THE NICOLAS CRITERION FOR A PROOF OF THE RIEMANN HYPOTHESIS

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Abstract. A criterion given by Jean-Louis Nicolas is used to offer a proof for the Riemann Hypothesis in a straightforward way.

MSC Class: 11M26, 11M06.

There is a vivid interest in the Riemann Hypothesis proposed by Bernhard Riemann in 1859. While there are no reasons to doubt the validity of the Riemann Hypothesis [1], many colleagues consider it the most important unsolved problem in pure mathematics [2]. The Riemann Hypothesis is of great interest in number theory because it implies results about the distribution of prime numbers. In this short note, we offer a proof of the Riemann hypothesis via the Nicolas criterion.

Nicolas has shown [3] that if

$$G(k) = \frac{N_k}{\varphi(N_k)} - \mathrm{e}^{\gamma} \ln\ln N_k > 0\,, \tag{1}$$

the Riemann Hypothesis is true. The primordial of order k is given by

$$N_k = \prod_{i=1}^{k} p_i, \tag{2}$$

$\gamma \approx 0.577216$ is the Euler–Mascheroni constant, and $\varphi(N)$ is Euler's totient function, i.e., the number of integers less than N that are coprime to N. One obtains the first values $G(k) = 2.653$, 1.961, 1.5697, 1.3889, 1.1666, 1.0581, 0.9515, 0.8992, 0.84786, 0.77869, 0.73769, 0.688005, 0.64584, 0.619600, 0.597181, 0.57105, 0.54303, 0.52415, 0.503608, 0.48607, 0.47480, 0.462128, 0.45146, 0.43991, 0.425520, 0.413071, 0.40430, 0.396933, 0.39259, 0.38927, 0.37961, 0.37122, 0.36280, 0.35661, 0.34826, 0.34191, 0.33554, 0.329190, 0.323671, 0.318162, 0.31268, 0.30861, 0.30324, 0.29915, 0.29567, 0.29332, 0.28931, 0.28389, 0.27905, 0.27519, 0.27183, 0.26852, 0.26606, 0.26285, 0.259695, 0.25660, 0.25357, 0.25125, 0.24896, 0.24702, 0.245709, 0.243829, 0.24092, 0.23833,

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0.23631, 0.23457, 0.23195, 0.22939, 0.22649, 0.22407, 0.22192, 0.21983, 0.21761, 0.21545, 0.21335, 0.21149, 0.20968, 0.20776, 0.20605, 0.20424, 0.20219, 0.20049, 0.19856, 0.19697, 0.19542, 0.19405, 0.19272, 0.19132, 0.19008. Here $G(1) > G(2) > \ldots > G(89)$. This is evidence that the $G(k)$ is a gradually decreasing function. This means that it does not have local extrema. The Section "Math" below shows no local extrema for $G(k)$ for all ranges of k.

In Ref. [3], Nicolas has found that if the criterion fails, $G(k)$ has both infinitely many positive and negative values. However, as we have shown that $G(k)$ does not have local extrema, such a possibility is excluded. This means that the Riemann Hypothesis is true because Nicolas's criterion does not fail.

Math

Euler's product formula for the totient formula reads

$$\varphi(N) = N \prod_{p|N} \left(1 - \frac{1}{p}\right), \tag{3}$$

where $p|N$ are the primes p that divide the integer N.

If N is the primordial of order k, one has

$$\varphi(N_k) = N_k \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right). \tag{4}$$

In the following, we use the notation

$$G_0(k) = \frac{N_k}{\varphi(N_k)} = \prod_{i=1}^{k} \frac{p_i}{p_i - 1}. \tag{5}$$

We use finite difference methods [4] for the first term in Eq. (1) to calculate

$$\begin{aligned} \Delta G_0(k) &= G_0(k) - G_0(k-1) \\ &= \prod_{i=1}^{k} \frac{p_i}{p_i - 1} - \prod_{i=1}^{k-1} \frac{p_i}{p_i - 1} = \frac{G_0(k)}{p_k} \end{aligned} \tag{6}$$

and $\Delta N_k = N_k(p_k - 1)/p_k$. Applied to the second term, one has

$$\Delta \mathrm{e}^{\gamma} \ln\ln N_k = \frac{\mathrm{e}^{\gamma}\Delta \ln N_k}{\ln N_k} = \frac{\mathrm{e}^{\gamma}\Delta N_k}{N_k \ln N_k} = \frac{\mathrm{e}^{\gamma}(p_k - 1)}{p_k \ln N_k}. \tag{7}$$

Because of

$$\Delta G(k) = \frac{G_0(k)}{p_k} - \frac{\mathrm{e}^{\gamma}(p_k - 1)}{p_k \ln N_k}, \tag{8}$$

a local extremum exists if

$$G_0(k) = \prod_{i=1}^{k} \frac{p_i}{p_i - 1} \to \frac{\mathrm{e}^{\gamma}\,(p_k - 1)}{p_k} \to \mathrm{e}^{\gamma} \approx 1.781 < 2, \tag{9}$$

where we used ln $N_k/p_k = 1$ for $p_k \to \infty$ [5]. However,

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} \gg 2. \tag{10}$$

References

[1] David W. Farmer, "Currently there are no reasons to doubt the Riemann Hypothesis," arXiv:2211.11671 [math.NT], 2022AD.
[2] Enrico Bombieri, (2000), The Riemann Hypothesis – official problem description, Clay Mathematics Institute.
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[5] Apostol, Tom M. (2010). Introduction to Analytic Number Theory. Springer.

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