Generalisation of \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \)

Y. Sakuma

July 24, 2024

Abstract

It is known that most of the formulae that hold for ordinary trigonometric functions hold for generalised trigonometric functions. In this study, we succeeded in generalizing \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \). This makes it possible to discuss the generalised case in unsolved problems involving trigonometric functions, such as the generalisation of the Flint Hills series.

1 Introduction

For \( p, q > 1 \), we define the function

\[
F_{p,q}(x) = \int_0^x (1 - t^q)^{-\frac{1}{p}} \, dt \quad (x \in [0,1]).
\]

Since this function is strictly increasing it has an inverse, which we denote by \( \sin_{p,q} x \)

\[
\sin_{p,q} x = F_{p,q}^{-1}(x) \quad (x \in \left[0, \frac{\pi_{p,q}}{2}\right]),
\]

where

\[
\pi_{p,q} = 2 \int_0^1 (1 - t^q)^{-\frac{1}{p}} \, dt.
\]

Note that \( \sin_{p,q} x \) is strictly increasing on \([0, \frac{\pi_{p,q}}{2}]\), we observe that \( \sin_{p,q} x \in [0, 1] \). We can extend \( \sin_{p,q} x \) to \([0, \pi_{p,q}]\) by defining

\[
\sin_{p,q} x = \sin_{p,q} \left( \pi_{p,q} - x \right) \quad (x \in \left[\frac{\pi_{p,q}}{2}, \pi_{p,q}\right]).
\]

Furthermore we can extend to \([-\pi_{p,q}, \pi_{p,q}]\) by defining

\[
\sin_{p,q}(-x) = -\sin_{p,q} x \quad (x \in [0, \pi_{p,q}]).
\]

Finally \( \sin_{p,q} x \) is extended to whole of \( \mathbb{R} \).
On the other hand, we define \( \cos_{p,q} x \) by

\[
\cos_{p,q} x = \frac{d}{dx} (\sin_{p,q} x).
\]

Generalising trigonometric function makes it possible to generalise various open problems. For example, Flint Hills series

\[
\sum_{n=1}^{\infty} \frac{1}{n^3 |\sin n|^2}.
\]

Meiburg \[2\] studied the convergence of the Flint Hills series by extending the problem by defining a new function called sine-like function.

In this study, the aim was to extend the \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \) to a generalised form as shown in Theorem 1.

\section{The value of \( \lim_{x \to 0} \frac{x}{\sin_{p,q} x} \)}

\[\text{Theorem 1.} \quad \lim_{x \to 0} \frac{x}{\sin_{p,q} x} = 1.\]

\[\text{Lemma 2.} \quad \text{If} \ x \in \left[0, \frac{\pi pq}{2}\right], \text{then} \ \sin_{p,q} x \leq x \leq \frac{\sin_{p,q} x}{\cos_{p,q} x}.\]

\[\text{Proof.} \quad \text{We defined the} \ f(x) \ \text{and} \ g(x) \ \text{as} \ f(x) = x - \sin_{p,q} x, \ g(x) = \frac{\sin_{p,q} x}{\cos_{p,q} x} - x. \ \text{The value of} \ f(0) \ \text{and} \ g(0) \ \text{is zero. Furthermore}
\]

\[
\frac{d}{dx} f(x) = 1 - \cos_{p,q} x = 1 - (1 - (\sin_{p,q} x)^q)^\frac{1}{p} \geq 1 - 1 = 0 \quad (1)
\]

\[
\frac{d}{dt} g(x) = \frac{q}{p} \left(\frac{\sin_{p,q} x}{\cos_{p,q} x}\right)^q \geq 0. \quad (2)
\]

\[\text{In (2) we used the fact that} \ \left(\sin_{p,q} x\right)^q + \left(\cos_{p,q} x\right)^p = 1 \ \quad (3)\]

\[\text{holds.} \] According to Edmunds et.al \[1\] (3) holds when \( x > 0 \) is close enough to zero. So (1) and (2) , both \( f(x) \) and \( g(x) \) are found to be monotonically increasing functions. Therefore, since \( f(x), g(x) > 0 \) whenever \( x > 0 \) , so

\[
\sin_{p,q} x \leq x \leq \frac{\sin_{p,q} x}{\cos_{p,q} x}
\]

holds.
Theorem 1. \( \lim_{x \to 0} \frac{x}{\sin_{p,q} x} = 1. \)

Proof. Since if \( x \in \left[0, \frac{\pi p}{2}\right] \), then \( \sin_{p,q} x > 0 \), the inequality in Lemma 2 can be transformed as follows that
\[
1 \leq \frac{x}{\sin_{p,q} x} \leq \frac{1}{\cos_{p,q} x}.
\] (4)

Since (5) holds, the squeeze theorem can be used in conjunction with (4).
\[
\lim_{x \to +0} \frac{1}{\cos_{p,q} x} = \lim_{x \to +0} \left(1 - \left(\sin_{p,q} x\right)^q\right)^{-\frac{1}{p}} = 1
\] (5)

Therefore
\[
\lim_{x \to +0} \frac{x}{\sin_{p,q} x} = 1.
\]

Next, we want to prove
\[
\lim_{x \to -0} \frac{x}{\sin_{p,q} x} = 1.
\]

Let \( x = -t \), then
\[
1 = \lim_{t \to -0} \frac{-t}{\sin_{p,q} (-t)} = \lim_{t \to -0} \frac{t}{\sin_{p,q} t}
\]
holds. Therefore
\[
\lim_{x \to -0} \frac{x}{\sin_{p,q} x} = 1.
\]

3 Conclusion

In this study, it was shown that \( \lim_{x \to 0} \frac{x}{\sin x} = 1 \) is also valid for generalised trigonometric functions. Edmunds et.al [1] also succeeded in generalising well-known formulas such as \( \sin^2 x + \cos^2 x = 1 \), so it is expected that many of the formulas that hold for ordinary trigonometric functions will hold in the generalised case.

References