Proof of Goldbach's Conjecture and Twin Prime Number Conjecture

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Abstract: Goldbach's conjecture has been around for more than 300 years and the twin prime conjecture for more than 160 years and both are still unsolved , both conjectures are important number theory conjectures for studying prime numbers,this article proposes a method of sequence shift to prove Goldbach's conjecture and the conjecture of twin primes, which may be a good method and seems quite easy to understand.

Keywords : Goldbach's conjecture, conjecture of twin primes, prime numbers, number theory.

1.Introduction:

Goldbach's conjecture [1]

In 1742, Goldbach proposed the Goldbach conjecture, which stumped the famous mathematician Euler and countless mathematicians over the next 300 years. Euler was unable to solve the problem until his death, and countless mathematicians since then have also been stumped by the problem.

Goldbach's original conjecture was that any integer greater than 5 could be written as the sum of three prime numbers. On his basis, Euler proposed that any even number greater than 2 can be written as the sum of 2 prime numbers. If Euler is right, it is easy to deduce that any odd number greater than 7 can be expressed as the sum of three odd prime numbers. This is known as the "weak Goldbach conjecture", and Euler proposed the "strong Goldbach conjecture"

Conjecture of twin primes [2]

The twin prime conjecture was proposed by Hilbert at the 1900 Congress of Mathematicians, that is, "there are an infinite number of pairs of prime numbers with a difference of 2"

In 1849, Alphonse de Polignac proposed a general conjecture: there are infinite prime number pairs (p, p+2k), for all natural numbers k. The case of k=1 is the twin prime conjecture.

The prime number theorem indicates that prime numbers become increasingly sparse as they approach infinity. Similarly, twin primes, like prime numbers, also exhibit this trend, but it is even more pronounced than for prime numbers.

2.Proof Process

Proof:

First step: Prove that any even number greater than or equal to 2 can be expressed as the difference of two prime numbers greater than or equal to 3.

Construct a sequence R that only contains prime numbers greater than or equal to 3, and sets all other numbers to 0;

R=[3,0,5,0,7,0,0,0,11, ...]

To prove the above proposition, it can be formulated as follows: if the sequence R is shifted right by M positions and the resulting sequence has the same prime numbers at the same positions as the original sequence R

$R(t)\ \&\ R(t+M) = 1\ (t\in(R(t)\neq 0))$.

At this point, the even number M is the difference between the two prime numbers at the same position in the shifted sequence and the original sequence R.

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Now using the method of proof by contradiction:

Let's assume that there is an even number greater than or equal to 2 that cannot be expressed as the difference of two prime numbers greater than or equal to 3, let's call it N. In this case, all other even numbers can be expressed in this way.

$R(t)\&R(t+2)=1$
$R(t)\&R(t+4)=1$
$R(t)\&R(t+6)=1$
...
$R(t)\&R(t+N+2)=1$
$R(t)\&R(t+N+4)=1$
$R(t)\&R(t+N+6)=1$
...

$R(t)\&R(t+N+N)=1$

……

Each of the equations can find a number to satisfy when t takes values of $R(t) \neq 0$.

However, $R(t) \& R(t+N) = 0$. Since there are no common positions in the R sequence that are both prime numbers after the R sequence is shifted to the right by N, each non-zero element in the R sequence corresponds to a zero element in the R sequence shifted by N.

Since the sequence R shifted right by N does not have the same prime numbers at the same positions as the original sequence R, then each non-zero element of the R sequence corresponds to a zero element after shifting the R sequence by N.

Below we prove that the above conditions $R(t) \& R(t+N) = 0$ and $R(t) \& R(t+N+2) = 1$, $R(t) \& R(t+N+4) = 1$, $R(t) \& R(t+N+6) = 1$, ... are impossible!

We define the RS sequence as the sequence obtained by removing the first S terms from the R sequence. Shifting the RS sequence to the right by P will result in the RSP sequence (where S and P are natural numbers).

It is easy to derive the following properties of the RSP sequence:

1.The RS sequence is a subsequence of the R sequence.

2.$RSP(t) = R(t+S+P)$

As follows: S=6,P=2

R = [3,0,5,0 ,7,0,0,0,11, ...]

RS = [0,0,11,…]

RSP = [0,0,11,…]

If the sequence R shifted to the right by N has no prime number intersection with R, then RS, as a subsequence of R, shifted to the right by N also has no prime number intersection with R.

That is, $RSN = R(t + S + N)$, $R(t) \& R(t + S + N) = 0$.

Then

$R(t)\&R(t+N)=0$

...

$R(t)\&R(t+2+N)=0$

$R(t)\&R(t+4+N)=0$

$R(t)\&R(t+6+N)=0$

...

R(t)&R(t+N+N)=0

……

Obtaining that the above R sequence shifted to the right by N+2, N+4, N+6, ... and other even numbers has no intersection with the R sequence indicates that the number of primes is finite. This conclusion is contradictory to Euclid's proof that there are infinitely many primes and also contradictory to the initial assumption, thus indicating that the assumption is wrong. Therefore, any even number greater than or equal to 2 can be expressed as the difference of two prime numbers greater than or equal to 3.

Here, we only consider the case where a single even number cannot be expressed as the difference between two prime numbers. The same logic applies to other multiples that cannot be expressed in this way. It is sufficient to consider the first even number that cannot be expressed as the difference between two prime numbers to arrive at a result that contradicts the assumption!

Because as shown above, If an even number N cannot be represented as the difference of two prime numbers, then any even number greater than it cannot be represented as the difference of two prime numbers.

Second step: Prove that any even number greater than 2 can be expressed as the sum of two prime numbers.

Construct a sequence R that only contains prime numbers greater than or equal to 3, and sets all other numbers to 0; R(0)=0, R(1)=0.

R=[0,0,0,3,0,5,0,7,0,0,0,11, ...]

To prove the above proposition, it can be formulated as follows: if the sequence R is rotated and shifted right by M positions, the resulting sequence has the same prime numbers at the same positions as the original sequence R

R(t) & R(−t+M) = 1 ($t \in (R(t) \neq 0)$,R(−t)=R(t)).

In this case, the even number M is the sum of the two prime numbers that are both

prime at the same position in the rotated and translated sequence and the original sequence R.

Now using the method of proof by contradiction：

Let's assume that there is an even number greater than or equal to 6 that cannot be expressed as the sum of two prime numbers greater than or equal to 3, let's call it N. In this case, all other even numbers can be expressed in this way.

R(t)&R(−t+N−2)=1
R(t)&R(−t+N−4)=1
R(t)&R(−t+N−6)=1
...
R(t)&R(−t+N−N)=1
……

Each of the equations can find a number to satisfy when t takes values of R(t) ≠ 0.

However, R(t) & R(−t+N) = 0. Since there are no common positions in the R sequence that are both prime numbers after the R sequence is rotated and shifted to the right by N, each non-zero element in the R sequence corresponds to a zero element in the R rotated and shifted by N.

Since the sequence R rotated and shifted right by N does not have the same prime numbers at the same positions as the original sequence R, then each non-zero element of the R sequence corresponds to a zero element after rotated and shifting the R sequence by N.

Below we prove that the above conditions R(t) & R(−t+N) = 0 and R(t) & R(−t+N−2) = 1, R(t) & R(−t+N−4) = 1, R(t) & R(−t+N−6) = 1, ... are impossible!

We define the RS sequence as the sequence obtained by removing the first S terms from the rotated R sequence. Shifting the RS sequence to the right by P will result in the RSP sequence (where S and P are natural numbers).

It is easy to derive the following properties of the RSP sequence:
1.The RS sequence is a subsequence of the R sequence.
2.RSP(−t) = R(−t−S+P)

As follows: S=6,P=2

R = [0,0,0,3,0,5,0 ,7,0,0,0,11, ...]
RS = [0,7,0,0,0,11, ...]
RSP = [0,7,0,0,0,11, ...]

R(−t) = […,11,0,0,0,7,0,5,0,3,0,0,0]
RS(−t) = […,11,0,0,0,7,0]
RSP(−t) = […,11,0,0,0,7,0]

If the sequence R rotated and shifted to the right by N has no prime number intersection with R, then RS, as a subsequence of R, rotated and shifted to the right by N also has no prime number intersection with R.

That is, RSN = R(−t −S + N), R(t) & R(−t − S + N) = 0.
Then

R(t)&R(−t+N)=0
...
R(t)&R(−t−2+N)=0
R(t)&R(−t−4+N)=0
R(t)&R(−t−6+N)=0
...
R(t)&R(−t−N+N)=0
……

Obtaining that the above R sequence rotated and shifted to the right by N−2, N−4, N−6, ... and other even numbers has no intersection with the R sequence.

Here, we only consider the case where a single even number cannot be expressed as the sum of two prime numbers. The same logic applies to other multiples that cannot be expressed in this way. It is sufficient to consider the first even number that cannot be expressed as the sum of two prime numbers to arrive at a result that contradicts the assumption!

Because as shown above, If an even number N cannot be represented as the sum of two prime numbers, then any even number smaller than it cannot be represented as the sum of two prime numbers.

Third step: Prove that there are infinite prime number pairs (p, p+2).

Here, we also using the method of proof by contradiction. Let's assume that there are only finite prime number pairs (p, p+2). Let U be a prime number greater than the largest prime pair p+2, and v∈(R(v) ≠0). At this point, we construct the sequence R=[U,0,…] as in the first step. Since there are no more prime number pairs (p, p+2), the sequence R shifted right by 2 will not have any prime numbers at the same positions as the original sequence R.
According to the derivation in the first step:

R(v)&R(v+2)=0
R(v)&R(v+4)=0
R(v)&R(v+6)=0
…

It can be reformulated as:

shifting right by 2,

R(v)&R(v+2)=0
R(v)&R(v+2+2)=0
R(v)&R(v+4+2)=0
……

shifting right by 4,

R(v)&R(v+4)=0
R(v)&R(v+2+4)=0
R(v)&R(v+4+4)=0
……

shifting right by 6,

R(v)&R(v+6)=0
R(v)&R(v+2+6)=0
R(v)&R(v+4+6)=0
……

The prime numbers greater than U will necessarily appear at the even positions after U. However, as we have deduced above, no matter how much the sequence R is shifted right, there will never be prime numbers at the same positions as the original sequence R. This means that there are only finite prime numbers!

However, Euclid has already proved that there are infinite prime numbers, so the above assumption is incorrect. In other words, there still exist prime number pairs (p, p+2) after the prime number U, so there are infinite prime number pairs (p, p+2).

Fourth step: Generalize the above reasoning to (p, p+2k) based on the above derivation process.

Here, we also using the method of proof by contradiction**.** Let's assume that there are only finite prime number pairs (p, p+2k). Let U be a prime number greater than the largest prime pair p+2k, and $v \in (R(v) \neq 0)$. At this point, we construct the sequence R=[U,0,…] as in the first step. Since there are no more prime number pairs (p, p+2k), the sequence R shifted right by 2k will not have any prime numbers at the same positions as the original sequence R. According to the derivation in the first step:

R(v)&R(v+2k)=0
R(v)&R(v+2k+2)=0
R(v)&R(v+2k+4)=0
…

It can be reformulated as:
shifting right by 2,

R(v)&R(v+2k+2)=0
R(v)&R(v+2k+2+2)=0
R(v)&R(v+2k+4+2)=0
……

shifting right by 4,

R(v)&R(v+2k+4)=0
R(v)&R(v+2k+2+4)=0
R(v)&R(v+2k+4+4)=0
……

shifting right by 6,

R(v)&R(v+2k+6)=0
R(v)&R(v+2k+2+6)=0
R(v)&R(v+2k+4+6)=0
……

The prime numbers greater than U will necessarily appear at the even positions after U. However, as we have deduced above, no matter how much the sequence R is shifted right, there will never be prime numbers at the same positions as the original sequence R. This indicates that there are only finite prime numbers!

However, Euclid has already proved that there are infinite prime numbers, so the above assumption is incorrect. In other words, there still exist prime number pairs (p, p+2k) after the prime number U, so there are infinite prime number pairs (p, p+2k).

3.Conclusion

Based on the above proofs, we can draw the following conclusions:

1. Any even number greater than or equal to 2 can be expressed as the difference of two prime numbers greater than or equal to 3.
2. Any even number greater than 2 can be expressed as the sum of two prime numbers.
3. There are infinite prime number pairs (p, p+2).
4. There are infinite prime number pairs (p, p+2k).

References:

[1] Baidu Encyclopedia, Goldbach conjecture entry
[2] Baidu Encyclopedia,Twin Prime Conjecture entry