Fransén-Robinson Constant

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6 August 2024

abstract

We give some formulas related to the Fransén-Robinson constant $F = 2.80777024 \ldots$

Keywords: Fransén-Robinson constant, Inverse Gamma integral

I. Introduction:Fransén-Robinson Constant

The Fransén-Robinson constant F is defined by

$$F = \int_0^\infty \frac{1}{\Gamma(x)} \, dx = 2.807770242028 \ldots \tag{1}$$

where $\Gamma(x)$ is the gamma function.

Notations:

Gamma function:

$$\Gamma(x) = \int_0^\infty e^{-u} \, u^{x-1} \, du \quad , \quad x > 0 \tag{2}$$

$$\Gamma(x+1) = x \, \Gamma(x) \tag{3}$$

Incomplete Gamma function:

$$\Gamma(x, y) = \int_y^\infty e^{-u} \; u^{x-1} \, du \tag{4}$$

Pochhammer symbol:

$$(a)_n = a\,(a+1)\,(a+2) \ldots (a+n-1) \quad , \quad (a)_0 = 1 \tag{5}$$

Psi function (Digamma function):

$$\psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{6}$$

Number Pi:

$$\pi = 4 \sum_{n=0}^{\infty} \frac{(-1)^n}{2\,n+1} = 3.141592 \ldots \tag{7}$$

Number e :

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281 \ldots \tag{8}$$

In this note we will give some formulas related with the Fransén-Robinson constant and the inverse Gamma function.

Continued fractions representations:

$$F = [2; 1, 4, 4, 1, 18, 5, 1, 3, 4, 1, 5, 3, \ldots] \tag{9}$$

$$F = e + [0;\, 11,\, 5,\, 1,\, 2,\, 1,\, 1,\, 1,\, 8,\, 2,\, 2,\, 1,\, 2,\, \ldots] \tag{10}$$

II. Related formulas

$$F = \int_1^\infty \frac{x-1}{\Gamma(x)}\, dx \tag{11}$$

$$F = \int_0^1 \frac{1-x}{x^3\, \Gamma(1/x)}\, dx \tag{12}$$

$$F = \frac{1}{2} \int_0^1 \frac{1}{\Gamma(x)}\, dx + \frac{1}{2} \int_1^\infty \frac{x}{\Gamma(x)}\, dx \tag{13}$$

$$F = \int_0^1 \left(\frac{1}{\Gamma(x)} + \frac{1}{x^2\, \Gamma(1/x)} \right) dx \tag{14}$$

$$F = \int_1^\infty \left(\frac{1}{\Gamma(x)} + \frac{1}{x^2\, \Gamma(1/x)} \right) dx \tag{15}$$

$$F = \int_n^\infty \frac{(x-1)\,(x-2)\, \ldots\, (x-n)}{\Gamma(x)}\, dx \quad , \quad n = 1,\, 2,\, 3,\, \ldots \tag{16}$$

$$F = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n - 1} \frac{(-1)^{k-1}\, 2^n\, (2^n - m)}{m^3\, \Gamma(2^n / m)} \tag{17}$$

$$F = e - i \int_0^\infty \frac{1}{e^{2\pi x} - 1} \left(\frac{1}{\Gamma(i\, x)} - \frac{1}{\Gamma(-i\, x)} \right) dx \quad , \quad i = \sqrt{-1} \tag{18}$$

$$F = e - \frac{i}{2} \int_{-\infty}^{\infty} \frac{e^{-\pi\, |x|}}{\Gamma(i\, x)\, \sinh(\pi\, x)}\, dx \quad , \quad i = \sqrt{-1} \tag{19}$$

$$\int_0^n \frac{1}{\Gamma(x)}\, dx + e - \sum_{k=0}^{n-1} \frac{1}{k!} < F < \int_0^n \frac{1}{\Gamma(x)}\, dx + e - \sum_{k=0}^{n-2} \frac{1}{k!} \quad , \quad n = 1,\, 2,\, 3,\, \ldots \tag{20}$$

$$F = \frac{1}{\pi} \int_0^s \Gamma(1-x)\, \sin(\pi\, x)\, dx + \int_s^\infty \frac{1}{\Gamma(x)}\, dx \quad , \quad 0 \le s \le 1 \tag{21}$$

$$F = \frac{1}{\pi} \int_{1-s}^1 \Gamma(x)\, \sin(\pi\, x)\, dx + \int_s^\infty \frac{1}{\Gamma(x)}\, dx \quad , \quad 0 \le s \le 1 \tag{22}$$

$$F = \int_0^1 \frac{1}{\Gamma(x)}\, dx + \sum_{n=0}^{\infty} \int_0^1 \frac{1}{\Gamma(x+1)\, (x+1)_n}\, dx \tag{23}$$

$$F = e + \int_0^\infty \frac{e^{-x}}{\pi^2 + (\ln(x))^2}\, dx \tag{24}$$

$$F = e + \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{\pi \tan(\theta) - e^{\pi \tan(\theta)}}\, d\theta \tag{25}$$

$$F = e + \int_{-\infty}^{\infty} \frac{e^{x - e^x}}{\pi^2 + x^2}\, dx \tag{26}$$

$$F = e + \int_{-\infty}^{\infty} \frac{e^{-x - e^{-x}}}{\pi^2 + x^2}\, dx \tag{27}$$

$$F = e + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{\pi x - e^{\pi x}}}{1 + x^2} \, dx \tag{28}$$

$$F = e + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-\pi x - e^{-\pi x}}}{1 + x^2} \, dx \tag{29}$$

$$F = e + \int_{0}^{\infty} \frac{e^{-x - e^{-x}} + e^{x - e^{x}}}{\pi^2 + x^2} \, dx \tag{30}$$

$$F = e + \frac{1}{\pi} \int_{-\infty}^{\infty} e^{\pi \sinh(x) - e^{\pi \sinh(x)}} \operatorname{sech}(x) \, dx \tag{31}$$

$$F = e + \int_{0}^{1} \frac{1}{\pi^2 + (\ln(-\ln(x)))^2} \, dx \tag{32}$$

$$F = e + \int_{1}^{\infty} \frac{1}{\left(\pi^2 + (\ln(\ln(x)))^2\right) x^2} \, dx \tag{33}$$

$$F = e + 2 \int_{-\infty}^{\infty} \frac{x \, e^{-e^{-x}}}{\left(\pi^2 + x^2\right)^2} \, dx \tag{34}$$

$$F = e - 2 \int_{-\infty}^{\infty} \frac{x \, e^{-e^{x}}}{\left(\pi^2 + x^2\right)^2} \, dx \tag{35}$$

$$F = e + \int_{0}^{1/\pi^2} \left(e^{-e^{-\sqrt{\frac{1}{x} - \pi^2}}} - e^{-e^{+\sqrt{\frac{1}{x} - \pi^2}}} \right) dx \tag{36}$$

$$F = e + \frac{1}{\pi} \int_{-\infty}^{\infty} (e^{x} - 1) \, e^{x - e^{x}} \tan^{-1}\left(\frac{x}{\pi}\right) dx \tag{37}$$

$$F = \int_{0}^{\infty} \frac{(x)_n}{\Gamma(x + n)} \, dx \quad , \; n = 0, 1, 2, 3, \ldots \tag{38}$$

$$f_n = -\frac{1}{2\pi} \int_{0}^{2\pi} \frac{e^{n + i x - e^{n + i x}}}{n + i (x - \pi)} \, dx \;, \; n = 1, 2, 3, \ldots; \; i = \sqrt{-1} \; ; \; \lim_{n \longrightarrow \infty} f_n = F \tag{39}$$

$$F = e - \frac{1}{2} + \int_{0}^{1} \frac{1}{\Gamma(x)} \, dx - \frac{1}{\pi} \int_{0}^{\infty} e^{-x} \tan^{-1}\left(\frac{\ln(x)}{\pi}\right) dx \tag{40}$$

$$F = \int_{0}^{n} \frac{1}{\Gamma(x)} \, dx + \int_{0}^{\infty} \frac{1}{\Gamma(x) \, (x)_n} \, dx \quad , \; n = 0, 1, 2, 3, \ldots \tag{41}$$

$$F = \int_{0}^{1} \frac{1}{\Gamma(x)} \, dx + \sum_{n=0}^{\infty} \int_{0}^{1} \frac{1}{\Gamma(x + 1) \, (x + 1)_n} \, dx \tag{42}$$

$$F = \int_{0}^{n} \frac{1}{\Gamma(x)} \left(\sum_{k=0}^{m} \frac{1}{(x)_{k n}} \right) dx + \int_{n}^{\infty} \frac{1}{\Gamma(n + m n)} \, dx \;, \; n, m = 0, 1, 2, 3, \ldots \tag{43}$$

$$F = \frac{1}{2} \int_{0}^{\infty} \frac{x}{\Gamma(x)} \, \psi(x + 1) \, dx \tag{44}$$

$$F = \int_{0}^{\infty} \frac{x}{\Gamma(x)} \, \psi(x) \, dx \tag{45}$$

$$F = \frac{a}{2 \, \Gamma(a)} + \frac{1}{2} \int_{0}^{a} \frac{x}{\Gamma(x)} \, \psi(x + 1) \, dx + \int_{a}^{\infty} \frac{1}{\Gamma(x)} \, dx \;, \; a \geq 0 \tag{46}$$

$$F = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^{\pi} x^{1-x} e^{x\left(y\cot(y)-\ln\left(\frac{y}{\sin(y)}\right)\right)} \, dy \, dx \tag{47}$$

$$F = \frac{2}{\pi} \int_0^\infty \int_0^\infty \operatorname{sech}(x\,\Gamma(y)) \, dx \, dy \tag{48}$$

$$F = \int_0^\infty \int_0^\infty (\operatorname{sech}(x\,\Gamma(y)))^2 \, dx \, dy \tag{49}$$

$$F = \sqrt{\int_0^\infty \int_0^\infty \frac{1}{\Gamma(x\,\Gamma(y))} \, dx \, dy} \tag{50}$$

$$F = \int_0^\infty \frac{2^{2-x}}{\Gamma(x)\,\Gamma(x+1)} \, {}_2F_1\left(1-x,\, x,\, 1+x,\, \frac{1}{2}\right) dx \tag{51}$$

$$F = 8 \sum_{n=0}^{\infty} \frac{a^{2n+1}}{n!\,(2n+1)} \int_0^\infty \frac{(1-x)_n\, 2^{-2x}}{(\Gamma(x))^2} \, dx + 4 \int_0^\infty \frac{2^{-x}\,(1-a)^x}{\Gamma(x)\,\Gamma(x+1)} \, {}_2F_1\left(1-x,\, x,\, 1+x,\, \frac{1-a}{2}\right) dx \;\; , \;\; 0 < a < 1 \tag{52}$$

Remark: ${}_2F_1$ is the Gauss hypergeometric function.

$$F = \int_0^\infty \frac{1}{\Gamma(x+i)} \, dx + \int_0^1 \frac{i}{\Gamma(x\,i)} \, dx \;\; , \;\; i = \sqrt{-1} \tag{53}$$

$$F = e + \int_0^1 \frac{1 - e\,\Gamma(x,\, 1)}{\Gamma(x)} \, dx \tag{54}$$

$$F = e + \int_0^1 \frac{1}{\Gamma(x)} \, dx - \int_{-\infty}^{\infty} \frac{e^{-e^x}}{\pi^2 + x^2} \, dx \tag{55}$$

$$F = \int_0^1 \frac{1}{\Gamma(x)} \, dx + \sum_{n=0}^{\infty} 2^n \int_0^1 \left(\frac{2}{\Gamma\left(2^{n+1}\,x\right)} - \frac{1}{\Gamma(2^n\,x)} \right) dx \tag{56}$$

$$I(n) = \frac{1}{n} \sum_{k=1}^{\infty} \frac{1}{\Gamma(k/n)} \;\; , \;\; n = 1,\, 2,\, 3,\, \ldots,\;\; I(\infty) = F \tag{57}$$

$$G(n) = \sum_{k=1}^{n!-1} \frac{n!\,(n!-k)}{k^3\,\Gamma(n!/k)} = \sum_{k=1}^{n!-1} \frac{n!}{k^2\,\Gamma((n!-k)/k)} \;\; , \;\; n = 2,\, 3,\, 4,\, \ldots,\;\; G(\infty) = F \tag{58}$$

$$F = e + \frac{1}{\pi} \int_{-n}^{n} \frac{e^{\pi x - e^{\pi x}}}{1+x^2} \, dx + \frac{1}{\pi^2} \, e^{-\pi n - e^{-\pi n}} \, \theta(n) \;\; , \;\; n \gg 1,\; 0 < \theta(n) < 1 \tag{59}$$

$$F = e + \frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{x\, e^{-e^{-\pi x}}}{\left(1+x^2\right)^2} \, dx \tag{60}$$

$$F = e + \frac{2}{\pi^2} \int_{-\infty}^{\infty} \frac{\sinh(x)\, e^{-e^{-\pi \sinh(x)}}}{(\cosh(x))^3} \, dx \tag{61}$$

$$F = e + \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} \sin(2x)\, e^{-e^{-\pi \tan(x)}} \, dx \tag{62}$$

$$F = \lim_{s \to 0} \int_1^\infty \frac{s}{\Gamma(s\,x)} \, dx \tag{63}$$

$$F\,\pi = 2 \int_0^\infty \int_0^\infty \frac{1}{\Gamma\left(1 + \sqrt{x^2+y^2}\right)} \, dx \, dy \tag{64}$$

$$F = \frac{1}{2\pi}\int_{x^2+y^2\le r^2}\frac{1}{\Gamma\left(1+\sqrt{x^2+y^2}\right)}\,dx\,dy + \int_r^\infty \frac{1}{\Gamma(x)}\,dx \quad,\ r \ge 0 \tag{65}$$

$$F = \int_0^\infty\int_0^\infty \frac{1}{\Gamma(1+x+y)}\,dx\,dy \tag{66}$$

$$F = \int_{-\infty}^\infty\int_{-\infty}^\infty \frac{e^{-x-y}}{\Gamma(1+e^{-x}+e^{-y})}\,dx\,dy = \int_{-\infty}^\infty\int_{-\infty}^\infty \frac{e^{x+y}}{\Gamma(1+e^{x}+e^{y})}\,dx\,dy \tag{67}$$

$$H(k) = \int_{-1}^1 \frac{\sinh(k)}{\Gamma(\cosh(k)+x\sinh(k))}\,dx \quad,\ k=1,2,3,\ \ldots;\ H(\infty) = F \tag{68}$$

III. Endnote

$$F = \int_0^s \frac{1}{\Gamma(x)}\,dx + s\int_s^\infty \frac{1}{\Gamma(1+x)}\,dx + \int_0^\infty \frac{1}{\Gamma(1+s+x)}\,dx \quad,\ s \ge 0 \tag{69}$$

$$F = s\int_1^\infty \frac{1}{\Gamma(s\,x)}\,dx + \int_0^s \frac{1}{\Gamma(x)}\,dx \quad,\ s > 0 \tag{70}$$

$$F = s\int_0^1 \frac{1}{\Gamma(s\,x)}\,dx + \int_0^\infty \frac{1}{\Gamma(s+x)}\,dx \quad,\ s \ge 0 \tag{71}$$

$$F = \int_0^1\int_0^1 \frac{x}{\Gamma(x\,y)}\,dx\,dy + \int_0^\infty\int_0^1 \frac{1}{\Gamma(x+y)}\,dx\,dy \tag{72}$$

$$F = \int_0^1\int_0^1 \frac{x}{\Gamma(x\,y)}\,dx\,dy + \int_0^1\int_0^1 \frac{1}{\Gamma(x+y)}\,dx\,dy + \int_0^1\int_0^1 \frac{1}{x^2\,\Gamma\left(\frac{1}{x}+y\right)}\,dx\,dy \tag{73}$$

$$F = \int_0^\infty\int_0^\infty \frac{1}{\Gamma(x+y)}\,dx\,dy - \int_0^\infty\int_0^\infty \frac{e^{-x}}{\Gamma(e^{-x}+y)}\,dx\,dy = \int_0^\infty\int_0^\infty \left(\frac{1}{\Gamma(x+y)} - \frac{e^{-x}}{\Gamma(e^{-x}+y)}\right)dx\,dy \tag{74}$$

IV. References

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