On the Arc Length of an Ellipse

Tai-choon Yoon[1]

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Abstract

Elliptic integral is an integral equation that appears in the process of calculating the length of an ellipse. It does not provide an exact solution, and the approximation equation for the solution is complicated. The arc length of an ellipse is given as $l = a\theta E(k)$. And the entire arc length of an ellipse is $l = 2a\pi E(k)$.

An ellipse refers to the trajectory when two fixed foci and a line longer than the distance between the two foci draws a triangular point in a plane where the sum of the distances is constant.

Expressing this as a Cartesian coordinate system,

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a. \quad (1)$$

If we remove the radical sign and reorganize it, we get

$$\frac{x^2}{a^2} + \frac{y^2}{(a^2 - c^2)} = 1. \quad (2)$$

Here, if $(a^2 - c^2) = b^2$, then b is called the semi-minor axis of the ellipse, and a is called the semi-major axis of the ellipse.

In general, the equation of an ellipse is expressed as follows.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad (3)$$

If we express this in polar coordinate system,

[1] Electronic address: tcyoon@hanmail.net

$$x = a\cos\theta,\ \ y = b\sin\theta. \tag{4}$$

This is a coordinate system that progresses in the anticlockwise direction based on the origin. If modified to a coordinate system that proceeds in the clockwise direction with the left vertex as the starting point, it is as follows.

$$x = a - \ a\cos\theta,\ \ y = b\sin\theta. \tag{5}$$

If equation (2) is rewritten in a clockwise rotating polar coordinate system,

$$x = a - \ a\cos\theta,\ \ y = \ \sqrt{a^2 -\ c^2}\sin\theta. \tag{6}$$

If we use Green's theorem to find the area drawn by the elliptical trajectory from (5), we have

$$\begin{aligned} A &= \int_0^{2\pi} y\,dx = \int_0^{2\pi} y\frac{dx}{d\theta}d\theta \\ &= \int_0^{2\pi}(b\sin\theta)(a\sin\theta)d\theta \\ &= ab\pi. \end{aligned} \tag{7}$$

We find the length of the trajectory of the ellipse from (6),

$$\begin{aligned} l &= \int_0^{2\pi}\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\ d\theta \\ &= \int_0^{2\pi}\sqrt{a^2 - c^2\cos^2\theta}\ d\theta \end{aligned} \tag{8}$$

If $a = c$, the solution can be easily obtained, but if $a \neq c$, a general solution cannot be obtained.

We can expand this elliptic integral into a power series as follows

$$\begin{aligned} l &= \ a\int_0^{2\pi}\sqrt{1 - k^2\cos^2\theta}\,d\theta\,, \quad 0 < \ k \ < 1 \\ &= a\int_0^{2\pi}(1 - \frac{2}{2^2}k^2\cos^2\theta - \frac{2}{2^4}k^4\cos^4\theta - \frac{4}{2^6}k^6\cos^6\theta - \frac{10}{2^8}k^8\cos^8\theta \ldots)d\theta \end{aligned} \tag{9}$$

where, $k = \frac{c}{a}$ is called the eccentricity of an ellipse.

Substituting the cosine square function as follows:

$$
\begin{aligned}
\cos^2\theta &= \frac{1}{2} + \frac{1}{2}\cos 2\theta \\
\cos^4\theta &= \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta \\
\cos^6\theta &= \frac{5}{16} + \frac{15}{32}\cos 2\theta + \frac{3}{16}\cos 4\theta + \frac{1}{32}\cos 6\theta \\
\cos^8\theta &= \frac{35}{128} + \frac{7}{16}\cos 2\theta + \frac{7}{32}\cos 4\theta + \frac{1}{16}\cos 6\theta + \frac{1}{128}\cos 8\theta
\end{aligned}
\tag{10}
$$

and substituting this into (9) and integrating, we get

$$
\begin{aligned}
l &= a\theta E(k), \\
E(k) &= 1 - \left[\left(\frac{1}{2^2}k^2 + \frac{3}{2^6}k^4 + \frac{20}{2^{10}}k^6 + \frac{175}{2^{14}}k^8 \dots\right)\theta\right]_0^{2\pi} \\
&- \left[\left(\frac{2}{2^4}k^2 + \frac{8}{2^8}k^4 + \frac{60}{2^{12}}k^6 + \frac{560}{2^{16}}k^8 \dots\right)\sin 2\theta\right]_0^{2\pi} \\
&- \left[\left(\frac{1}{2^8}k^4 + \frac{12}{2^{12}}k^6 + \frac{140}{2^{16}}k^8 + \frac{1680}{2^{20}}k^{10} \dots\right)\sin 4\theta\right]_0^{2\pi} \\
&- \left[\left(\frac{1}{3072}k^6 + \frac{5}{12288}k^8 + \frac{105}{262144}k^{10} + \frac{385}{1048576}k^{12} \dots\right)\sin 6\theta\right]_0^{2\pi} \\
&- \cdots .
\end{aligned}
\tag{11}
$$

As for the length of the circumferential orbit of an ellipse, everything below the second term of $E(k)$ becomes 0, so the value of the first term determines the length of an elliptical orbit. In the case, we get

$$
\begin{aligned}
E(k) &\cong 1 - \frac{1}{2^2}k^2 - \frac{3}{2^6}k^4 - \frac{20}{2^{10}}k^6 - \frac{175}{2^{14}}k^8 - \frac{1764}{2^{18}}k^{10} \dots \\
&= 1 - \sum_{n=1}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \frac{k^{2n}}{2n-1}.
\end{aligned}
\tag{12}
$$

where $(2n-1)!!$ is the factorial for odd numbers, and $(2n)!!$ is the factorial for even numbers.

The length of the arc formed by two angles (θ_1, θ_2) of the ellipse is given as $a(\theta_2 - \theta_1)E(k)$.

As we can see the equation (11), $E(k)$ of the entire elliptic arc is simply an infinite series with respect to eccentricity k of an ellipse. However, when θ has a very small value, sine function makes a slight change. But since it can be ignored, we can get an arc length of an ellipse $l = a\theta E(k)$.

Additionally, if we take the limit for k in equation (11), we get

$$\lim_{k\to+0} l = \lim_{k\to+0}(2a\pi E(k)) = 2a\pi,$$
$$\lim_{k\to+1} l = \lim_{k\to+1}(2a\pi E(k)) \cong 1.31122a\pi. \tag{13}$$

If an elliptic integral equation is given as follows,

$$l = \int_0^{2\pi}\sqrt{a^2\cos^2\theta\ + b^2\sin^2\theta}\ d\theta \tag{14}$$

It can be rewritten as

$$l = a\int_0^{2\pi}\sqrt{1-k^2\sin^2\theta}\ d\theta \tag{15}$$

where $k = \sqrt{\frac{a^2 - b^2}{a^2}}$.

This is called the incomplete elliptic integral of the second kind. This provides the same approximate solution of Eq. (12).

$$\begin{aligned} l &= a\theta E(k), \\ E(k) &= 1 - \left[\left(\frac{1}{2^2}k^2 + \frac{3}{2^6}k^4 + \frac{20}{2^{10}}k^6 + \frac{175}{2^{14}}k^8 \ldots\right)\theta\right]_0^{2\pi} \\ &+ \left[\left(\frac{2}{2^4}k^2 + \frac{8}{2^8}k^4 + \frac{60}{2^{12}}k^6 + \frac{560}{2^{16}}k^8 \ldots\right)\sin 2\theta\right]_0^{2\pi} \\ &- \left[\left(\frac{1}{2^8}k^4 + \frac{12}{2^{12}}k^6 + \frac{140}{2^{16}}k^8 + \frac{1680}{2^{20}}k^{10} \ldots\right)\sin 4\theta\right]_0^{2\pi} \\ &+ \left[\left(\frac{1}{3072}k^6 + \frac{5}{12288}k^8 + \frac{105}{262144}k^{10} + \frac{385}{1048576}k^{12} \ldots\right)\sin 6\theta\right]_0^{2\pi} \\ &- \cdots \end{aligned} \tag{16}$$

References

[1] Elliptic integral, https://en.wikipedia.org/wiki/, Access Date: Apr. 25, 2024

[2] Ellipse, https://en.wikipedia.org/wiki/, Access Date: Apr. 25, 2024