ADDITIVE PROPERTY OF GENERALIZED CORE-EP INVERSE IN BANACH *-ALGEBRA

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ABSTRACT. We present new necessary and sufficient conditions under which the sum of two generalized core-EP invertible elements in a Banach *-algebra has generalized core-EP inverse. As an application, the generalized core-EP invertibility for the matrices with generalized core-EP invertible entries is investigated.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra with an involution $\ast$. An element $a \in \mathcal{A}$ has group inverse provided that there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$ 

Such $x$ is unique if exists, denoted by $a^\#$, and called the group inverse of $a$. Evidently, a square complex matrix $A$ has group inverse if and only if $\text{rank}(A) = \text{rank}(A^2)$.

An element $a \in \mathcal{A}$ has core inverse if there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, (ax)^* = ax.$$ 

If such $x$ exists, it is unique, and denote it by $a^{\#\#}$. As is well known, an element $a \in \mathcal{A}$ has core inverse if and only if $a \in \mathcal{A}$ has group inverse and it has $(1,3)$-inverse. Here, $a \in \mathcal{A}$ has $(1,3)$ inverse provided that there exists some $x \in \mathcal{A}$ such that $axa = a$ and $(ax)^* = ax$.

In [10], Gao and Chen extended the core inverse and introduced the core-EP inverse (i.e., pseudo core inverse). An element $a \in \mathcal{A}$ has core-EP inverse if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$ 

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If such $x$ exists, it is unique, and denote it by $a^\mathbb{D}$. Evidently, $a \in \mathcal{A}$ has core-EP inverse if and only if $a^n$ has core inverse for some $n \in \mathbb{N}$.

Many authors have investigated group, core and core-EP inverses from many different views, e.g., [1, 9, 11, 12, 13, 16, 17, 18, 19, 20, 22]. The additive properties of generalized inverses mentioned above are attractive.

We use $\mathcal{A}^\#$, $\mathcal{A}^\oplus$ and $\mathcal{A}^\mathbb{D}$ to denote the set of all group invertible, core invertible and core-EP invertible elements in $\mathcal{A}$, respectively.

Let $a, b \in \mathcal{A}^\#$. In [20, Theorem 4.3], Xue, Chen and Zhang proved that $a + b \in \mathcal{A}^\#$ if $ab = 0$ and $a^*b = 0$. In [22, Theorem 4.1], Zhou et al. considered the core inverse of $a + b$ under the conditions $a^2a^\oplus b = baa^\oplus$, $ab^\oplus b = a^*b$. In [7, Theorem 2.5], the authors studied the additive property of core inverses under the wider condition $ab(1 - aa^\#) = 0$ (see [6, Theorem 2.3]).

Let $a, b \in \mathcal{A}^\oplus$. In [20, Theorem 4.3], Xue, Chen and Zhang proved that $a + b \in \mathcal{A}^\oplus$ if $ab = 0$ and $a^*b = 0$. In [22, Theorem 4.1], Zhou et al. considered the core inverse of $a + b$ under the conditions $a^2a^\oplus b = baa^\oplus$, $ab^\oplus b = a^*b$. In [7, Theorem 2.5], the authors studied the additive property of core inverses under the wider condition $ab(1 - aa^\#) = 0$ (see [6, Theorem 2.3]).

Let $a, b \in \mathcal{A}^\mathbb{D}$. In [10, Theorem 4.4], Gao and Chen proved that $a + b$ has core-EP inverse if $ab = 0$ and $a^*b = 0$.

As a natural generalization of core-EP invertibility, the authors introduced the generalized core-EP inverse in Banach algebra with an involution (see [4, 5]). An element $a \in \mathcal{A}$ is generalized core-EP invertible if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, \lim_{n \to \infty} ||a^n - xa^{n+1}||^\frac{1}{n} = 0.$$ 

If such $x$ exists, it is unique, and denote it by $a^\mathbb{D}$.

Recall that an element $a \in \mathcal{A}$ has generalized Drazin inverse if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$ 

Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1}\}$. Such $x$ is unique, if exists, and denote it by $a^d$. The generalized Drazin inverse plays an important role in ring and matrix theory (see [3]).

We use $\mathcal{A}^d$, $\mathcal{A}^\oplus$ and $\mathcal{A}^{(1,3)}$ to denote the set of all generalized Drazin invertible, generalized core-EP invertible and $(1,3)$-invertible elements in $\mathcal{A}$, respectively. We list several characterizations of generalized core-EP inverse.

**Theorem 1.1.** (see [4, 5, 8]) Let $\mathcal{A}$ be a Banach $*$-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:

1. $a \in \mathcal{A}^\oplus$. 
(2) There exist \( x, y \in \mathcal{A} \) such that 
\[
a = x + y, \quad x^*y = yx = 0, \quad x \in \mathcal{A}^\oplus, \quad y \in \mathcal{A}^\text{qnil}.\]

(3) There exists a projection \( p \in \mathcal{A} \) such that 
\[
a + p \in \mathcal{A}^{-1}, \quad pa = pap \in \mathcal{A}^\text{qnil}.\]

(4) \( a \in \mathcal{A}^d \) and \( a^d \in \mathcal{A}^\oplus \). In this case, \( a^\oplus = (a^d)^2(a^d)^\oplus \).

(5) \( a \in \mathcal{A}^d \) and \( a^d \in \mathcal{A}^{(1,3)} \). In this case, \( a^\oplus = (a^d)^2(a^d)^{(1,3)} \).

Let \( a, b \in \mathcal{A}^\oplus \). In [8, Theorem 3.4], the authors proved that \( a + b \in \mathcal{A}^\oplus \) provided that \( ab = 0, a^*b = 0 \) and \( ba = 0 \). The motivation of this paper is to present new additive results for the generalized core-EP inverses. We shall give necessary and sufficient conditions under which the sum of two generalized core-EP invertible elements has generalized core-EP inverse. As an application, the generalized core-EP invertibility for the matrices with generalized core-EP invertible entries is investigated.

Throughout the paper, all Banach *-algebras are complex with an identity. An element \( p \in \mathcal{A} \) is a projection if \( p^2 = p = p^* \). Let \( a^\pi = 1 - aa^d \) and \( a^\sigma = 1 - aa^\oplus \) for \( a \in \mathcal{A}^\oplus \). Let \( a, p^2 = p \in \mathcal{A} \). Then \( a \) has the Pierce decomposition relative to \( p \), and we denote it by \( (a_{11} \ a_{12}) \ a_{21} \ a_{22})_p \).

2. KEY LEMMAS

To prove the main results, some lemmas are needed. We begin with

**Lemma 2.1.** ([8, Lemma 3.2]) Let \( a, b \in \mathcal{A}^\oplus \). If \( ab = ba \) and \( a^*b = ba^* \), then \( a^\oplus b = ba^\oplus \).

**Lemma 2.2.** ([8, Theorem 3.3]) Let \( a, b \in \mathcal{A}^\oplus \). If \( ab = ba \) and \( a^*b = ba^* \), then \( ab \in \mathcal{A}^\oplus \) and \( (ab)^\oplus = a^\oplus b^\oplus \).

**Lemma 2.3.** Let \( a \in \mathcal{A}^\oplus \) and \( b \in \mathcal{A}^\text{qnil} \). If \( a^*b = 0 \) and \( ba = 0 \), then \( a + b \in \mathcal{A}^\oplus \). In this case, 
\[
(a + b)^\oplus = a^\oplus.
\]

**Proof.** Since \( a \in \mathcal{A}^\oplus \), by virtue of Theorem 1.1, there exist \( x \in \mathcal{A}^\oplus \) and \( y \in \mathcal{A}^\text{qnil} \) such that \( a = x + y, x^*y = 0, yx = 0 \). As in the proof of [5, Theorem 2.1], \( x = aa^\oplus a \) and \( y = a - aa^\oplus a \). Then \( a = x + (y + b) \). Since \( by = b(a - aa^\oplus a) = 0 \), it follows by [14, Theorem 2.2] that \( y + b \in \mathcal{A}^\text{qnil} \). We directly verify that 
\[
\begin{align*}
x^*(y + b) &= x^*y + x^*b = (a^\oplus a)^*(a^*b) = 0, \\
(y + b)x &= yx + (ba)a^\oplus a = 0.
\end{align*}
\]
In light of Theorem 1.1, \( a + b \in \mathcal{A}^\oplus \). In this case,
\[
(a + b)^\oplus = x^\oplus = a^\oplus,
\]
as asserted. \(\square\)

**Lemma 2.4.** Let \( a \in \mathcal{A}^\oplus \) and \( m \in \mathbb{N} \). Then \( a^\oplus a^m a^\oplus = a^{m-1} a^\oplus \).

**Proof.** Since \( a(a^\oplus)^2 = a^\oplus \), we see that \( a^\oplus = a^{n-m+1}(a^\oplus)^{n-m} \) for any \( n \geq m + 1 \). Then
\[
(a^{m-1} - a^\oplus a^m)a^\oplus = (a^n - a^\oplus a^{n+1})(a^\oplus)^{n-m}.
\]
Hence,
\[
||(a^{m-1} - a^\oplus a^m)a^\oplus||^\frac{1}{n} \leq ||a^n - a^\oplus a^{n+1}||^\frac{1}{n} ||a^\oplus||^\frac{n-m}{n}.
\]
Since \( \lim_{n \to \infty} ||a^n - a^\oplus a^{n+1}||^\frac{1}{n} = 0 \), we deduce that
\[
\lim_{n \to \infty} ||(a^{m-1} - a^\oplus a^m)a^\oplus||^\frac{1}{n} = 0.
\]
Therefore \( a^{m-1}a^\oplus = a^\oplus a^m a^\oplus \). \(\square\)

**Lemma 2.5.** Let \( a \in \mathcal{A}^\oplus \) and \( b \in \mathcal{A} \). Then the following are equivalent:

1. \( (1 - a^\oplus) b = 0 \).
2. \( (1 - a a^\oplus) b = 0 \).
3. \( a^\tau b = 0 \).

**Proof.** (1) \(\Rightarrow\) (3) Since \( (1 - a^\oplus) b = 0 \), we have \( b = a^\oplus ab \). In view of Theorem 1.1, \( a^\oplus = (a^d)^2(a^d)^\oplus \). Thus, \( a^\tau b = (1 - aa^d) b = (1 - aa^d)(a^d)^2(a^d)^\oplus ab = 0 \).

(3) \(\Rightarrow\) (2) Since \( a^d = (a^d)^2 a = a^d a^d a^\oplus a^d a = (a^d)^2 a^\oplus a^d a^\oplus a^d a \), then \( b = aa^d b = a^\oplus a^2 a^d b \); and so \( (1 - aa^\oplus) b = (1 - aa^\oplus) a^\oplus a^2 a^d b = 0 \), as desired.

(2) \(\Rightarrow\) (1) In view of Lemma 2.4, \( aa^\oplus = a^\oplus a^2 a^\oplus \). Since \( (1 - aa^\oplus) b = 0 \), we get \( b = aa^\oplus b \). Therefore \( (1 - a^\oplus) b = (1 - a^\oplus) a a^\oplus b = (a - a^\oplus a^2) a^\oplus b = 0 \), as asserted. \(\square\)

Let \( \mathcal{A} \) be a Banach *-algebra. Then \( M_2(\mathcal{A}) \) is a Banach *-algebra with *-transpose as the involution. We come now to generalized EP-inverse of a triangular matrix over \( \mathcal{A} \).

**Lemma 2.6.** Let \( p \in \mathcal{A} \) be a projection and \( x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \).
(1) If $a, d \in \mathcal{A}^d$, then $x \in M_2(\mathcal{A}_p^d)$ and $x^d = \left( \begin{array}{c} a^d \\ 0 \\ d^d \end{array} \right)_p$, where
\[ z = \sum_{i=0}^{\infty} (a^d)^{i+2}bd^d \pi + \sum_{i=0}^{\infty} a^i a^\pi b(d^d)^{i+2} - a^d bd^d. \]

(2) If $a, d \in \mathcal{A}^{\oplus}$ and $a^\pi b = 0$, then $x \in M_2(\mathcal{A}_p^{\oplus})$ and
\[ x^{\oplus} = \left( \begin{array}{c} a^{\oplus} \\ -a^{\oplus}bd^{\oplus} \\ 0 \end{array} \right)_p, \]
where $z = -a^d bd^{\oplus}$.

**Proof.** See [23, Lemma 2.1] and [19, Theorem 2.5].

We are ready to prove the following lemma which is repeatedly used in the sequel.

**Lemma 2.7.** Let $p \in \mathcal{A}$ be a projection and $x = \left( \begin{array}{c} a \\ b \\ 0 \end{array} \right)_p \in M_2(\mathcal{A}_p)$ with $a, d \in \mathcal{A}^{\oplus}$. If
\[ \sum_{i=0}^{\infty} a^i a^\pi b(d^d)^{i+2} = 0, \]
then $x \in M_2(\mathcal{A}_p^{\oplus})$ and
\[ x^{\oplus} = \left( \begin{array}{c} a^{\oplus} \\ -a^{\oplus}bd^{\oplus} \\ 0 \end{array} \right)_p, \]
where $z = -a^d bd^{\oplus}$.

**Proof.** In view of Theorem 1.1, $a, d \in \mathcal{A}^d$ and $a^d, d^d \in \mathcal{A}^{\oplus}$. By virtue of Lemma 2.6, we have
\[ x^d = \left( \begin{array}{c} a^d \\ 0 \\ d^d \end{array} \right)_p, \]
where
\[ s = \sum_{i=0}^{\infty} (a^d)^{i+2}bd^d \pi + \sum_{i=0}^{\infty} a^i a^\pi b(d^d)^{i+2} - a^d bd^d. \]
By hypothesis, we get $s = \sum_{i=0}^{\infty} (a^d)^{i+2}bd^d \pi - a^d bd^d$. Since $(a^d)^{\pi} s = (1 - a^d a^2 a^d) s = p^\pi s = a^\pi [\sum_{i=0}^{\infty} (a^d)^{i+2}bd^d \pi - a^d bd^d] = 0$. In view of [19, Lemma
we have \([1 - a^d(a^d)^\oplus]s = 0\). Then it follows by Lemma 2.6 that

\[
(x^d)^\oplus = \begin{pmatrix} (a^d)^\oplus & t \\ 0 & (d^d)^\oplus \end{pmatrix},
\]

where \(t = -(a^d)^\oplus s(d^d)^\oplus\). Hence, \(t = -(a^d)^\oplus \sum_{i=0}^{\infty} (a^d)^{i+2}bd_i d^\pi - a^d bd^d (d^d)^\oplus = (a^d)^\oplus a^d bd^d (d^d)^\oplus\). Then we have

\[
(x^d)^2 = \begin{pmatrix} (a^d)^2 & w \\ 0 & (d^d)^2 \end{pmatrix},
\]

where \(w = \sum_{i=0}^{\infty} (a^d)^{i+3}bd_i d^\pi - (a^d)^2 bd^d - a^d b(d^d)^2\). Therefore

\[
x^\otimes = \begin{pmatrix} (a^d)^2 (x^d)^\oplus \\ 0 \end{pmatrix} = \begin{pmatrix} (a^d)^2 & w \\ 0 & (d^d)^2 \end{pmatrix} \begin{pmatrix} (a^d)^\oplus & t \\ 0 & (d^d)^\oplus \end{pmatrix} = \begin{pmatrix} a^\otimes z \\ 0 \end{pmatrix},
\]

where

\[
z = (a^d)^2 t + w(d^d)^\oplus = (a^d)^2 [(a^d)^\oplus a^d bd^d (d^d)^\oplus] - [(a^d)^2 bd^d + a^d b(d^d)^2] (d^d)^\oplus = (a^d)^2 bd^d(d^d)^\oplus - a^d(a^d b + bd^d)d^d(d^d)^\oplus = (a^d)^2 bd^d(d^d)^\oplus - (a^d)^2 bd^d (d^d)^\oplus - a^d [b(d^d)^2 (d^d)^\oplus] = -a^d bd^\otimes.
\]

This completes the proof. \(\square\)

**Lemma 2.8.** Let \(\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A})_p\) with \(a, d \in \mathcal{A}^\otimes\). If \(a^\pi bd^\otimes = 0\), then \(\alpha \in M_2(\mathcal{A})^\otimes\) and

\[
\alpha^\otimes = \begin{pmatrix} a^\otimes & -a^\otimes bd^\otimes \\ 0 & d^\otimes \end{pmatrix}_p.
\]

**Proof.** Since \(a^\pi bd^\otimes = 0\), it follows by Theorem 1.1 that \(a^\pi b(d^d)^2 (d^d)^\oplus = 0\); hence,

\[
a^\pi bd^d = [a^\pi b(d^d)^2 (d^d)^\oplus]b^d b = 0.
\]

By using Lemma 2.5, we have \((1 - a^a^\otimes)bd^\otimes = 0\), and so \(bd^\otimes = a^a^\otimes bd^\otimes\). Then

\[
a^d bd^\otimes = a^a^d a^a^\otimes bd^\otimes = a^a^\otimes bd^\otimes.
\]
In light of Lemma 2.7, 
\[ \alpha^@ = \begin{pmatrix} a^@ & -a^@b^@d^@ \\ 0 & d^@ \end{pmatrix}, \]

as asserted. \(\Box\)

3. MAIN RESULTS

This section is devoted to investigate the generalized core-EP inverse of the sum of two generalized core-EP invertible elements in a Banach *-algebra. We come now to establish additive property of generalized core-EP inverse under orthogonal conditions.

**Theorem 3.1.** Let \(a, b, a^@b \in A^@\). If 
\[ a^@ab = 0, a^@ba = 0 \text{ and } a^@b^*a = 0, \]
then the following are equivalent:

1. \(a + b \in A^@\) and \(a^@((a + b)^@)aa^@ = 0.\)
2. \((a + b)aa^@ \in A^@\) and 
\[ \sum_{i=0}^{\infty} (a + b)^i(a + b)^@aa^@ (a + b)^@b^@d^@ = 0. \]

In this case,
\[ (a + b)^@ = [(a + b)aa^@]^@ + (a^@b)^@ - (a + b)^d aa^@ (a + b)(a^@b)^@. \]

**Proof.** (1) \(\Rightarrow\) (2) Let \(p = aa^@\). By hypothesis and Lemma 2.5, we have 
\[ p^@ab = 0, p^@ba = 0 \text{ and } p^@b^*a = 0. \]
Hence, \(p^@bp = (p^@ba)a^@ = 0, \)
\[ p^@ap = (1 - aa^@)a^2a^@ = 0 \]
and
\[ pap^@ = aa^@a(1 - aa^@) = aa^@a - a^2a^@. \]

Then we have
\[ a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}_p. \]

Hence
\[ a + b = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ 0 & a_4 + b_4 \end{pmatrix}_p. \]

Here, \(a_1 = aa^@a^2a^@ = a^2a^@\) and \(b_1 = aa^@b^@a^@ = b^@a^@. \)
Since $a^\pi(a+b)^@aa^@ = 0$, it follows by Lemma 2.5 that $p^\pi(a+b)^@aa^@ = 0$. Write

$$(a+b)^@ = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}.$$ 

Then

$$(a_1+b_1)\alpha^2 = \alpha, \quad [(a_1+b_1)\alpha]^* = (a_1+b_1)\alpha, \lim_{n \to \infty} ||(a_1+b_1)^n - \alpha(a_1+b_1)^{n+1}||^1 = 0.$$ 

We infer that $(a_1+b_1)^@ = \alpha$, as required.

$(2) \Rightarrow (1)$ Let $p = aa^@$. Construct $a_i, b_i (i = 1, 2, 4)$ as in $(1) \Rightarrow (2)$. Then

$$a + b = \begin{pmatrix} a_1 + b_1 & a_2 + b_2 \\ 0 & a_4 + b_4 \end{pmatrix}.$$ 

Hence $a_1 + b_1 = (a+b)aa^@$. Since $p^\pi(a+b) = a^\pi a + p^\pi b$ and $(p^\pi b)(p^\pi a) = 0$, it follows by [3, Lemma 15.2.2] that $p^\pi(a+b) \in A^d$. As $p^\pi(a+b)aa^@ = 0$, by using [21, Lemma 2.2],

$$(a_1 + b_1)^d = [(a+b)aa^@]^d = (a+b)^d aa^@.$$ 

Moreover, we have

$$(a_1 + b_1)^\pi = aa^@ - (a+b)^d aa^@(a+b) aa^@ 
= aa^@ - (a+b)^d (a+b) aa^@ 
= (a+b)^\pi aa^@.$$ 

We see that

$$a_1 + b_1 = (a+b)aa^@ \in A^@.$$ 

Also we have $a_4 = p^\pi ap^\pi = p^\pi a$ and $b_4 = p^\pi bp^\pi = p^\pi b$, and so

$$a_4 + b_4 = p^\pi a + p^\pi b.$$ 

We claim that

$$(p^\pi a)(p^\pi b) = p^\pi ab = 0,$$

$$(p^\pi b)^*(p^\pi a) = (p^\pi bp^\pi)^*(p^\pi a) 
= (1 - aa^@)b^*(1 - aa^@)(p^\pi a) 
= p^\pi b^*(p^\pi a) = 0.$$ 

As in the proof of [5, Theorem 2.1], $a - a^@a^2 \in A^{qnil}$. By using Cline’s formula, $p^\pi a = a - aa^@ \in A^{qnil}$. Thus, $a_4 + b_4 \in A^@$ and $(a_4 + b_4)^@ = (p^\pi b)^@$ by Lemma 2.3.

We check that

$$(a_4 + b_4)^d = p^\pi b^d,$$

$$(a_4 + b_4)^\pi = p^\pi b^\pi.$$
Moreover, we see that
\[
\sum_{i=0}^{\infty} (a_1 + b_1)^i (a_2 + b_2) [(a_4 + b_4)^d]^{i+2} = 0.
\]

According to Lemma 2.7, \( a + b \in \mathcal{A}^\oplus \). Furthermore, we have
\[
(a + b)^\oplus = (a_1 + b_1)^\oplus + (a_4 + b_4)^\oplus + z = [(a + b)a_4^\oplus] + [(1 - a_4^\oplus)b]^\oplus + z,
\]
where
\[
z = -(a_1 + b_1)^d (a_2 + b_2) (a_4 + b_4)^\oplus = -(a + b)^d a_4^\oplus (a + b) [(1 - a_4^\oplus)b]^\oplus,
\]
as asserted.

\[\Box\]

**Corollary 3.2.** ([8, Theorem 3.4]) Let \( a, b \in \mathcal{A}^\oplus \). If \( a^*b = 0 \) and \( ab = ba = 0 \), then \( a + b \in \mathcal{A}^\oplus \). In this case,
\[
(a + b)^\oplus = a^\oplus + b^\oplus.
\]

**Proof.** This is immediate from Theorem 3.1. \[\Box\]

**Corollary 3.3.** Let \( a, b \in \mathcal{A}^\oplus \). If \( a^*b = 0 \) and \( a^*b^* = 0 \), then the following are equivalent:

1. \( a + b \in \mathcal{A}^\oplus \) and \( a^\sigma (a + b)^\oplus a^{\sigma^\oplus} = 0 \).
2. \( (a + b) a a^{\sigma^\oplus} \in \mathcal{A}^\oplus \).

In this case,
\[
(a + b)^\oplus = [(a + b)a a^{\sigma^\oplus}]^\oplus.
\]

**Proof.** By hypothesis, we see that \( a^\sigma ab = a(a^\sigma b) = 0, a^\sigma ba = (a^\sigma b)a = 0, a^\sigma b^* a = (a^\sigma b^*)a = 0 \). Since \( a^\sigma b = 0 \), it follows by Lemma 2.5 that \( a^\sigma b^d = [(1 - a a^{\sigma^\oplus})b](b^d)^2 = 0 \). In light of Theorem 3.1, \( a + b \in \mathcal{A}^\oplus \) and \( a^\sigma (a + b)^\oplus a a^{\sigma^\oplus} = 0 \) if and only if \( (a + b) a a^{\sigma^\oplus} \in \mathcal{A}^\oplus \). In this case, \( a^\sigma = 0 \), and therefore \( (a + b)^\oplus = [(a + b)a a^{\sigma^\oplus}]^\oplus \). \[\Box\]

**Corollary 3.4.** Let \( a, b \in \mathcal{A}^\oplus \). If \( a^\sigma b = 0, a^\sigma b^* = 0 \) and \( ba^d = 0 \), then \( a + b \in \mathcal{A}^\oplus \). In this case, \( (a + b)^\oplus = a^\oplus \).
Proof. We easily verify that \((a^2a^\oplus)a^\oplus = aa^\oplus\); hence, \([(a^2a^\oplus)a^\oplus]^* = (a^2a^\oplus)a^\oplus\). Moreover, we have \((a^2a^\oplus)a^\oplus(a^\oplus)^2 = a^\oplus\). By induction, we prove that \((a^2a^\oplus)^n = a^{n+1}a^\oplus\) and \((a^2a^\oplus)^{n+1} = a^{n+2}a^\oplus\). Therefore \((a^2a^\oplus)^n - a^\oplus(a^2a^\oplus)^{n+1} = [a^n] - a^\oplus a^{n+1}]aa^\oplus\).

Since \(\lim_{n \to \infty} \|a^n - a^\oplus a^{n+1}\|_\frac{1}{n} = 0\), we deduce that \(\lim_{n \to \infty} \|a^2(a^2a^\oplus)^n - a^\oplus(a^2a^\oplus)^{n+1}\|_\frac{1}{n} = 0\).

Hence, \((a^2a^\oplus)^\oplus = a^\oplus\). Therefore we complete the proof by Corollary 3.3. \(\square\)

We next present the additive property of generalized core-EP inverse under commutative conditions. For the detailed formula of the generalized core-EP inverse of the sum, we leave to the readers as it can be derived by the straightforward computation according to our proof.

**Theorem 3.5.** Let \(a, b \in A^\oplus\). If \(ab = ba\) and \(a^*b = ba^*\), then the following are equivalent:

1. \(a + b \in A^\oplus\) and \(a^x(a + b)^\oplus aa^\oplus = 0\).
2. \(1 + a^\oplus b \in A^\oplus\) and

\[
\sum_{i=0}^{\infty} (1 + a^\oplus b)^i a^\oplus (1 + a^\oplus b)^x aa^\oplus a[(1 - aa^\oplus)b^\oplus(1 + (1 - aa^\oplus)ab^\oplus)^{-1}]^{i+1} = 0
\]

Proof. Since \(ab = ba\) and \(a^*b = ba^*\), it follows by Lemma 2.1 that \(a^\oplus b = ba^\oplus\). Let \(p = aa^\oplus\). Then \(p^xbp = (1 - aa^\oplus)baa^\oplus = (1 - aa^\oplus)aa^\oplus b = 0\). Moreover, we have \(pbp^x = aa^\oplus b(1 - aa^\oplus) = aba^\oplus(1 - aa^\oplus) = 0\). In light of Lemma 2.4, we have

\[
p^xap = (1 - aa^\oplus)aa^\oplus
=p^2a^\oplus - aa^\oplus a^2a^\oplus
=0.
\]

So we get

\[
a = \begin{pmatrix} a_1 & a_2 \\ 0 & a_4 \end{pmatrix}_p, \quad b = \begin{pmatrix} b_1 & 0 \\ 0 & b_4 \end{pmatrix}_p.
\]

Hence

\[
a + b = \begin{pmatrix} a_1 + b_1 & a_2 \\ 0 & a_4 + b_4 \end{pmatrix}_p.
\]

Moreover,

\[
a_1 = aa^\oplus a^2a^\oplus = a^2a^\oplus.
\]
Theorem 15.2.16, This implies that \( a_1 + b_1 = (1 + a^{\oplus} b) a^2 a^{\oplus} \in A^{\oplus} \).

This implies that

\[
(a_1 + b_1)^i = (1 + a^{\oplus} b)^i (a^2 a^{\oplus})^i = (1 + a^{\oplus} b)^i a^{i+1} a^{\oplus}.
\]

Furthermore,

\[
(a_1 + b_1)^d = (1 + a^{\oplus} b)^d a^{\oplus}.
\]

Thus

\[
(a_1 + b_1)^{\pi} = 1 - (1 + a^{\oplus} b)(1 + a^{\oplus} b)^d a a^{\oplus}.
\]

Clearly, we have \((1 - a a^{\oplus}) a a^{\oplus} = a^2 a^{\oplus} - a a^{\oplus} a a^{\oplus} = 0\). Then

\[
a_4 = (1 - a a^{\oplus}) a (1 - a a^{\oplus}) = a - a a^{\oplus} a.
\]

As in the proof of [5, Theorem 2.1], \( a - a a^{\oplus} a^2 \in A^{mid} \). By using Cline’s formula, \( a_4 \in A^{mid} \). Moreover,

\[
b_4 = (1 - a a^{\oplus}) b (1 - a a^{\oplus}) = (1 - a a^{\oplus}) b.
\]

Since \( b p^{\pi} = p^{\pi} b, b^{*} p^{\pi} = (p^{\pi} b)^{*} = p^{\pi} b^{*} \). In light of Lemma 2.2, \( b_4 = p^{\pi} b \in A^{\oplus} \) and \( b_4^{\oplus} = p^{\pi} b^{\oplus} \). Furthermore,

\[
(a_4 + b_4) = (1 - a a^{\oplus})(a + b)
\]

\[
(a_4 + b_4)^i = (1 - a a^{\oplus})(a + b)^i.
\]

(1) \( \Rightarrow \) (2) We have

\[
(a + b)^{\oplus} = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}^p.
\]

As in the proof of Theorem 3.1, \([p(a + b)p]^{\oplus} = \alpha\). That is, \( (a + b) a a^{\oplus} \in A^{\oplus} \).

We observe that

\[
1 + a^{\oplus} b = [1 - a a^{\oplus}] + [a a^{\oplus} + a^{\oplus} b]
= [1 - a a^{\oplus}] + [a a^{\oplus} + b a^{\oplus}]
= [1 - a a^{\oplus}] + [a + b] a^{\oplus}
\]

We easily check that \([a + b] a a^{\oplus} a^{\oplus} = a^{\oplus} [(a + b) a a^{\oplus}]\). In view of [3, Theorem 15.2.16], \( (a + b) a^{\oplus} = [(a + b) a a^{\oplus}] a^{\oplus} \in A^{d} \) and

\[
[a + b] a^{\oplus}]^d = [(a + b) a a^{\oplus}]^d [a^{\oplus}]^d.
\]
In view of Theorem 1.1, \([(a + b)aa^\oplus]^d\) has (1, 3)-inverse. Then there exists \(y \in A\) such that
\[
[(a + b)aa^\oplus]^d = [(a + b)aa^\oplus]^d y \cdot [(a + b)aa^\oplus]^d y.
\]
We verify that
\[
\begin{align*}
[(a + b)aa^\oplus]^d & \cdot [(a + b)aa^\oplus]^d y = \left( [(a + b)aa^\oplus]^d y \right) \cdot \left( [(a + b)aa^\oplus]^d y \right), \\
[(a + b)aa^\oplus]^d & = [(a + b)aa^\oplus]^d y. \\
\end{align*}
\]
Clearly, \([a^2a^\oplus](a^\oplus)^d = aa^\oplus\). Then
\[
\begin{align*}
\left( [(a + b)aa^\oplus]^d y \right) & = \left( [(a + b)aa^\oplus]^d y \right) \cdot \left( [(a + b)aa^\oplus]^d y \right), \\
[(a + b)aa^\oplus]^d & = \left( [(a + b)aa^\oplus]^d y \right). \\
\end{align*}
\]
Therefore \([(a + b)a^\oplus]^d\) has (1, 3)-inverse \((a^2a^\oplus)y\). In light of Theorem 1.1, \((a + b)a^\oplus \in A^\oplus\).

Obviously, we have
\[
[1 - aa^\oplus](a + b)a^\oplus = [1 - aa^\oplus][a + b]a^\oplus = [a + b]a^\oplus [1 - aa^\oplus] = 0.
\]
According to Corollary 3.2, \(1 + a^\oplus b \in A^\oplus\).

In view of Lemma 2.6,
\[
(a + b)^d = \begin{pmatrix} (a_1 + b_1)^d & z \\ 0 & (a_4 + b_4)^d \end{pmatrix}_p,
\]
where
\[
\begin{align*}
z & = \sum_{i=0}^{\infty} [(a_1 + b_1)^d]^{i+2} a_2 (a_4 + b_4)^i (a_4 + b_4)^z \\
& + \sum_{i=0}^{\infty} (a_1 + b_1)^i (a_1 + b_1)^z a_2 [(a_4 + b_4)^d]^{i+2} \\
& - (a_1 + b_1)^d a_2 (a_4 + b_4)^d. \\
\end{align*}
\]
By virtue of Theorem 1.1,
\[
(a + b)^\oplus = [(a + b)^d]^2 [(a + b)^d]^\oplus
\]
Hence,
\[ [(a + b)^d]^{\oplus} = (a + b)(a + b)^d[(a + b)^d]^{\oplus} = (a + b)^2(a + b)^{\oplus}. \]

Since \( p^\pi(a + b)^2p = p^\pi(a + b)^dp = 0 \), we see that \( p^\pi[(a + b)^d]^{\oplus}p = 0 \). As in the proof of [19, Theorem 2.5], \( [(a_1 + b_1)^d]^{\pi}z = 0 \). Thus, we have \( (a_1 + b_1)^{\pi}z = 0 \); hence,
\[ \sum_{i=0}^{\infty} (a_1 + b_1)^i(a_1 + b_1)^{\pi}a_2[(a_4 + b_4)^d]^{i+2} = 0 \]

Thus,
\[ (a_4 + b_4)^d = (1 - aa^{\oplus})b^{\oplus}[1 + (1 - aa^{\oplus})ab^d]^{-1}. \]

Therefore
\[ \sum_{i=0}^{\infty} (1 + a^{\oplus})^i a^{\pi}a^{\oplus}[1 - (1 + a^{\oplus})b(1 + a^{\oplus})^d aa^{\oplus}][1 - (1 - aa^{\oplus})ab^d]^{-1}]^{i+2} = 0. \]

Accordingly,
\[ \sum_{i=0}^{\infty} (1 + a^{\oplus})^i a^{\pi}a^{\oplus}(1 + a^{\oplus})^d aa^{\oplus}a[(1 - aa^{\oplus})b^{\oplus}(1 + (1 - aa^{\oplus})ab^d]^{-1}]^{i+2} = 0. \]

(2) \( \Rightarrow \) (1) Step 1. Since \( (1 + a^{\oplus}b)aa^{\oplus} = aa^{\oplus}(1 + a^{\oplus}b) \) and \( (aa^{\oplus})^{\ast} = aa^{\oplus} \), it follows by Lemma 2.2 that
\[ (1 + a^{\oplus}b)aa^{\oplus} \in A^{\oplus}. \]

Then
\[ [(1 + a^{\oplus}b)aa^{\oplus}]^d = (1 + a^{\oplus}b)^d aa^{\oplus} \in A^{(1,3)}. \]

Thus, we can find a \( y \in A \) such that
\[ (1 + a^{\oplus}b)^d aa^{\oplus} = (1 + a^{\oplus}b)^d aa^{\oplus}y(1 + a^{\oplus}b)^d aa^{\oplus}, \]
\[ ((1 + a^{\oplus}b)^d aa^{\oplus}y)^{\ast} = (1 + a^{\oplus}b)^d aa^{\oplus}y. \]

We easily verify that
\[ (1 + a^{\oplus}b)^d a^{\oplus} = (1 + a^{\oplus}b)^d a^{\oplus}z(1 + a^{\oplus}b)^d a^{\oplus}, \]
\[ ((1 + a^{\oplus}b)^d a^{\oplus}z)^{\ast} = (1 + a^{\oplus}b)^d a^{\oplus}z, \]
where \( z = a^2 a^{\oplus}y \).

Clearly, \( [(1 + a^{\oplus}b)a^2 a^{\oplus}]^d = (1 + a^{\oplus}b)^d a^{\oplus} \in A^{(1,3)} \). By virtue of Theorem 1.1, \( (a + b)aa^{\oplus} = (1 + a^{\oplus}b)a^2 a^{\oplus} \in A^{\oplus} \).
Step 2. Obviously, \(a_4b_4 = b_1a_4\). Since \(1 + a_4^d b_4 = 1\), it follows by [23, Theorem 3.3] that \((a_4 + b_4)^d = \sum_{i=0}^{\infty} (b_4^d)^i (-a_4)^i = b_4^d (1 + a_4b_4^d)^{-1}\). Since \(b_4 \in \mathcal{A}^\oplus\), by virtue of Theorem 1.1 that \(b_4^d \in \mathcal{A}^{(1,3)}\). Then we can find a \(y \in \mathcal{A}\) such that
\[
b_4^d = b_4^d y b_4^d = (b_4^d y)^* = b_4^d y.
\]

Set \(z = (1 + a_4b_4^d)y\). Then we verify that
\[
b_4^d (1 + a_4b_4^d)^{-1} = b_4^d (1 + a_4b_4^d)^{-1} z b_4^d (1 + a_4b_4^d)^{-1},
\]
\[
(b_4^d (1 + a_4b_4^d)^{-1} z)^* = (b_4^d y)^* = b_4^d y = b_4^d (1 + a_4b_4^d)^{-1} z.
\]

Hence, \(b_4^d (1 + a_4b_4^d)^{-1} \in \mathcal{A}^{(1,3)}\). In light of Theorem 1.1., \(a_4 + b_4 \in \mathcal{A}^\oplus\).

Step 3. By virtue of Theorem 1.1, \(a_1 + b_1, a_4 + b_4 \in \mathcal{A}^d\). By virtue of Lemma 2.6,
\[
(a + b)^d = \begin{pmatrix} (a_1 + b_1)^d & z \\ 0 & (a_4 + b_4)^d \end{pmatrix},
\]
where
\[
z = \sum_{i=0}^{\infty} [(a_1 + b_1)^d]^i a_2 (a_4 + b_4)^i (a_4 + b_4)^\pi
\]
\[
+ \sum_{i=0}^{\infty} (a_1 + b_1)^i (a_1 + b_1)^\pi a_2 [(a_4 + b_4)^d]^i + 2
\]
\[
- (a_1 + b_1)^d a_2 (a_4 + b_4)^d
\]
By hypothesis, we have
\[
\sum_{i=0}^{\infty} (1 + a^\oplus b)^i a^\oplus a^\oplus (1 + a^\oplus b)^\pi a a^\oplus a [(1 - a a^\oplus) b^\oplus (1 + (1 - a a^\oplus) a b^\oplus)^{-1}]^i + 2 = 0.
\]
This implies that
\[
\sum_{i=0}^{\infty} (a_1 + b_1)^i (a_1 + b_1)^\pi a_2 [(a_4 + b_4)^d^i]^i + 2 = 0.
\]
Then \((a_1 + b_1)^\pi z = 0\); and so \([a_1 + b_1)^d]^{\pi} z = 0\). In light of Lemma 2.8, \(a + b \in \mathcal{A}^\oplus\). Moreover, we have \(p^\pi (a + b)^\oplus p = 0\). In view of Lemma 2.5, \(a^\pi (a + b)^\oplus a a^\oplus = 0\). This completes the proof.

Corollary 3.6. Let \(a, b \in \mathcal{A}^\oplus\). If \(ab = ba, a^*b = b a^*\) and \(1 + a^\oplus b \in \mathcal{A}^{-1}\), then \(a + b \in \mathcal{A}^\oplus\).

**Proof.** Since \(1 + a^\oplus b \in \mathcal{A}^{-1}\), we have \((1 + a^\oplus b)^\pi = 0\). This completes the proof by Theorem 3.5. \(\square\)
4. APPLICATIONS

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{A})$. The aim of this section is to present the generalized core-EP invertibility of the square matrix $M$ by using the generalized core-EP invertibility of its entries.

**Lemma 4.1.** Let $b, c \in \mathcal{A}$. If $bc, cb \in \mathcal{A}^\oplus$, then $Q := \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$ has generalized core-EP inverse. In this case,

$$Q^\oplus = \begin{pmatrix} 0 & b(cb)^\oplus \\ c(bc)^\oplus & 0 \end{pmatrix}.$$

**Proof.** Since $Q^2 = \begin{pmatrix} bc & 0 \\ 0 & cb \end{pmatrix}$, we see that $Q^2$ has generalized core-EP inverse and

$$(Q^2)^\oplus = \begin{pmatrix} (bc)^\oplus & 0 \\ 0 & (cb)^\oplus \end{pmatrix}.$$ 

In light of [4, Lemma 3.4], $Q$ has generalized core-EP inverse and

$$Q^\oplus = Q(Q^2)^\oplus = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \begin{pmatrix} (bc)^\oplus & 0 \\ 0 & (cb)^\oplus \end{pmatrix} = \begin{pmatrix} 0 & b(cb)^\oplus \\ c(bc)^\oplus & 0 \end{pmatrix},$$

as asserted. □

We are now ready to prove:

**Theorem 4.2.** Let $a, d, bc, cb \in \mathcal{A}^\oplus$. If

$$bd = 0, ca = 0, a^\pi b = 0, d^\pi c = 0, a^\pi c^* = 0, d^\pi b^* = 0,$$

then $M$ has generalized core-EP inverse.

**Proof.** Write $M = P + Q$, where

$$P = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, Q = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.$$

Since $a$ and $d$ have generalized core-EP inverses, so has $P$, and that

$$P^d = \begin{pmatrix} a^d & 0 \\ 0 & d^d \end{pmatrix}, P^\pi = \begin{pmatrix} a^\pi & 0 \\ 0 & d^\pi \end{pmatrix}.$$
In view of Lemma 4.1, \( Q \) has generalized core-EP inverse. By hypothesis, we check that

\[
P^\pi Q = \begin{pmatrix} 0 & a^\pi b \\ d^\pi c & 0 \end{pmatrix} = 0,
\]

\[
P^\pi Q^* = \begin{pmatrix} 0 & a^\pi c^* \\ d^\pi b^* & 0 \end{pmatrix} = 0,
\]

\[
QP^d = \begin{pmatrix} 0 & bd^d \\ ca^d & 0 \end{pmatrix} = 0.
\]

According to Corollary 3.4, \( M \) has generalized core-EP inverse. \( \square \)

**Corollary 4.3.** Let \( a, d, bc, cb \in A^\oplus \). If

\[
a^d b = 0, d^d c = 0, bd^\pi = 0, ca^\pi = 0, b^*a^\pi = 0, c^*d^\pi = 0,
\]

then \( M \) has generalized core-EP inverse.

**Proof.** Obviously, \( M^* = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \). By hypothesis, we have

\[
c^*(d^*)^d = 0, b^*(a^*)^d = 0, (a^*)^\pi c^* = 0, (d^*)^\pi b^* = 0, (a^*)^\pi b = 0, (d^*)^\pi c = 0.
\]

Applying Theorem 4.2 to the operator \( M^* \), we prove that \( M^* \) has generalized core-EP inverse. Therefore \( M \) has generalized core-EP inverse, as asserted. \( \square \)

**Theorem 4.4.** Let \( a, d, bc, cb \in A^\oplus \). If

\[
ab = bd, dc = ca, a^*b = bd^*, d^*c = ca^*
\]

and \( a^\oplus bd^\oplus c \in A^{nil} \), then \( M \) has generalized core-EP inverse.

**Proof.** Write \( M = P + Q \), where

\[
P = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, Q = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}.
\]

As in the proof of Theorem 4.2, \( P \) and \( Q \) have generalized core-EP inverses.

It is easy to verify that

\[
PQ = \begin{pmatrix} 0 & ab \\ dc & 0 \end{pmatrix} = \begin{pmatrix} 0 & bd \\ ca & 0 \end{pmatrix} = QP.
\]

Likewise, we verify that \( P^*Q = QP^* \). Moreover, we check that

\[
I_2 + P^\oplus Q = \begin{pmatrix} 1 & a^\oplus b \\ d^\oplus c & 1 \end{pmatrix}.
\]
ADDITIVE PROPERTY OF GENERALIZED CORE-EP INVERSE IN BANACH *-ALGEBRA

Obviously, we have
\[
\begin{pmatrix}
1 & a \oplus b \\
\overline{d \oplus c} & 1
\end{pmatrix}
= \begin{pmatrix}
1 - a \oplus bd \oplus c & a \oplus b \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\overline{d \oplus c} & 1
\end{pmatrix}.
\]

As \(a \oplus bd \oplus c \in \mathcal{A}^{qni}, 1 - a \oplus bd \oplus c \in \mathcal{A}^{-1}\). This implies that \(\begin{pmatrix}
1 & a \oplus b \\
\overline{d \oplus c} & 1
\end{pmatrix}\) is invertible. This implies that \(I_2 + P \oplus Q\) is invertible. By using Corollary 3.6, \(M\) has generalized core-EP inverse. □

**Corollary 4.5.** Let \(a, d, bc, cb \in \mathcal{A}^{\oplus}\). If
\[
ab = bd, ca = dc, a^*b = bd^*, ac^* = c^*d
\]
and \(bd \oplus ca \oplus \in \mathcal{A}^{qni}\), then \(M\) has generalized core-EP inverse.

**Proof.** Analogously to Corollary 4.3, we complete the result by applying Theorem 4.4 to \(M^* = \begin{pmatrix}
a^* & c^* \\
b^* & d^*
\end{pmatrix}\). □

**References**


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