12870	24310
1	10	55	220	715	2002	5005	11440	24310	48620

1.2 Variant of C(n, k)

The second family of sequences that we can derive from the triangle is a variant of C(n, k) where each sequence is constructed from the sum C(n, k) + C(n-1, k) which we denote as S(n, k).

$$S(n,\ k) = \frac{n!}{k!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!},\quad 0 \leqslant k \leqslant n$$

Table-S(n, k)

S(n, 0)	S(n, 1)	S(n, 2)	S(n, 3)	S(n, 4)	S(n, 5)	S(n, 6)	S(n, 7)	S(n, 8)	S(n, 9)
1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11
2	5	9	14	20	27	35	44	54	65
2	7	16	30	50	77	112	156	210	275
2	9	25	55	105	182	294	450	660	935
2	11	36	91	196	378	672	1122	1782	2717
2	13	49	140	336	714	1386	2508	4290	7007
2	15	64	204	540	1254	2640	5148	9438	16445
2	17	81	285	825	2079	4719	9867	19305	35750
2	19	100	385	1210	3289	8008	17875	37180	72930

1.3 Narayana Numbers

Narayana numbers are the third family of sequences that we can derive from the triangle, where each sequence is constructed from the product of C(n, k)·C(n-1, k) and divided by C(2, k), which we denote as N(n, k), named after the Indian-Canadian mathematician Tadepalli Venkata Narayana(1930-1987).

$$N(n,\ k)\ =\ \frac{1}{n}\binom{n}{k}\binom{n}{k+1},\ 1\leqslant k\leqslant n$$

Table-N(n, k)

N(n, 0)	N(n, 1)	N(n, 2)	N(n, 3)	N(n, 4)	N(n, 5)	N(n, 6)	N(n, 7)	N(n, 8)	N(n, 9)
1	1	1	1	1	1	1	1	1	1
1	3	6	10	15	21	28	36	45	55
1	6	20	50	105	196	336	540	825	1210
1	10	50	175	490	1176	2520	4950	9075	15730
1	15	105	490	1764	5292	13860	32670	70785	143143
1	21	196	1176	5292	19404	60984	169884	429429	1002001
1	28	336	2520	13860	60984	226512	736164	2147145	5725720
1	36	540	4950	32670	169884	736164	2760615	9202050	27810640
1	45	825	9075	70785	429429	2147145	9202050	34763300	...
1	55	1210	15730	143143	1002001	5725720	27810640	…	...

1.4 Methodology

The methodology that we will follow during the development of the article can be separated into two phases. The first one is based on the differences between terms: multiplying by n and adding each term. The second phase is based on pattern recognition, where we will use strategies to try to clean up the sequence and arrive at one of the three families of sequences: C(n, k), S(n, k) and N(n, k).

1.4.1 Phase 1 and 2

Phase 1 can be seen implicitly in telescoping sums, which is a purely algebraic approach. However, in this text we will use phase 1 explicitly. We will not only multiply by n, but any exponent, with the only condition that it is smaller than the exponent we are dealing with. For example, if we are looking for a formula for sum of the first n cubes, we cannot multiply by n^3, since this would result in n^4(since the only exponents less than 3 are 2 and 1, omitting 0), which is greater than the exponent we are initially considering.

Since we express all the formulas in combinatorial sequences, we can apply them to any sequence. This versatility will help us to use strategies for comparing sequences to arrive at sequences that are easily recognizable. This is the main difference with many other approaches.

2 Sums of Powers.

The sum of powers of the first n natural numbers has been a classical problem of great interest to mathematicians specializing in number theory. Some of the mathematicians who have contributed significantly are Johann Faulhaber(1580-1635) and Jacob Bernoulli(1654-1705). For a more detailed discussion of this topic, see references [1] and [2].

The sum of the p-th powers of the first n natural numbers

$$\sum_{k=1}^{n} k^p = 1^p + 2^p + 3^p + \dots + n^p$$

2.1 Trivial Cases of P

The first 3 values(0, 1, 2) taken by p can be considered trivial cases, since it is evident to recognize the sequences by contrasting with the tables previously presented. For p = 3, it can also be argued that it is a trivial case, since the nth term coincides with the square of the nth triangular number. However, this pattern cannot be considered as an absolute truth, but rather as an initial observation that will be justified later.

2.1.1 P = 0

Using the property($a^0 = 1$), we have that the sum of the first n natural numbers raised to an exponent 0 is:

$$\sum_{k=1}^{n} k^0 = C(n, 1)$$

2.1.2 P = 1

$$\sum_{k=1}^{n} k^1 = C(n, 2)$$

2.1.3 P = 2

$$\sum_{k=1}^{n} k^2 = S(n, 3) = C(n, 3) + C(n-1, 3)$$

2.1.4 P = 3

One of the first strategies we will use is to calculate the differences of the power we are considering with respect to the natural numbers:

$$1^p - 1, 2^p - 2, 3^p - 3, \dots, n^p - n$$

We calculate the differences for p = 3 until n = 6:

$$1 - 1,\ 8 - 2,\ 27 - 3,\ 64 - 4,\ 125 - 5,\ 216 - 6$$

$$0,\ 6,\ 24,\ 60,\ 120,\ 210$$

We note a divisibility by 6, so we will express the numbers in the form 6n:

$$6(0),\ 6(1),\ 6(4),\ 6(10),\ 6(20),\ 6(35)$$

Since we start with $n^3 - n$, we clear the n:

$$n^3 = 6 \cdot C(n-1,\ 3) + C(n,\ 1)$$

Therefore, we conclude that sum of first n cubic numbers is:

$$\sum_{k=1}^{n} k^3 = 6 \cdot C(n-1,\ 4) + C(n,\ 2) = C(n,\ 2)^2$$

The new formula can be easily proven, since the reduced formula for the tetrahedral numbers is $(n^3 - n)/6$. Only a few algebraic manipulations are necessary. On the other hand, the formula for the squared triangular numbers is a fact established by the Greek mathematician Nicomachus of Gerasa(60-120). The new formula will be used to calculate other sequences, while Nicomachus's formula will be used to simplify them.

2.2 P > 3

2.2.1 P = 4

The second strategy is based on the differences between consecutive terms, denoted by Δ, previously described in (1.4), which we apply in C(n-1, 4) and C(n, 2):

C(n-1, 4)	ΔC(n-1, 4)	C(n, 2)	ΔC(n, 2)
0	0	1	1
1	1	3	2
5	4	6	3
15	10	10	4
35	20	15	5
70	35	21	6
126	56	28	7
210	84	36	8
330	120	45	9
495	165	55	10

Since the sequence C(n, k) is the partial sum of C(n, k-1) we have that:

$$\Delta C(n,\ k)\ =\ C(n,\ k-1) \qquad \Delta S(n,\ k)\ =\ S(n,\ k-1)$$

The next step is to multiply by the set of natural numbers {1,2,3,4,…,n}:

ΔC(n-1, 4)·n	ΔC(n, 2)·n
0·1 = 0	1·1 = 1
1·2 = 2	2·2 = 4
4·3 = 12	3·3 = 9
10·4 = 40	4·4 = 16
20·5 = 100	5·5 = 25
35·6 = 210	6·6 = 36
56·7 = 392	7·7 = 49
84·8 = 672	8·8 = 64
120·9 = 1080	9·9 = 81
165·10 = 1650	10·10 = 100

We observe a clear pattern in column 2 that will end up simplifying in S(n, 3), and in column 1 we notice a divisibility by 2 that will allow us to sort the sequences to identify a pattern:

ΔC(n-1, 4)·(n/2)
0/2 = 0
2/2 = 1
12/2 = 6
40/2 = 20
100/2 = 50
210/2 = 105
392/2 = 196
672/2 = 336
1080/2 = 540
650/2 = 825

Initially C(n-1, 5) is begin multiplied by 6, although in the process constants can be left out. However, it is important to keep track since in this case the constant is multiplied by 2 at the end:

$$6\cdot 2\cdot S(n-1,\ 4)\ +\ S(n,\ 2)$$

Therefore, we conclude that the sum of the first n numbers to the fourth power is:

$$\sum_{k=1}^{n} k^4\ =\ 12\cdot S(n-1,\ 5)\ +\ S(n,\ 3)$$

2.2.2 P = 5

Since we already calculated S(n, 3) in (2.1.4), which is equal to $C(n, 2)^2$, we will now focus on calculating S(n-1, 5):

ΔS(n-1, 5)·n	ΔS(n-1, 5)·(n/2)	∑
0·1 = 0	0/2 = 0	0
1·2 = 2	2/2 = 1	1
6·3 = 18	18/2 = 9	10
20·4 = 80	80/2 = 40	50
50·5 = 250	250/2 = 125	175
105·6 = 630	630/2 = 315	490
196·7 = 1372	1372/2 = 686	1176
336·8 = 2688	2688/2 = 1344	2520
540·9 = 4860	4860/2 = 2430	4950
825·10 = 8250	8250/2 = 4125	9075

We note that the nth partial sum is equal to N(n-1, 3), Therefore, we conclude that the sum of the first n numbers to the fifth power is:

$$\sum_{k=1}^{n} k^5 = 24 \cdot N(n-1, 3) + C(n, 2)^2$$

2.2.3 P = 6

Since we already calculated $C(n, 2)^2$ in (2.2.1), which is equal to 12·S(n-1, 5) + S(n, 3), we will now focus on calculating N(n-1, 3):

ΔN(n-1, 3)·n	(3x + y)/2	∑: x, y
0·1 = 0	$(3\cdot0^2 + 0)/2 = 0$	0, 0
1·2 = 2	$(3\cdot1^2 + 1)/2 = 2$	1, 1
9·3 = 27	$(3\cdot4^2 + 6)/2 = 27$	17, 7
40·4 = 160	$(3\cdot10^2 + 20)/2 = 160$	117, 27
125·5 = 625	$(3\cdot20^2 + 50)/2 = 625$	517, 77
315·6 = 1890	$(3\cdot35^2 + 105)/2 = 1890$	1742, 182
686·7 = 4802	$(3\cdot56^2 + 196)/2 = 4802$	4878, 378
1344·8 = 10752	$(3\cdot84^2 + 336)/2 = 10752$	11934, 714
2430·9 = 21870	$(3\cdot120^2 + 540)/2 = 21870$	26334, 1254
4125·10 = 41250	$(3\cdot165^2 + 825)/2 = 41250$	53559, 2079

As we take higher and higher values of p, the difficulty of begin able to observe patterns in the sequences increases, so we will need more and more sophisticated tools.

Special Function

$$\Gamma(n,\ p) = \frac{n^p + n^{p-4} - 2n^{p-2}}{2^{p-4} \cdot 9},\quad 6 \leqslant p \leqslant 10$$

The derivation of Γ is related to a pattern in the recurrence of the formulas for the exponents(6, 7, 8, 9, 10), which will be very helpful for us to distinguish the new sequence from the already known ones.

Table-Γ(n, p)

Γ(n, 6)	Γ(n, 7)	Γ(n, 8)	Γ(n, 9)	Γ(n, 10)
0	0	0	0	0
1	1	1	1	1
16	24	36	54	81
100	200	400	800	1600
400	1000	2500	6250	15625
1225	3675	11025	33075	99225
3136	10976	38416	134456	470596
7056	28224	112896	451584	1806336
14400	64800	291600	1312200	5904900
27225	136125	680625	3403125	17015625

We note the nth term of Γ(n, 6) is equal to the square of the nth tetrahedral number.

Therefore, we conclude that the sum of the first n numbers to the sixth power is:

$$\sum_{k=1}^{n} k^6 = 36 \cdot \sum_{i=1}^{n} C(i-1,\ 3)^2 + 24 \cdot S(n-1,\ 5) + S(n,\ 3)$$

2.2.4 P = 7

Recall that ∑ is the inverse operation of Δ, so:

$$\Delta \sum_{k=1}^{n} S_k = S_n$$

Since we already calculated S(n-1, 5) and S(n, 3), respectively in (2.2.2) and (2.1.4), which is equal to $2 \cdot N(n-1,\ 3) + C(n,\ 2)^2$, we will now focus on calculating $\sum C(i-1,\ 3)^2$:

C(n-1, 3)²·n	C(n-1, 3)²·(n/2)	∑
0·1 = 0	0/2 = 0	0
1·2 = 2	2/2 = 1	1
16·3 = 48	48/2 = 24	25
100·4 = 400	400/2 = 200	225
400·5 = 2000	2000/2 = 1000	1225
1225·6 = 7350	7350/2 = 3675	4900
3136·7 = 21952	21952/2 = 10976	15876
7056·8 = 56448	56448/2 = 28224	44100
14400·9 = 129600	129600/2 = 64800	108900
27225·10 = 272250	272250/2 = 136125	245025

We note that the nth partial sum is equal to C(n-1, 4)². Therefore, we conclude that sum of the firs n numbers to the seventh power is:

$$\sum_{k=1}^{n} k^7 = 72 \cdot C(n-1,\ 4)^2 + 48 \cdot N(n-1,\ 3) + C(n,\ 2)^2$$

2.2.5 P = 8

Since we already calculated N(n-1, 3) and C(n, 2)², respectively in (2.2.3) and (2.2.1), which is equal to (3·∑C(i-1, 3)² + S(n-1, 5))/2 + 12·S(n-1, 5) + S(n, 3), we will now focus on calculating C(n-1, 4)²:

ΔC(n-1, 4)²·n	ΔC(n-1, 4)²·(n/2)	∑
0·1 = 0	0/2 = 0	0
1·2 = 2	2/2 = 1	1
24·3 = 72	72/2 = 36	37
200·4 = 800	800/2 = 400	437
1000·5 = 5000	5000/2 = 2500	2937
3675·6 = 22050	22050/2 = 11025	13962
10976·7 = 76832	76832/2 = 38416	52378
28224·8 = 225792	225792/2 = 112896	165274
64800·9 = 583200	583200/2 = 291600	456874
136125·10 = 1361250	1361250/2 = 680625	1137499

We note that by dividing by 2 we obtain the squares of S(n-1, 4). Therefore, we conclude that the sum of the first n number to the eighth power is:

$$\sum_{k=1}^{n} k^8 = 144 \cdot \sum_{i=1}^{n} S(i-1,\ 4)^2 + 72 \cdot \sum_{i=1}^{n} C(i-1,\ 3)^2 + 36 \cdot S(n-1,\ 5) + S(n,\ 3)$$

2.2.6 P = 9

Since we already calculated ∑C(i-1, 3)², S(n-1, 5) and S(n, 3), respectively in (2.2.4), (2.2.2) and (2.1.4), which is equal to 2·C(n-1, 4)² + 2·N(n-1, 3) + C(n, 2)², we will now focus on calculating ∑S(i-1, 4)²:

S(n-1, 4)²·n	S(n-1, 4)²·(n/2)	∑
0·1 = 0	0/2 = 0	0
1·2 = 2	2/2 = 1	1
36·3 = 108	108/2 = 54	55
400·4 = 1600	1600/2 = 800	855
2500·5 = 12500	12500/2 = 6250	7105
11025·6 = 66150	66150/2 = 33075	40180
38416·7 = 268912	268912/2 =134456	174636
112896·8 = 903168	903168/2 = 451584	626220
291600·9 = 2624400	2624400/2 = 1312200	1938420
680625·10 = 6806250	6806250/2 = 3403125	5341545

Combined Sequences

Combined sequences are those composed of two or more distinct sequences where there is harmony(the sequence contains the 1), which makes them more difficult to find a pattern. One of the strategies we can apply for this type of sequences is to use two sequences with known patterns that approach each other from the left and from the right. This is the case for Γ(n. 9), where Γ(n, 8) is approximated by the left and Γ(n, 10) is approximated by the right. These help us to understand the possible behavior of the sequence.

∑Γ(n, 9)	x + 6y
0	0² + 6(0) = 0
1	1² + 6(0) = 1
55	7² + 6(1) = 55
855	27² + 6(21) = 855
7105	77² + 6(196) = 7105
40180	182² + 6(1176) = 40180
174636	378² + 6(5292) = 174636
626220	714² + 6(19404) = 626220
1938420	1254² + 6(60984) = 1938420
5341545	2079² + 6(169884) = 5341545

The first sequence, which is S(n-1, 5)², can be derived by a simple divisibility criterion. The other sequences that satisfy the inequality do not have constant divisibility, which would not allow us to know the other sequence.

$$\sum_{i=1}^{n} \Gamma(i,\ 8) < S_n < \sum_{i=1}^{n} \Gamma(i,\ 9)$$

Therefore, we conclude that the sum of the first n numbers to the ninth power is:

$$\sum_{k=1}^{n} k^9 = 288 \cdot S(n-1,\ 5)^2 + 1728 \cdot N(n-2,\ 5) + 144 \cdot C(n-1,\ 4)^2 + 72 \cdot N(n-1,\ 3) + C(n,\ 2)^2$$

2.2.7 P = 10

Since we already calculated C(n-1, 4)², N(n-1, 3), C(n. 2)², respectively in (2.2.5), (2.2.3) and (2.2.1), which is equal to 2·∑S(n-1, 4)² + (3·∑C(i-1, 3)² + S(n-1, 5))/2 + 12·S(n-1, 5) + S(n, 3), we will now focus on calculating ∑Γ(n, 9):

Γ(n, 9)·n	Γ(n, 9)·(n/2)	∑
0·1 = 0	0/2 = 0	0
1·2 = 2	2/2 = 1	1
54·3 = 162	162/2 = 81	82
800·4 = 3200	3200/2 = 1600	1682
6250·5 = 31250	31250/2 = 15625	17307
33075·6 = 198450	198450/2 = 99225	116532
134456·7 = 941192	941192/2 = 470596	587128
451584·8 = 3612672	3612672/2 = 1806336	2393464
1312200·9 = 11809800	11809800/2 = 5904900	8298364
3403125·10 = 34031250	34031250/2 = 17015625	25313989

We note that Γ(n, 9)·(n/2) = Γ(n, 10) = (N(n-1, 3) – N(n-2, 3))². Therefore, we conclude that the sum of the first n numbers to the tenth power is:

$$\sum_{k=1}^{n} k^{10} = 576 \cdot \sum_{i=1}^{n} \left(N(i-1,\ 3) - N(i-2,\ 3)\right)^2 + 288 \cdot \sum_{i=1}^{n} S(i-1,\ 4)^2 + 108 \cdot \sum_{i=1}^{n} C(i-1,\ 3)^2 + 48 \cdot S(n-1,\ 5) + S(n,\ 3)$$

2.2.8 P = 11

Since we already calculated ∑S(i-1, 4)², ∑C(i-1, 3)², S(n-1, 5) and S(n, 3), respectively in (2.2.6), (2.2.4), (2.2.2) and (2.1.4), which is equal to 2·S(n-1, 5)² + 12·N(n-2, 5) + 2·C(n-1, 4)² + 2·N(n-1, 3) + C(n, 2)², we will now focus on calculating ∑(N(n-1, 3) – N(n-2, 3))²:

(N(n-1, 3) – N(n-2, 3))²·n	∑	3x – y
0·1 = 0	0	3·0² – 0 = 0
1·2 = 2	2	3·1² – 1 = 2
81·3 = 243	245	3·10² – 55 = 245
1600·4 = 6400	6645	3·50² – 855 = 6645
15625·5 = 78125	84770	3·175² – 7105 = 84770
99225·6 = 595350	680120	3·490² – 40180 = 680120
470596·7 = 3294172	3974292	3·1764² – 174636 = 3974292
1806336·8 = 14450688	18424980	3·5292² – 626220 = 18424980
5904900·9 = 53144100	71569080	3·13860² – 1938420 = 71569080
17015625·10 = 170156250	241725330	...

We can deduce one thing from our initial sequence: since it starts at 2 and there is no clear divisibility by 2, it must necessarily contain at least two sequences starting at 1. By trial and error, we can derive the two sequences by contrasting with already known sequences by calculating sequences in (2.2.6).

Odd Pattern

The main sequences for odd values of p can be encapsulated in the following pattern:

$$\sigma(n,\ p) = \frac{S(n,\ 4)^p}{C(n,\ 3)},\ 2 \leqslant p \leqslant 4$$

Table-σ(n, p)

σ(n, 2)	σ(n, 3)	σ(n, 4)
1	1	1
9	54	324
40	800	16000
125	6250	312500
315	33075	3472875
686	134456	26353376
1344	451584	151732224
2430	1312200	708588000
4125	3403125	2807578125
6655	8052550	9743585500

By means of σ(n, p) we can calculate the differences between terms of the new sequences for odd values of p.

We note that the sequences $S(n-1, 5)^2$ and $6\cdot N(n-2, 5)$ cancel. Therefore, we conclude that the sum of the first n numbers to the eleventh power is:

$$\sum_{k=1}^{n} k^{11} = 1728\cdot N(n-1,\ 3)^2 + 216\cdot C(n-1,\ 4)^2 + 96\cdot N(n-1,\ 3) + C(n,\ 2)^2$$

2.2.9 P = 12

Since we already calculated $C(n-1, 4)^2$, $N(n-1, 3)$, $C(n. 2)^2$, respectively in (2.2.5), (2.2.3) and (2.2.1), which is equal to $2\cdot\sum S(n-1, 4)^2 + (3\cdot\sum C(i-1, 3)^2 + S(n-1, 5))/2 + 12\cdot S(n-1, 5) + S(n, 3)$, we will now focus on calculating $N(n-1, 3)^2$:

$\Delta N(n-1, 3)^2\cdot n$	x + y	∑: x, y
0·1 = 0	$0^3 + 0^2$	0, 0
1·2 = 2	$1^3 + 1^2$	1, 1
99·3 = 297	$6^3 + 9^2$	217, 82
2400·4 = 9600	$20^3 + 40^2$	8217, 1682
28125·5 = 140625	$50^3 + 125^2$	133217, 17307
209475·6 = 1256850	$105^3 + 315^2$	1290842, 116532
1142876·7 = 8000132	$196^3 + 686^2$	8820378, 587128
4967424·8 = 39739392	$336^3 + 1344^2$	46753434, 2393464
18152100·9 = 163368900	$540^3 + 2430^2$	204217434, 8298364
57853125·10 = 578531250	$825^3 + 4125^2$	765733059, 25313989

Following the same strategy in (2.2.8), we can derive the two sequences from an already known sequence. Therefore, we conclude that the sum of first n numbers to the twelfth power is:

$$\sum_{k=1}^{n} k^{12} = 1728\cdot\sum_{i=1}^{n} S(i-1,\ 4)^3 + 1728\cdot\sum_{i=1}^{n}\left(N(i-1,\ 3) - N(i-2,\ 3)\right)^2 + 432\cdot\sum_{i=1}^{n} S(i-1,\ 4)^2$$
$$+ 144\cdot\sum_{i=1}^{n} C(i-1,\ 3)^2 + 60\cdot S(n-1,\ 5) + S(n,\ 3)$$

2.3 OEIS: Online Encyclopedia of Integer Sequences

The OEIS is a database with more than 370,000 documented numerical sequences, useful for identifying and studying mathematical and scientific patterns. It allows searching sequences by number or keywords, facilitating the identification of recurring patterns and the validation of previous discoveries. It also helps to discover new relationships between seemingly distinct sequences and provides tools for detailed analysis, including explicit formulas and mathematical properties.

A great resource we can resort to when working with combinatorial sequences, where I have had the opportunity to contribute with several formulas, some of which are already documented in this article: (2.2.2), (2.2.3), (2.2.4), (2.2.6), (2.2.8), which can be consulted respectively in the references [3], [4], [5], [6], [7].

Note: The formula presented in (2.2.6) is not found in reference [6], so the formula is compacted due to a pattern that unifies three sequences.

$$144 \cdot C(n-1,\ 4)^2 + 72 \cdot N(n-1,\ 3) + C(n,\ 2)^2 = \left(n^4 - (n-1)^4 + (n-2)^4 - \ldots\ 0^4\right)^2$$

3 Open Problem: P ≥ 13.

Something that has been emphasized throughout the article is that as the value of p increases, the patterns become more complex and, consequently, more sophisticated tools are required. As a result, there are no known formulas for p-values equal to or greater than 13. In addition, there is little mathematical literature available on this subject that is easily accessible.

3.1 P = 13

The problem for p = 13 is equivalent to finding a pattern in $\sum\sigma(n,4)$, since we have already calculated the other sequences.

$\sum\sigma(n, 4)$
1
325
16325
328825
3801700
30155076
181887300
890475300
3698053425
13441638925

Acknowledgments. This article could not have been written without the help of my grandmother Maria Esther.

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