Application of Hooke's law and occurrence of relativistic orthogonal harmonic oscillators in leptons

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Abstract

In 1915 Parson proposed the so-called *ring model* for the electron. This flat geometry for the electron can also be interpreted as a superposition of two orthogonal harmonic oscillators in the same plane. For these two harmonic oscillators a new relativistic Lagrangian is conjectured from which Hooke's law follows. Explicit expressions for the spring constant and electron energy are deduced from this simple ring model.

Spring constants and energies for all leptons can also be deduced from the recently postulated more complex toroidal model for leptons. The *ring torus model* appears to apply to charged leptons and the electron neutrino, whereas the *spindle torus model* may apply to the muon and tauon neutrino.

It appears that the magnetic dipole moments of the charged leptons predicted by the toroidal model agree with the observed ones, first order anomalous corrections included. Furthermore, explicit expressions for the magnetic dipole moments of all neutrinos are also obtained.

Moreover, a comparison is made between the magnitude of the electromagnetic and elastic contribution to the energy of the electron. It is found that the elastic energy may be dominant.

1. Introduction

The geometric shape of leptons is still uncertain. For example, do they possess a *flat* or a *spherical* structure? By introducing the so-called *ring model* Parson [1] already introduced a flat geometry for the electron in 1915. This approach has recently revisited by Consa [2]. In this model it is assumed that the charge e flows through a ring of radius r at the speed of light. The resulting electric current generates an associated magnetic field. In first order the following equations are postulated

$$x(t) = r\cos\omega t, \quad y(t) = r\sin\omega t, \tag{1.1}$$

where r is the radius of a circle in the x-y plane and ω is an angular frequency. Both r and ω do not depend on time. It is noted that the ring model can be interpreted as a superposition of two orthogonal harmonic oscillators in the x-y plane. In terms of the so-called toroidal model for the electron, more complex basic equations than (1.1) have been postulated, among others by Consa [2] and Biemond [3]. However, we first consider the simple ring model

From (1.1) the following relations can be calculated

$$r^2 = x(t)^2 + y(t)^2 = x^2 + y^2, \tag{1.2}$$

$$\dot{x}(t)^2 + \dot{y}(t)^2 = \dot{x}^2 + \dot{y}^2 = \omega^2 r^2 = v_r^2, \tag{1.3}$$

where x equals $x(t)$ and $\dot{x}$ is a short-hand notation for the first time derivative of $x(t)$. The quantity v_r, or shortly v, is the rotational velocity of the charge in the ring.

It is noticed that two other differential equations directly follow from (1.1)

$$\ddot{x}+\omega^2 x=0 \quad \text{and} \quad \ddot{y}+\omega^2 y=0. \tag{1.4}$$

Here $\ddot{x}$ and $\ddot{y}$ denote the second time derivative of x and y, respectively. So, the superposition of two orthogonal harmonic oscillators in the x- and y-direction with the same amplitude r and the same angular frequency ω corresponds to a circular orbit of charge e.

Usually, it is assumed that the electromagnetic forces are dominant in the electron (see, e.g., Moylan [4], Blinder [5], and Morozov [6]). The relations of (1.4), however, suggest that an elastic force is at work in the electron, i.e., Hooke's law[1]. In modern notation his law can be represented by the forces F_x and F_y in the x- and y-direction

$$F_x = m\ddot{x} = -kx \quad \text{and} \quad F_y = m\ddot{y} = -ky, \tag{1.5}$$

where mass m is the rest mass of the electron and k is the spring constant k. As is argued in section 2.1, these relations may retain their validity in the relativistic domain. Next differential equations directly follow from (1.5)

$$\ddot{x}+\frac{k}{m}x=0 \quad \text{and} \quad \ddot{y}+\frac{k}{m}y=0. \tag{1.6}$$

Combination of (1.4) and (1.5) shows that the spring constant k for both harmonic oscillators is equal to

$$k = m\omega^2. \tag{1.7}$$

In addition, utilizing (1.3) and (1.7), the total energy E for the ring model may be written as

$$E = m(\dot{x}^2+\dot{y}^2) = mv_r^{\,2} = m\omega^2 r^2 = \tfrac{1}{2}m\omega^2 r^2 + \tfrac{1}{2}kr^2. \tag{1.8}$$

It is argued below that this expression may apply to every value of the speed v between zero and the light speed c.

Furthermore, it is postulated (compare to, e.g., refs. [2, 3]) that the radius of the ring denoted by r in (1.1) equals the Compton wavelength λ_C

$$r = \frac{\hbar}{mc} = \lambda_C, \tag{1.9}$$

where $\hbar$ is the reduced Planck constant, m is the rest mass of the electron and c the speed of light. For the electron $r = 3.8616\times10^{-11}$ cm. Several alternative choices for the internal structure and radius of an electron have been proposed in the past. For example, Williamson and van der Mark [7] postulated "a state of a self-confined single-wavelength photon" for the internal structure of the electron. They also suggested a radius comparable to the Compton wavelength λ_C of (1.9), but argued that the apparent size of the object will be much smaller in energetic scattering events. Hu [8] proposed another model for the internal structure of an electron. He characterized the electron as a "circulating massless particle at the speed of light". These attempts illustrate that there is no generally accepted internal structure for an electron. Moreover, the correct choice of a radius r of the electron remains uncertain.

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[1] Robert Hooke (1635-1703) published this law in 1676 in the form of the alphabetical Latin anagram "ceiiinosssttuv", which decipher to "ut tension sic vis". It translates to "as the extension, so the force".

Following refs. [2, 3], the following relation is adopted for the energy E for all leptons

$$E = \hbar\omega = m v_r^{\,2}. \tag{1.10}$$

In the case of (1.10), as in the case of (1.3) and (1.8), the speed v_r may possess every value between zero and the light speed c. When the rotational velocity v_r of all leptons will match the speed of light c, the angular frequency ω approaches to

$$\omega = \frac{mc^2}{\hbar}. \tag{1.11}$$

Combination of (1.7) and (1.11) yields the following expression for the spring constant k

$$k = m\omega^2 = \frac{m^3c^4}{\hbar^2}, \quad \text{or} \quad k_l = \frac{m_l^3c^4}{\hbar^2}, \quad k_1 = \frac{m_1^3c^4}{\hbar^2}. \tag{1.12}$$

Here the spring constant k_l applies to all charged leptons, electron, muon and tauon, denoted by the subscript $l = e, \mu, \tau$, respectively. The constant k may also be generalized to the electron neutrino or neutrino 1 leading to the spring constant k_1.

It is noticed that the following important result for the z-component of the magnetic dipole moment $\boldsymbol{\mu}(l)$ of the electron, muon or tauon ($l = e, \mu, \tau$) can be deduced from the ring model embodied in (1.1) (see, e.g., refs. [2], [3])

$$\mu_z(l) = \frac{e\hbar}{2m_l c}, \tag{1.13}$$

where m_l is the mass of the considered charged lepton. It is noted that Gaussian units are used throughout this paper. To my knowledge, no alternative explanation is available for the observed approximate validity of (1.13). So, the ring model and its underlying assumptions predict the observed magnetic dipole moments of all charged leptons in first order.

In section 2 a new Lagrangian L, depending on two harmonic oscillators in the x- and y-direction, respectively, is conjectured. This Lagrangian predicts a circular motion in the x-y plane with angular frequency ω and possesses a number of remarkable properties. In addition, our approach will be compared to the standard special relativistic treatment for two ortho-normal harmonic oscillators.

In section 3 the basic equations of the toroidal model of leptons [2, 3], more complex than the simple equations of (1.1), are considered. In that case the energy E of all charged leptons (and neutrino 1) can be deduced from a two-dimensional harmonic oscillator in the x-y plane and a third harmonic oscillator in the z-direction, resulting in the so-called *ring torus model*. In section 4 the toroidal model is extended to neutrinos. It appears that neutrinos 2 and 3 can be described by the so-called *spindle torus model*. The predicted spring constants for these neutrinos are smaller than for neutrino 1. In section 5 the magnitude of the elastic energy of the electron is compared to the electromagnetic energy. Finally, in section 6 a summary of the results is given and some final remarks are added.

2. Discussion of two different relativistic Lagrangians

Two Lagrangian formulations of relativistic mechanics will be investigated in this section (compare with, e.g., Goldstein [9, chapters 6 and 8])

2.1 Lagrangian formulation based on a new metric

Instead of proposing the forces F_x and F_y of (1.5), they may also be deduced from a postulated Lagrangian L depending on the two orthogonal harmonic oscillators of (1.1). The new Lagrangian possesses many remarkable properties. It predicts simple canonical moments, important for their possible application to quantum-mechanics. Moreover, simple expressions for the forces F_x and F_y, and the Hamiltonian H are found. In section 2.2 the obtained results are compared with the results of the standard special relativistic approach (see, e.g., Goldstein [9, chapter 6]). The following Lagrangian L is conjectured here

$$L=-mc^2\frac{d\tau}{dt}=-mc^2\left(1+\frac{kx^2}{mc^2}+\frac{ky^2}{mc^2}-\frac{\dot{x}^2}{c^2}-\frac{\dot{y}^2}{c^2}\right)^{1/2}\equiv -mc^2X^{1/2}, \tag{2.1}$$

where X is a short-hand notation for the quantity between brackets and the proper time τ is given by the metric

$$d\tau^2=dt^2\left\{1+\frac{1}{c^2}\left(\frac{k}{m}x^2+\frac{k}{m}y^2-\frac{dx^2}{dt^2}-\frac{dy^2}{dt^2}\right)\right\}. \tag{2.2}$$

Subsequently, the following momenta can be defined and calculated from (2.1)

$$p_x\equiv\frac{\partial L}{\partial\dot{x}}=mX^{-1/2}\dot{x}\quad\text{and}\quad p_y\equiv\frac{\partial L}{\partial\dot{y}}=mX^{-1/2}\dot{y}. \tag{2.3}$$

In addition, a force F_x and F_y can be defined and calculated from (2.1)

$$F_x\equiv\frac{\partial L}{\partial x}=-X^{-1/2}kx\quad\text{and}\quad F_y\equiv\frac{\partial L}{\partial y}=-X^{-1/2}ky. \tag{2.4}$$

Application of the Euler-Lagrange equation to the x-component then yields

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}}-\frac{\partial L}{\partial x}=mX^{-1/2}\left(\ddot{x}+\frac{k}{m}x\right)-mX^{-3/2}\frac{\dot{x}}{c^2}\left\{\frac{k}{m}(x\dot{x}+y\dot{y})-(\dot{x}\ddot{x}+\dot{y}\ddot{y})\right\}=0. \tag{2.5}$$

Likewise, an analogous Euler-Lagrange equation for the y-component can be found. Furthermore, by making use of (2.1) and (2.3), the following Hamiltonian H can be calculated (compare to, e.g., Goldstein [9])

$$H=p_x\dot{x}+p_y\dot{y}-L=mX^{-1/2}(\dot{x}^2+\dot{y}^2)+mc^2X^{1/2}=mX^{-1/2}v^2+mc^2X^{1/2}. \tag{2.6}$$

From the simple relations of (1.1) the following remarkable results can be deduced

$$x\dot{x}+y\dot{y}=0\quad\text{and}\quad\dot{x}\ddot{x}+\dot{y}\ddot{y}=0. \tag{2.7}$$

Insertion of these relations into (2.5) shows that the last term between braces on the right-side of (2.5) becomes zero. The validity of the Euler-Lagrange equation for the x-component of (2.5) then requires that relation (1.6a) is valid. Likewise, the Euler-Lagrange equation for the y-component only reduces to zero value when relation (1.6b) is valid. So, equation (1.6) is valid when the Euler-Lagrange equations apply, whereas the postulated equations of (1.1) imply that (1.2), (1.3) and (1.4) are valid.

Insertion of (1.2), (1.3) and (1.7) into quantity X of (2.1) shows that X reduces to unity value

$$X = 1 + \frac{kr^2}{mc^2} - \frac{v^2}{c^2} = 1, \tag{2.8}$$

where all values for $v^2 = \dot{x}^2 + \dot{y}^2$ between $v = 0$ and $v = c$ are allowed. As a consequence, the relativistic momenta p_x and p_y of (2.3) simplify to their corresponding non-relativistic counterparts. In addition, the forces F_x and F_y in (2.4) reduce to the modern expressions for Hooke's law in the x- and y-direction

$$F_x = m\ddot{x} = -kx \quad \text{and} \quad F_y = m\ddot{y} = -ky. \tag{2.9}$$

Note that these forces coincide to those of (1.5). Furthermore, combination of (2.2) and (2.8) shows that the proper time interval $d\tau$ reduces to the time interval dt. So, time becomes absolute in this exceptional case.

Finally, insertion of $X = 1$ from (2.8) into the expression for the Hamiltonian H of (2.6) yields the result

$$H = mc^2\left(1 + \frac{v^2}{c^2}\right) = mc^2 + mv^2 = mc^2 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}kr^2. \tag{2.10}$$

This equation shows that for $v = 0$, the Hamiltonian H may be identified as the Einstein formula for the rest energy $E_0 = mc^2$. See for a discussion of this result, e.g., Okun [10]. In the other limiting case of $v = c$, the Hamiltonian H equals to value $H = 2\,mc^2$. This result is *relativistic* and *finite*. A new energy difference E can be defined by $E \equiv H - E_0$. This energy equals to $E = mv^2$; it may be put equal to the energy E of (1.10). So, the energy E for all values of v between zero and c can be written as

$$E = mv^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kr^2. \tag{2.11}$$

Summing up, the results for the forces F_x and F_y, the energy E and the Hamiltonian H in this section all rest on the particular combination of the basic equations (1.1) and the conjectured Langrangian L of (2.1).

2.2 Two-dimensional oscillator as an extension of special relativity

For comparison, the following Lagrangian formulation of relativistic mechanics will be considered in this section (compare with, e.g., Goldstein [9, chapters 6 and 8]). In absence of oscillation, the Minkowski metric in two dimensions applies

$$d\tau^2 = \left(1 - \frac{\dot{x}^2}{c^2} - \frac{\dot{y}^2}{c^2}\right)dt^2 = \left(1 - \frac{v^2}{c^2}\right)dt^2. \tag{2.12}$$

As a consequence, the following Lagrangian L and Hamiltonian H are then obtained

$$L = -mc^2\frac{d\tau}{dt} = -mc^2\left(1 - \frac{v^2}{c^2}\right)^{1/2}, \quad H = mc^2\left(1 - \frac{v^2}{c^2}\right)^{-1/2}. \tag{2.13}$$

Usually, when a harmonic oscillator is present, a potential energy term like $\tfrac{1}{2}kr^2$ is added to the Lagrangian L and Hamiltonian H, so that (2.13) changes into

$$L = -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - \tfrac{1}{2}kr^2, \quad H = mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} + \tfrac{1}{2}kr^2. \qquad (2.14)$$

For two orthogonal harmonic oscillators in the x- and y-direction, respectively, the potential energy term equals $\tfrac{1}{2}kr^2 = \tfrac{1}{2}k(x^2+y^2)$. From the Lagrangian (2.14) and the corresponding Euler-Lagrange equation in the x-direction the following force F_x can be calculated

$$F_x = -kx = m\ddot{x}\left(1-\frac{v^2}{c^2}\right)^{-1/2} + \frac{m\dot{x}(\dot{x}\ddot{x}+\dot{y}\ddot{y})}{c^2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}. \qquad (2.15)$$

This expression can be compared with the corresponding result of (2.9).

The obtained Hamiltonian H of (2.14) has the property that it approaches to infinity for $v = c$. In the limiting case $v << c$ one obtains the familiar standard expression for the Hamiltonian H

$$H \approx mc^2 + \tfrac{1}{2}mv^2 + \tfrac{1}{2}kr^2. \qquad (2.16)$$

Contrary to the Hamiltonian H of (2.16), the Hamiltonian H of (2.10) applies to every value of v in the closed interval $[0, c]$ and becomes equal to the value $H = 2\,mc^2$ in the limiting case of $v = c$. Further evidence is necessary to find the correct choice of H for leptons.

3. Toroidal model of leptons

Following the toroidal solenoid model of Consa [2], the basic equations (1.1) of the ring model of the electron have been generalized to all charged leptons and neutrinos by Biemond [3]. For charged leptons the electric charge is assumed to be concentrated in a single point, whereas for the neutrinos the mass is thought to be concentrated in a single point. The topology of this point charge or point mass is described by postulating a set of Cartesian coordinates, depending on two angular frequencies, i.e., ω and $N\omega$ and two constant parameters r_1 and r_2. So, the following basic equations are postulated

$$\begin{aligned} x(t) &= (r_1 + r_2\cos N\omega t)\cos\omega t, \\ y(t) &= (r_1 + r_2\cos N\omega t)\sin\omega t, \\ z(t) &= -r_2\sin N\omega t, \end{aligned} \qquad (3.1)$$

where r_1 is the radius of the torus and r_2 is the radius of the tube.

The following speed squared, applicable to all considered leptons, can be calculated from (3.1)

$$\dot{r}(t)^2 \equiv \dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2 = \omega^2 r_1^2\left\{1 + 2\frac{r_2}{r_1}\cos N\omega t + \frac{r_2^2}{r_1^2}(\cos N\omega t)^2 + N^2\frac{r_2^2}{r_1^2}\right\}. \qquad (3.2)$$

It is postulated that the integrated value of $\dot{r}(t)^2$ of (3.2) over a period $T = 2\pi/\omega$ will match the speed of light squared. The integrated value of $\dot{r}(t)^2$ then becomes

$$\frac{1}{T}\int_0^T \dot{r}(t)^2 dt = \omega^2 r_1^2\left(1 + N^2\frac{r_2^2}{r_1^2} + \tfrac{1}{2}\frac{r_2^2}{r_1^2}\right) = c^2. \qquad (3.3)$$

Utilizing speeds v_1 and v_2 defined by $v_1 \equiv \omega r_1$ and $v_2 \equiv \omega r_2$, respectively, the time averaged expression of (3.3) can be rewritten as

$$v_1^2 + \left(N^2 + \tfrac{1}{2}\right) v_2^2 = c^2. \qquad (3.4)$$

Multiplication of both sides of (3.4) with mass m_l of the charged lepton l (or mass m_i of neutrino $i = 1, 2, 3$) yields the time averaged energy $\bar{E}(l)$ (or $\bar{E}(i)$)

$$\bar{E}(l) = m_l v_1^2 + (N^2 + \tfrac{1}{2}) m_l v_2^2 = m_l c^2. \qquad (3.5)$$

In the evaluation of $\bar{E}(l)$ two limiting cases will now be distinguished: $r_1 >> Nr_2$ and $Nr_2 >> r_1$. The first limiting case $r_1 >> Nr_2$ with $N = 1$ appears to be applicable to all charged leptons and to the electron neutrino or neutrino 1. It has been shown [3] that for the choice $N = 1$ the calculated magnetic dipole moments of all charged leptons are compatible with the observed ones, anomalous corrections included. Moreover, the same ratio $r_2/r_1 = (\alpha/\pi)^{1/2} = 0.04820$ is obtained for all charged leptons (α is the fine-structure constant). Furthermore, for the neutrino 1 the ratio r_2/r_1 might be put equal to $(\alpha_W/\pi)^{1/2}$, where α_W is the electroweak coupling constant at low energy, as has been discussed in refs. [11, 12]. When the illustrative value $\alpha_W = 1/32$ is chosen, one obtains a value of $r_2/r_1 = (\alpha_W/\pi)^{1/2} = 0.10$.

For $N = 1$ and the limiting case $r_1 >> r_2$, figure 1 is given as an illustration of the so-called *ring torus model* for charged leptons. In this case the basic equations (3.1) imply an orbit of the point charge e that follows the surface of the so-called *ring torus*, sometimes colloquially referred as a *doughnut*. When a positive charge e is chosen, a positive sign is obtained from the set of basic equations (3.1) for the z-component of the magnetic dipole moment $\boldsymbol{\mu}_z(l)$. In addition, the y-component $\boldsymbol{\mu}_y(l)$ of the magnetic dipole moment and the total dipole moment $\boldsymbol{\mu}(l)$ are shown in figure 1. The numbers 1, 2, 3 and 4 denote the location of charge e at time $t = 0$, $t = \frac{1}{4}T$, $t = \frac{1}{2}T$ and $t = \frac{3}{4}T$, respectively (T is given by $T = 2\pi/\omega$). Note that the speeds vary at these different times, e.g., at position 1: $\dot{\mathbf{x}}(t) = \dot{\mathbf{x}} = 0$, $\dot{\mathbf{y}}(t) = \dot{\mathbf{y}} = \mathbf{v}_1 + \mathbf{v}_2$ and $\dot{\mathbf{z}}(t) = \dot{\mathbf{z}} = -\,\mathbf{v}_2$, and so on. Although the positions 1, 2, 3 and 4 are lying in the same plane, the orbit of charge e (drawn in red) is not completely flat.

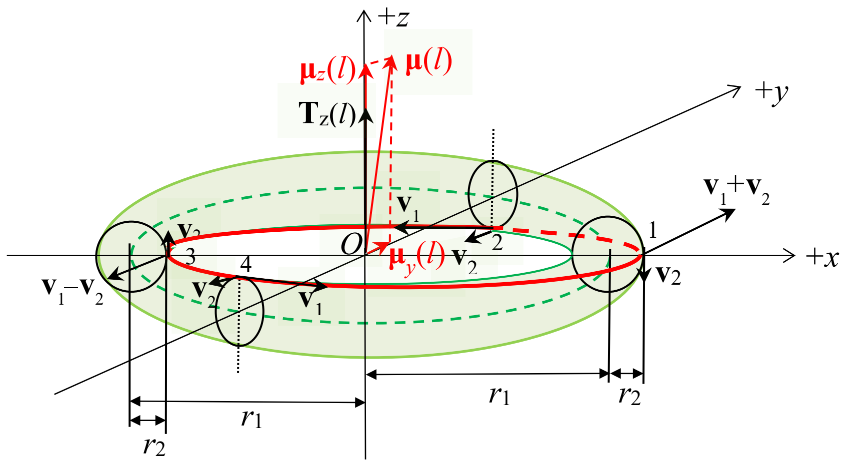

Figure 1. *Ring torus model* of charged leptons, according to eq. (3.1) for $N = 1$ and $r_1 >> r_2$. When O is the origin of the coordinate system, the location of a positive charge e is fixed by the Cartesian coordinates $x(t) = x$, $y(t) = y$ and $z(t) = z$. The positive charge e moves with an average speed v_1 in a ring of radius r_1 and a speed v_2 ($v_1 >> v_2$) in a circle of radius r_2. The green blocked line is a circle with radius r_1 in the x-y plane and the orbit of e is drawn in red. For clarity reasons the values of r_1, r_2, v_1 and v_2 are not drawn to scale. The vectors of the y- and z-component of the magnetic dipole moment $\boldsymbol{\mu}(l)$ of charged lepton l are also shown. In addition, the direction of z-component the toroidal moment $\mathbf{T}_z(l)$ for lepton l with positive charge is denoted. See section 4 of ref. [3] for further comment.

In the limiting case of $r_1 >> Nr_2$ with $N = 1$ the energy $E(l)$ is written in the form of two orthogonal harmonic oscillators in the x-y plane and a third harmonic oscillator in the z-direction. In line with this choice, the energy $E(l)$ can be split up in the following way

$$E(l) = m_l(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \{\tfrac{1}{2}m_l(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}k_{l,1}(x^2 + y^2)\} + (\tfrac{1}{2}m_l\dot{z}^2 + \tfrac{1}{2}k_{l,2}z^2), \quad (3.6)$$

where two different spring constants $k_{l,1}$ and $k_{l,2}$ have been introduced. It is noticed that the energy of the harmonic oscillator in the z-direction is small compared to the energy of the two orthogonal harmonic oscillators in the x-y plane. Therefore, the results of the two-dimensional harmonic oscillator discussed in section 2.1 will approximately retain their validity in the case of (3.6). The first two terms on the right-hand side of (3.6) may then be taken equal. Further evaluation of (3.6) by substitution of the speeds $\dot{x}$, $\dot{y}$, $\dot{z}$ calculated from (3.1) and the coordinates x, y, z yields

$$\begin{aligned} E(l) = \tfrac{1}{2}m_l\omega^2(r_1^2 + 2r_1r_2\cos\omega t + r_2^2) + \tfrac{1}{2}k_{l,1}(r_1^2 + 2r_1r_2\cos\omega t + r_2^2\cos^2\omega t) + \\ \tfrac{1}{2}m_l\omega^2r_2^2\cos^2\omega t + \tfrac{1}{2}k_{l,2}r_2^2\sin^2\omega t). \end{aligned} \quad (3.7)$$

For the time averaged value $\bar{E}(l)$ follows from (3.7)

$$\bar{E}(l) = \tfrac{1}{2}m_l\omega^2(r_1^2 + r_2^2) + \tfrac{1}{2}k_{l,1}(r_1^2 + \tfrac{1}{2}r_2^2) + \tfrac{1}{4}m_l\omega^2r_2^2 + \tfrac{1}{4}k_{l,2}r_2^2. \quad (3.8)$$

When the first two terms and the last two terms on the right-hand side of (3.8) are put equal, one obtains the following expressions for the spring constants $k_{l,1}$ and $k_{l,2}$

$$k_{l,1} = \frac{r_1^2 + r_2^2}{r_1^2 + \tfrac{1}{2}r_2^2}m_l\omega^2 \approx \left(1 + \tfrac{1}{2}\frac{r_2^2}{r_1^2}\right)m_l\omega^2, \quad \text{and} \quad k_{l,2} = m_l\omega^2. \quad (3.9)$$

Since the ratio r_2/r_1 is much smaller than unity value, the value of $k_{l,1}$ approaches to the value of $k_{l,2}$ and to the single spring constant k_l of (1.12). Insertion of (3.9) into $\bar{E}(l)$ of (3.8) yields

$$\bar{E}(l) \approx m_l\omega^2(r_1^2 + r_2^2) + \tfrac{1}{2}m_l\omega^2r_2^2. \quad (3.10)$$

Substitution of the speeds $v_1 \equiv \omega r_1$ and $v_2 \equiv \omega r_2$ into (3.5) shows that the right-hand sides of energies $\bar{E}(l)$ of (3.10) and $\bar{E}(l)$ of (3.5) coincide for $N = 1$. Furthermore, it is noted that the last term on the right-hand side of (3.10), $\tfrac{1}{2}m_l\omega r_2^2$ corresponds to the sum of the last two terms on the right-hand side of (3.8).

It appears that for $N = 1$ the calculated result for the z-component of magnetic dipole moment $\mu_z(l)$ from the toroidal model [3] coincides with the standard expression for electron, muon and tau lepton ($l = e, \mu, \tau$), first order anomalous correction included

$$\mu_z(l) = \left(1 + \tfrac{1}{2}\frac{r_2^2}{r_1^2}\right)\frac{e\hbar}{2m_lc} = \left(1 + \tfrac{1}{2}\frac{\alpha}{\pi}\right)\frac{e\hbar}{2m_lc} = (1 + 0.0011614)\frac{e\hbar}{2m_lc}. \quad (3.11)$$

Equation (3.11) implies that $\tfrac{1}{2}r_2^2/r_1^2 = \alpha/(2\pi) = 0.0011614$, where α is the fine-structure constant. Moreover, the same ratio for $r_2/r_1 = (\alpha/\pi)^{1/2} = 0.04820$ is found [3] for all charged leptons. In addition, the toroidal model provides an explanation of the first order anomalous correction of the magnetic dipole moment of all charged leptons.

As an alternative, the energy $\bar{E}(l)$ of (3.10) can also be split up as follows

$$\bar{E}(l) = m_l\omega^2 r_1^2 + {}^3\!/_2\, m_l\omega^2 r_2^2 \equiv \bar{E}(r_1) + \bar{E}(r_2), \tag{3.12}$$

so that

$$\bar{E}(r_2)/\bar{E}(r_1) = {}^3\!/_2 \frac{r_2^2}{r_1^2} = {}^3\!/_2 \frac{\alpha}{\pi} = 0.0034842. \tag{3.13}$$

This result implies that only a small part of the total energy $\bar{E}(l)$ is connected to radius r_2.

4. Toroidal model of neutrinos

An expression for the magnetic dipole moment of massive Dirac neutrinos has previously been deduced by Lee and Shrock [13] and Fujikawa and Shrock [14], in the context of electroweak interactions at the one-loop level. The predicted magnetic dipole moment $\mu(i)$ (i = 1, 2, 3) of the neutrino i was found to be proportional to its mass m_i. As an alternative, another expression for the magnetic dipole moment $\mu(i)$, the so-called Wilson-Blackett formula, can be deduced by application of the so-called gravitomagnetic approach [3, 11, 12]. By combination of the corresponding z-components of the magnetic dipole moment $\mu_z(1)$ of neutrino 1 a value of 1.530 meV/c^2 is obtained for mass m_1. Moreover, from recent observed values of $\Delta m_{21}^2 \equiv m_2^2 - m_1^2$ and $\Delta m_{32}^2 \equiv m_3^2 - m_2^2$ the values of the other two masses m_2 and m_3 can also be calculated (see, e.g., ref. [12])

$$m_1 = 1.530 \text{ meV}/c^2, \quad m_2 = 8.79 \text{ meV}/c^2 \quad \text{and} \quad m_3 = 50.5 \text{ meV}/c^2. \tag{4.1}$$

These results have been deduced for normal ordering.

In addition, it is conjectured[2] that the rotational velocity v_r of equation (1.3) of neutrino 1 can match the speed of light c. All formulas analogous to (1.4) through (1.12) of the *ring model* may also be applied to neutrino 1. For the sake of convenience, it is assumed that the neutrino as a whole displays no translational motion.

Analogous to the charged leptons, it is postulated for the *ring model* that the radius r of the ring is also given by the Compton wavelength λ_C

$$r = \frac{\hbar}{m_1 c} = \lambda_C. \tag{4.2}$$

Utilizing (4.2) and the simple equations (1.1) of the ring model, the following expression for the + z-component of magnetic dipole moment $\boldsymbol{\mu}(1)$, $\mu_z(1)$, of neutrino 1 can be deduced

$$\mu_z(1) = \frac{G^{1/2}\hbar}{2c}, \tag{4.3}$$

where G is the gravitational constant. It is stressed that relation (4.3) for $\mu_z(1)$ needs further observational conformation.

The z-component of the magnetic dipole moment $\mu_z(i)$ of all neutrinos (i = 1, 2, 3) can also be deduced from the more complex basic equations (3.1) of the toroidal model [3]. In that case the full set of Cartesian coordinates of (3.1) are used for the location of the mass point of every neutrino. As a first example, one obtains for $\mu_z(1)$ of neutrino 1

$$\mu_z(1) = \left(1 + {}^1\!/_2 \frac{r_2^2}{r_1^2}\right)\frac{G^{1/2}\hbar}{2c} = \left(1 + {}^1\!/_2 \frac{\alpha_W}{\pi}\right)\frac{G^{1/2}\hbar}{2c} \equiv g_1{}' \frac{G^{1/2}\hbar}{2c}. \tag{4.4}$$

[2] This conjecture may be valid for all charged leptons. In that case the point charge and point mass coincide.

The value of the ratio r_2/r_1 of neutrino 1 in (4.4) might be put equal to $(\alpha_W/\pi)^{½}$, as has been discussed in refs. [3, 12]. The quantity α_W is the electroweak coupling constant at low energy. When an illustrative value of α_W = 1/32.0 is chosen, one obtains the values r_2/r_1 = 0.10 and $\alpha_W/2\pi$ = 0.0050. In that case the quantity g_1' becomes equal to g_1' = 1.0050. So, the value of the term ½ r_2^2/r_1^2 in (4.4) is small compared to the unity term for neutrino 1. As shown in (3.11), the ratios r_2/r_1 of all charged leptons l ($l = e, \mu, \tau$) are equal and small. Therefore, the *ring torus model* does not only apply to charged leptons, but also to neutrino 1. It is noted that for the *ring model* the formula $\mu_z(1)$ of (4.3) has been deduced. In this result the contribution of $\alpha_W/2\pi$ is absent.

Analogous to (4.4), the following relation can be deduced for the neutrinos 2 and 3 from the gravitomagnetic approach [3]

$$\mu_z(i) = \left(1 + ½\frac{r_2^2}{r_1^2}\right)\frac{G^{½}\hbar}{2c} \equiv g_i'\frac{G^{½}\hbar}{2c}, \quad i = 2,3. \tag{4.5}$$

Moreover, according to the theoretical prediction from refs. [13, 14], the z-components of the magnetic dipole moment $\mu_z(i)$ are proportional to m_i. When a value $g_1' = 1$ is chosen for neutrino 1 and masses m_i from (4.1), the magnetic dipole moments $\mu_z(2)$ and $\mu_z(3)$ of neutrino 2 and 3 can be written as

$$\mu_z(2) = \frac{m_2}{m_1}\frac{G^{½}\hbar}{2c} = 5.75\frac{G^{½}\hbar}{2c} \quad \text{and} \quad \mu_z(3) = \frac{m_3}{m_1}\frac{G^{½}\hbar}{2c} = 33.0\frac{G^{½}\hbar}{2c}. \tag{4.6}$$

Comparison of (4.5) and (4.6) shows that g_2' = 5.75 and $(r_2/r_1)^2$ = 9.5, and g_3' = 33.0 and $(r_2/r_1)^2$ = 64, respectively. These results imply that the limiting case $Nr_2 >> r_1$ applies to neutrinos 2 and 3. As a consequence, the basic equations (3.1) then lead to the so-called *spindle torus*.

For the limiting case $Nr_2 >> r_1$ an additional condition for the quantity g_i' (i =2, 3) has previously been deduced [3]

$$g_i' = N\left(1 + \tfrac{1}{2}\frac{1}{N^2}\frac{r_1^2}{r_2^2} + \tfrac{1}{4}\frac{1}{N^2} - \tfrac{3}{8}\frac{1}{N^4}\frac{r_1^2}{r_2^2} - \tfrac{1}{8}\frac{1}{N^4}\frac{r_1^4}{r_2^4} - \tfrac{3}{64}\frac{1}{N^4} + \ldots\right). \tag{4.7}$$

Combination of (4.5), (4.6) and (4.7) yields a value N = 5.7 for neutrino 2, whereas a value N = 33 is obtained for neutrino 3. It is noted that Sbitnev [15], starting from basic equations comparable with those of (3.1), also discussed "vortex balls" with values $N > 1$ and $r_2 > r_1$.

Analogous to the calculation of the energy $E(l)$ of (3.6) and the time averaged value $\bar{E}(l)$ of (3.8) for charged leptons, the corresponding energies $E(1)$ and $\bar{E}(1)$ for neutrino 1 can be calculated in the limiting case of N = 1 and $r_1 >> r_2$. In addition, the spring constants $k_{1,1}$ and $k_{1,2}$ for neutrino 1 also follow from a calculation, analogous to that of (3.9). For a summary, see table 1.

For neutrinos 2 and 3 the limiting case $N > 1$ and $r_2 > r_1$ applies. Utilizing the relations $v_1 \equiv \omega r_1$ and $v_2 \equiv \omega r_2$ for the speeds v_1 and v_2, the averaged energy $\bar{E}(l)$ of (3.5) transforms into

$$\bar{E}(i) = m_i\omega^2 r_1^2 + m_i(N^2 + ½)\omega^2 r_2^2 = \bar{E}(r_1) + \bar{E}(r_2) = m_i c^2, \quad i = 2,3. \tag{4.8}$$

so that

$$\bar{E}(r_2)/\bar{E}(r_1) = (N^2 + ½)\frac{r_2^2}{r_1^2}. \tag{4.9}$$

It is noted that for neutrinos 2 and 3 the energy $\bar{E}(r_1)$ is relatively small compared to $\bar{E}(r_2)$.

In addition, the energy $\overline{E}(r_1)$ is also much smaller than m_ic^2, so that this non-relativistic energy may be split up into two parts: a classical kinetic energy $½m_i\omega r_1^2$ and potential energy $½k_{i,1}r_1^2$ of equal magnitude. The energy $\overline{E}(r_2)$, however, is relativistic, for the speed $(N^2 + 1)^{½}\omega r_2$ in (4.8) approaches to speed c. The latter energy is due to N revolutions, where each revolution resembles a circle of radius r_2. Each partial orbit, however, is not completely closed and the plane of the orbit is not completely flat. Nevertheless, as described in section 2.1, each of these nearly circular orbits may approximately be considered as the superposition of two orthogonal relativistic harmonic oscillators. Corresponding to the equal terms $½m\omega r^2$ and $½kr^2$ of the energy E in (2.11), the energy $\overline{E}(r_2)$ may also be split up into two equal parts. So, the energies $\overline{E}(r_1)$ and $\overline{E}(r_2)$ may be written as

$$\overline{E}(r_1) = ½m_i\omega^2 r_1^2 + ½k_{i,1}r_1^2 \quad \text{and} \quad \overline{E}(r_2) = ½m_i(N^2+½)\omega^2 r_2^2 + ½k_{i,2}r_2^2. \tag{4.10}$$

The following spring constants then follow from (4.10)

$$k_{i,1} = m_i\omega^2 \quad \text{and} \quad k_{i,2} = (N^2+½)m_i\omega^2. \tag{4.11}$$

Note that the spring constant $k_{i,1}$ is a factor $(N^2 + ½)$ smaller than $k_{i,2}$. Insertion of the formulas for ω and g (see eq. (1.13) and eq. (1.4), respectively, in ref. [3]), then leads to the following expressions for the spring constants $k_{i,1}$ and $k_{i,2}$

$$k_{i,1} = m_i\omega^2 = \frac{m_i^3c^4}{g(i)^4\hbar^2}, \quad k_{i,2} = (N^2+½)m_i\omega^2 = (N^2+½)\frac{m_i^3c^4}{g(i)^4\hbar^2}. \tag{4.12}$$

As examples, the magnitudes of spring constant $k_{2,2}$ of neutrino 2 and spring constant $k_{3,2}$ of neutrino 3 will be compared. Insertion of the calculated values $N = 5.7$, $r_2^2/r_1^2 = 9.5$ ($r_2/r_1 = 3.1$) and $m_2 = 5.75\ m_1$ for neutrino 2, and $N = 33$, $r_2^2/r_1^2 = 64$ ($r_2/r_1 = 8$) and $m_3 = 33\ m_1$ for neutrino 3 into (4.12), respectively, yields (see also table 1)

$$\begin{aligned} k_{2,2} &= (N^2+½)\frac{m_2^3c^4}{g(2)^4\hbar^2} = (5.7^2+½)\frac{5.75^3 m_1^3c^4}{g(2)^4\hbar^2} = 0.063\frac{m_1^3c^4}{\hbar^2}, \\ k_{3,2} &= (N^2+½)\frac{m_3^3c^4}{g(3)^4\hbar^2} = (33^2+½)\frac{33^3 m_1^3c^4}{g(3)^4\hbar^2} = 0.0081\frac{m_1^3c^4}{\hbar^2}. \end{aligned} \tag{4.13}$$

These spring constants are much smaller than the single spring constant $k_1 = m_1^3c^4/\hbar^2$ of neutrino 1, deduced for the *ring model* and given in (1.12).

As an example of the limiting case with $Nr_2 >> r_1$, the so-called *spindle torus* of neutrino 3 with $N = 33$ and ratio $r_2/r_1 = 8$ is illustrated in figure 2. For clarity reasons, only one open orbit of mass m_3 is sketched from point P until point T passing through the points Q, R and S. The coordinates of all these points can be calculated from (3.1). For point P the coordinates are: $x = (r_1 + r_2) = +9\ r_1$, $y = 0$, $z = 0$, for Q: $x = r_1 \cos(90°/33) = +0.999\ r_1$, $y = r_1 \sin(90°/33) = +0.0476\ r_1$, $z = -r_2$, for R: $x = (r_1 - r_2)\cos(180°/33) = -6.97\ r_1$, $y = (r_1 - r_2)\sin(180°/33) = -0.665\ r_1$, $z = 0$, For S: $x = r_1\cos(270°/33) = +0.990\ r_1$, $y = r_1 \sin(270°/33) = +0.142\ r_1$, $z = +r_2$ and for T: $x = (r_1 + r_2)\cos(360°/33) = +8.84\ r_1$, $y = (r_1 + r_2)\sin(360°/33) = +1.70\ r_1$, $z = 0$. The points P, T and the auxiliary point U (with coordinates $x = -r_1 - r_2 = -9\ r_1$, $y = 0$, $z = 0$) are all lying on a circle with radius $r_1 + r_2 = 9\ r_1$, so that $OP = OT = OU = 9\ r_1$, where the origin O is the centre of the circle. In addition, the point R is lying in the same plane, but the distance $OR = |r_1 - r_2| = 7\ r_1$ is smaller than radius OP.

From (3.1) it follows that the orbit from P to T passes through the z-axis when $r_1 + r_2 \cos N\omega t = 0$ or $\cos 33\omega t = -r_1/r_2 = -⅛$. In that case the z-coordinate can be calculated from $z = -r_2 \sin 33\omega t = \pm 0.992\, r_2$. These z-values can be compared with the value $z = +r_2$ of point S and $z = -r_2$ of point Q. The depression $d = r_2 - 0.992\, r_2 = 0.008\, r_2$ of the orbit at the poles resembles the shape of an apple. Therefore, the surface of the spindle torus is sometimes denoted as the *apple torus*. It is noted that for large values of r_2/r_1 the surface of the *spindle torus* approaches the surface of a *sphere*. Summing up, the form of neutrino 1 can approximately be described by the *ring model* of Parson or more accurately by the *ring torus model*, whereas neutrino 3 can be described by the *spindle torus model* or approximately by a *sphere*. Neutrino 2 can also be characterized as a *spindle torus*.

The next open orbit of m_3 starts at point T and ends on the equatorial circle of radius $9r_1$ in the direction of U. After insertion of a total of 33 such partial orbits, the last orbit ends in the starting point P, completing one closed orbit. Note that the arc length PT is given by $2\pi\,(r_1 + r_2)(360/33)$. It is noticed that the first (open) orbit beginning in P and ending in T is approximately flat and resembles a circle of radius r_2.

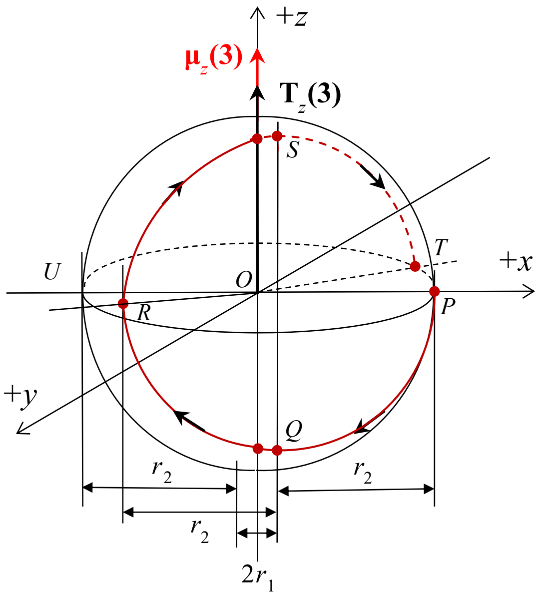

Figure 2. *Spindle torus model* of neutrino 3 with $N = 33$ and $r_1 = 8\, r_2$. For clarity reasons, only one open orbit of mass m_3 is sketched from point P until point T passing through the points Q, R and S. After insertion of a total of 33 such partial orbits, the last orbit ends in the starting point P, completing one closed orbit. The directions of the magnetic dipole moment $\boldsymbol{\mu}_z(3)$ and the toroidal moment $\mathbf{T}_z(3)$ are also shown. See text for further comment.

The values $N = 5.7$ and $r_2/r_1 = 3.1$ for neutrino 2 (see comment to (4.12)), show that the limiting case $Nr_2 >> r_1$ also applies to this neutrino. The resulting *spindle torus* is characterized by 5.7 revolutions in a complete closed orbit, whereas the ratio $r_2/r_1 = 3.1$.

In addition, the z-component of the magnetic dipole moment, $\boldsymbol{\mu}_z(3)$ is denoted in figure 2. The magnitude of $\boldsymbol{\mu}_z(1)$ is shown in (4.4), whereas the values of $\boldsymbol{\mu}_z(2)$ and $\boldsymbol{\mu}_z(3)$ are given in (4.6). The values of the z-component of the toroidal moments $\mathbf{T}_z(1)$, $\mathbf{T}_z(2)$ and $\mathbf{T}_z(3)$ are calculated in ref. [3] and summarized in table 1 of that reference. As an example, the vector $\mathbf{T}_z(3)$ is given by

$$T_z(3) \approx N\left(g_3'-1\right)g(3)\frac{G^{1/2}\hbar}{2c}\frac{\hbar}{m_3 c} = 33\times 32\times 264\,\frac{G^{1/2}\hbar}{2c}\frac{\hbar}{33\,m_1 c} = 8450\,\frac{G^{1/2}\hbar}{2c}\frac{\hbar}{m_1 c}. \quad (4.14)$$

It is stressed that these results rest on assumptions that require conformation.

5. Comparison of electromagnetic and elastic energy of the electron

In this section the magnitudes of the electromagnetic energy E_{em} and the elastic energy E_{elast} of the electron will be compared. Usually, it is assumed that the mass of the electron is exclusively due to electromagnetic origin, as has been pointed out by many authors (see, e.g., [4, 5, 6]). An estimate of the magnitude of E_{em} for the electron will now be given. For reasons of simplicity, only the electromagnetic energy outside a uniformly charged sphere of radius r_0 is considered. Apart from the electric energy E_e, due to the charge of the electron, a contribution from magnetic origin, E_m, will be added. Therefore, the total electromagnetic energy E_{em} can be written as

$$E_{em} = E_e + E_m = \frac{1}{8\pi}\int \mathbf{E}\cdot\mathbf{E}\,dV + \frac{1}{8\pi}\int \mathbf{B}\cdot\mathbf{B}\,dV. \tag{5.1}$$

Here the electric field $\mathbf{E}$ and the magnetic induction field $\mathbf{B}$ for $r \geq r_0$ are given by, respectively

$$\mathbf{E} = \frac{e}{r^3}\mathbf{r}, \qquad \mathbf{B} = \frac{3\boldsymbol{\mu}\cdot\mathbf{r}}{r^5}\mathbf{r} - \frac{\boldsymbol{\mu}}{r^3}, \tag{5.2}$$

where $\boldsymbol{\mu} = \boldsymbol{\mu}(e)$ is the magnetic dipole moment of the electron. It is assumed that both $\mathbf{E}$ and $\mathbf{B}$ may be neglected in first order for $r < r_0$.

Insertion of (5.2a) into (5.1) yields for energy E_e

$$E_e = \frac{1}{8\pi}\int_{r_0}^{\infty} \frac{e^2}{r^4} 4\pi r^2 dr = \frac{1}{2}\frac{e^2}{r_0}. \tag{5.3}$$

Note that the energy E_e becomes infinite when r_0 approaches zero. If the so-called classical radius for the electron

$$r_0 = \frac{e^2}{m_e c^2} = 2.818\times 10^{-13}\,\text{cm} \tag{5.4}$$

is substituted into (5.3), one obtains $E_e = ½\, m_e c^2$ for energy E_e. On the other hand, when the radius r_0 would be equal to the Compton wavelength λ_C of (1.9), the electric energy E_e becomes

$$E_e = \tfrac{1}{2}\frac{e^2}{\hbar c} m_e c^2 = \tfrac{1}{2}\alpha m_e c^2 = 0.003649\, m_e c^2. \tag{5.5}$$

The choice of λ_C for r_0, however, leads to the predicted formula of (1.13) for the z-component of the magnetic dipole moment $\boldsymbol{\mu}(e)$, $\mu_z(e)$. In first order, the latter result has been confirmed by many observations.

Substitution of (5.2b) into (5.1) yields for energy E_m

$$E_m = \frac{1}{8\pi}\int \left\{ \frac{3(\boldsymbol{\mu}\cdot\mathbf{r})^2}{r^8} + \frac{\boldsymbol{\mu}\cdot\boldsymbol{\mu}}{r^6} \right\} dV = \frac{\mu^2}{8\pi}\int_{r_0}^{\infty}\int_0^{\pi} \frac{3\cos^2\theta + 1}{r^6} 2\pi r^2 \sin\theta\, d\theta\, dr = \frac{1}{3}\frac{\mu^2}{r_0^3}. \tag{5.6}$$

Substitution of $\mu = \mu_z(e)$ from (1.13) and the classical radius r_0 of (5.4) into (5.6), yields the result $E_m = \tfrac{1}{12}\,\alpha^{-2} m_e c^2$, much larger than $E_e = ½\, m_e c^2$. When the Compton wavelength λ_C of (1.9) is substituted as radius r_0 into (5.6), however, one obtains for the magnetic energy E_m

$$E_m = \tfrac{1}{12}\frac{e^2}{\hbar c} m_e c^2 = \tfrac{1}{12}\alpha m_e c^2 = 0.000608\, m_e c^2. \tag{5.7}$$

Combination of (5.5) and (5.7) leads to the following expression of the electromagnetic energy E_{em}

$$E_{em} = E_e + E_m = \tfrac{1}{2}\alpha m_e c^2 + \tfrac{1}{12}\alpha m_e c^2 = \tfrac{7}{12}\alpha m_e c^2. \tag{5.8}$$

In the latter case the energy of the electron must be attributed to additional causes, for example, to the elastic energy E_{elast} of (2.11) based on Hooke's law of (2.9).

6. Summary and final remarks

In 1915 Parson [1] already proposed a *flat* (instead of a *spherical*) geometric shape for the electron by introduction of the so-called *ring model*. This model has recently been extended by Consa [2] and Biemond [3]. All these authors assume that the elementary charge e may flow through the ring of radius r at the speed of light c. An important result of the ring model is that it predicts the correct z-component of magnetic dipole moment $\mu_z(l)$ of all charged leptons l ($l = e, \mu, \tau$) in first order. It is noticed that this result is obtained by the additional assumption that the radius r of the ring is given by the Compton wavelength λ_C of (1.9). It appears that the ring model can also be extended to the electron neutrino, or neutrino 1, as has been discussed in section 4.

Furthermore, the *ring model* can be interpreted as a superposition of two orthogonal, relativistic harmonic oscillators in the x-y plane. The two simple differential equations in the x- and y-direction of (1.4) are then obtained, suggesting the validity of Hooke's law (1.5) in both directions. In addition, according (1.8), one obtains for the energy E

$$E = m v_r^{\,2} = m\omega^2 r^2 = \tfrac{1}{2} m\omega^2 r^2 + \tfrac{1}{2} k r^2 = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2} k(x^2 + y^2). \tag{6.1}$$

Moreover, for both harmonic oscillators the same formal relation for the spring constant k for the charged lepton l and for neutrino 1 is obtained in (1.12)

$$k_l = \frac{m_l^3 c^4}{\hbar^2}, \quad l = e, \mu, \tau; \qquad k_1 = \frac{m_1^3 c^4}{\hbar^2}, \quad i = 1. \tag{6.2}$$

The general applicability of the spring constant k indicates that Hooke's law may be a key element in the explanation of the structure of all charged leptons l and neutrino 1. It is noticed that the spring constants k_l depend on the third power of mass m_l, whereas they are independent of charge e. A summary of the results for the *ring model* is given in table 1.

In section 2.1 a new Lagrangian L, depending on two harmonic oscillators in the x- and y-direction, is conjectured. The latter Lagrangian predicts a circular motion in the x-y plane with angular frequency ω. Remarkable results for momentum, force, Hamiltonian H and energy E are obtained. In section 2.2, the new approach is compared to the usually applied special relativistic treatment for two orthogonal harmonic oscillators.

In section 3 the more complex basic equations (3.1) of the toroidal model for the electron from Consa [2] and Biemond [3] are applied to charged leptons. In the limiting case $r_1 >> Nr_2$ and $N = 1$ this model leads to the *ring torus model* as illustrated in figure 1. Results for this model are also summarized in table 1. In this case the energy $E(l)$ of the charged lepton l can be written as the sum of two orthogonal harmonic oscillators in the x- and y-direction, respectively, and a third harmonic oscillator in the z-direction according to (3.6), whereas the time averaged energy $\bar{E}(l)$ is given by (see (3.8))

$$\bar{E}(l) = \tfrac{1}{2} m_l \omega^2 (r_1^2 + r_2^2) + \tfrac{1}{2} k_{l,1}(r_1^2 + \tfrac{1}{2} r_2^2) + \tfrac{1}{4} m_l \omega^2 r_2^2 + \tfrac{1}{4} k_{l,2} r_2^2. \tag{6.3}$$

Table 1. Theoretical results for the *ring model* and the *ring torus model* for charged leptons ($l = e, \mu, \tau$) and neutrino 1 are given. For neutrino 2 and 3 the *spindle torus model* applies. Results for this model are also shown. See ref. [3] and text for further comment.

Ring model (Parson's model)	
Charged lepton l, $l = e, \mu, \tau$	Neutrino i =1
$N = 1$	$N = 1$
$E = \tfrac{1}{2} m_l \omega^2 r^2 + \tfrac{1}{2} k_l r^2, \quad E = \hbar\omega = m_l c^2$	$E = \tfrac{1}{2} m_1 \omega^2 r^2 + \tfrac{1}{2} k_1 r^2, \quad E = \hbar\omega = m_1 c^2$
$r = \dfrac{\hbar}{m_l c}$	$r = \dfrac{\hbar}{m_1 c}$
$\omega = \dfrac{m_l c^2}{\hbar}$	$\omega = \dfrac{m_1 c^2}{\hbar}$
$k_l = m_l \omega^2 = \dfrac{m_l^3 c^4}{\hbar^2}$	$k_1 = m_1 \omega^2 = \dfrac{m_1^3 c^4}{\hbar^2}$
$\mu_z(l) = \dfrac{e\hbar}{2 m_l c}$	$\mu_z(1) = \dfrac{G^{1/2}\hbar}{2c}$
Ring torus model	
Charged lepton l, $l = e, \mu, \tau$	Neutrino i =1
$N = 1$	$N = 1$
$E(l) = \left\{\tfrac{1}{2} m_l(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2} k_{l,1}(x^2+y^2)\right\} +$ $(\tfrac{1}{2} m_l \dot{z}^2 + \tfrac{1}{2} k_{l,2} z^2),$ $\bar{E}(l) = \tfrac{1}{2} m_l \omega^2 (r_1^2+r_2^2) + \tfrac{1}{2} k_{l,1}(r_1^2 + \tfrac{1}{2} r_2^2) +$ $\tfrac{1}{4} m_l \omega^2 r_2^2 + \tfrac{1}{4} k_{l,2} r_2^2$	$E(1) = \left\{\tfrac{1}{2} m_1(\dot{x}^2+\dot{y}^2) + \tfrac{1}{2} k_{1,1}(x^2+y^2)\right\} +$ $(\tfrac{1}{2} m_1 \dot{z}^2 + \tfrac{1}{2} k_{1,2} z^2),$ $\bar{E}(1) = \tfrac{1}{2} m_1 \omega^2 (r_1^2+r_2^2) + \tfrac{1}{2} k_{1,1}(r_1^2 + \tfrac{1}{2} r_2^2) +$ $\tfrac{1}{4} m_1 \omega^2 r_2^2 + \tfrac{1}{4} k_{1,2} r_2^2$
$r_1 = g\dfrac{\hbar}{m_l c}, \quad g = \sqrt{1 + \tfrac{3}{2}\dfrac{r_2^2}{r_1^2}}$	$r_1(1) = g(1)\dfrac{\hbar}{m_1 c}, \quad g(1) = \sqrt{1 + \tfrac{3}{2}\dfrac{r_2^2}{r_1^2}}$
$\mu_z(l) = \dfrac{e\hbar}{2 m_l c}\left(1 + \tfrac{1}{2}\dfrac{r_2^2}{r_1^2}\right), \quad \dfrac{r_2^2}{r_1^2} \approx \dfrac{\alpha}{\pi}$	$\mu_z(1) = \dfrac{G^{1/2}\hbar}{2c}\left(1 + \tfrac{1}{2}\dfrac{r_2^2}{r_1^2}\right), \quad \dfrac{r_2^2}{r_1^2} \approx \dfrac{\alpha_W}{\pi}$
$\omega = \dfrac{m_l c^2}{g^2 \hbar} \approx \dfrac{m_l c^2}{\hbar}$	$\omega = \dfrac{m_1 c^2}{g(1)^2 \hbar}, \quad \omega \approx \dfrac{m_1 c^2}{\hbar}$
$k_{l,1} = \dfrac{r_1^2 + r_2^2}{r_1^2 + \tfrac{1}{2} r_2^2} m_l \omega^2 \approx \dfrac{m_l^3 c^4}{\hbar^2},$ $k_{l,2} = m_l \omega^2 = \dfrac{m_l^3 c^4}{g^4 \hbar^2} \approx \dfrac{m_l^3 c^4}{\hbar^2}$	$k_{1,1} = \dfrac{r_1^2 + r_2^2}{r_1^2 + \tfrac{1}{2} r_2^2} m_1 \omega^2 \approx \dfrac{m_1^3 c^4}{\hbar^2},$ $k_{1,2} = m_1 \omega^2 = \dfrac{m_1^3 c^4}{g(1)^4 \hbar^2} \approx \dfrac{m_1^3 c^4}{\hbar^2}$
Spindle torus model	
Neutrino $i = 2$	Neutrino $i = 3$
$N = 5.7$	$N = 33$
$\bar{E}(2) = \bar{E}(r_1) + \bar{E}(r_2),$ $\bar{E}(r_1) = \tfrac{1}{2} m_2 \omega^2 r_1^2 + \tfrac{1}{2} k_{2,1} r_1^2,$ $\bar{E}(r_2) \approx \tfrac{1}{2} m_2 N^2 \omega^2 r_2^2 + \tfrac{1}{2} k_{2,2} r_2^2$	$\bar{E}(3) = \bar{E}(r_1) + \bar{E}(r_2),$ $\bar{E}(r_1) = \tfrac{1}{2} m_3 \omega^2 r_1^2 + \tfrac{1}{2} k_{3,1} r_1^2,$ $\bar{E}(r_2) \approx \tfrac{1}{2} m_3 N^2 \omega^2 r_2^2 + \tfrac{1}{2} k_{3,2} r_2^2$
$\mu_z(2) = \dfrac{G^{1/2}\hbar}{2c}\left(1 + \tfrac{1}{2}\dfrac{r_2^2}{r_1^2}\right), \quad \dfrac{r_2^2}{r_1^2} = 9.5, \ \dfrac{r_2}{r_1} = 3.1$	$\mu_z(3) = \dfrac{G^{1/2}\hbar}{2c}\left(1 + \tfrac{1}{2}\dfrac{r_2^2}{r_1^2}\right), \quad \dfrac{r_2^2}{r_1^2} = 64, \ \dfrac{r_2}{r_1} = 8$
$r_1(2) = g(2)\dfrac{\hbar}{m_2 c}, \quad g(2) = \sqrt{1 + (N^2 + \tfrac{1}{2})\dfrac{r_2^2}{r_1^2}} = 18$	$r_1(3) = g(3)\dfrac{\hbar}{m_3 c}, g(3) = \sqrt{1 + (N^2 + \tfrac{1}{2})\dfrac{r_2^2}{r_1^2}} = 264$
$\omega = \dfrac{m_2 c^2}{g(2)^2 \hbar} = \dfrac{5.75 m_1 c^2}{g(2)^2 \hbar} = 0.018\dfrac{m_1 c^2}{\hbar}$	$\omega = \dfrac{m_3 c^2}{g(3)^2 \hbar} = \dfrac{33 m_1 c^2}{g(3)^2 \hbar} = 0.00047\dfrac{m_1 c^2}{\hbar}$
$k_{2,1} = m_2 \omega^2 = \dfrac{5.75^3 m_1^3 c^4}{g(2)^4 \hbar^2} = 0.0019\dfrac{m_1^3 c^4}{\hbar^2},$ $k_{2,2} = (5.7^2 + \tfrac{1}{2})\dfrac{5.75^3 m_1^3 c^4}{g(2)^4 \hbar^2} = 0.063\dfrac{m_1^3 c^4}{\hbar^2}$	$k_{3,1} = m_3 \omega^2 = \dfrac{33^3 m_1^3 c^4}{g(3)^4 \hbar^2} = 0.0000074\dfrac{m_1^3 c^4}{\hbar^2},$ $k_{3,2} = (33^2 + \tfrac{1}{2})\dfrac{33^3 m_1^3 c^4}{g(3)^4 \hbar^2} = 0.0081\dfrac{m_1^3 c^4}{\hbar^2}$

Calculation yields the following spring constants $k_{l,1}$ and $k_{l,2}$ (see (3.9) and table 1)

$$k_{l,1} = \frac{r_1^2 + r_2^2}{r_1^2 + \frac{1}{2} r_2^2} m_l \omega^2 \approx \frac{m_l^3 c^4}{\hbar^2}, \quad k_{l,2} = m_l \omega^2 \approx \frac{m_l^3 c^4}{\hbar^2}. \tag{6.4}$$

Since the ratio r_2/r_1 is much smaller than unity value for charged leptons, the value of $k_{l,1}$ is approximately equal to the spring constant $k_{l,2}$ and to the spring constant k_l of the *ring model* of (1.12). See also (3.11) and added comment for the value of the ratio $r_2/r_1 = (\alpha/\pi)^{1/2} = 0.04820$ for all charged leptons, where α is the fine-structure constant.

In section 4 it is shown that the *ring torus model* for charged leptons can also be applied to neutrino 1. The spring constants $k_{1,1}$ and $k_{1,2}$ for neutrino 1 are analogous to that of (3.9). In addition, in ref. [3] a value $r_2/r_1 = (\alpha_W/\pi)^{1/2} = 0.10$ is proposed for the ratio r_2/r_1 of neutrino 1, where α_W is the electroweak coupling constant at low energy. As a consequence, the spring constants $k_{1,1}$ and $k_{1,2}$ for neutrino 1 also nearly coincide to the spring constant of the *ring model* given in (1.12).

For neutrinos 2 and 3 the limiting case $N > 1$ and $r_2 > r_1$ applies. This limiting case leads to the *spindle torus model* and is also discussed in section 4. For neutrino 3 the spindle torus is illustrated in figure 2. Note that the surface of spindle torus approaches the surface of a sphere. Results for the spindle torus model are also summarized in table 1. Instead of (3.8), the approximated value of the time averaged energy $\bar{E}(i)$ can be written as a combination of (4.8) and (4.10)

$$\bar{E}(i) = \bar{E}(r_1) + \bar{E}(r_2) \approx (\tfrac{1}{2} m_i \omega^2 r_1^2 + \tfrac{1}{2} k_{i,1} r_1^2) + (\tfrac{1}{2} m_i N^2 \omega^2 r_2^2 + \tfrac{1}{2} k_{i,2} r_2^2). \quad i = 2,3 \tag{6.5}$$

When $Nr_2 >> r_1$, the following spring constants are shown in (4.12)

$$k_{i,1} = m_i \omega^2 = \frac{m_i^3 c^4}{g(i)^4 \hbar^2}, \quad k_{i,2} = N^2 m_i \omega^2 = N^2 \frac{m_i^3 c^4}{g(i)^4 \hbar^2}. \tag{6.6}$$

In table 1 the values of the spring constants $k_{2,1}$ and $k_{2,2}$ for neutrino 2 and $k_{3,1}$ and $k_{3,2}$ for neutrino 3 are given. See also (4.13) for $k_{2,2}$ and $k_{3,2}$. It is noted that the spring constants k_i for the *spindle torus model* are much smaller than the reference value $m_1^3 c^4/\hbar^2$ for neutrino 1. So, spring constants are found for all charged leptons $l = e, \mu, \tau$ and all neutrinos $i = 1, 2, 3$. These results illustrate that Hooke's classical force law may be a key element in the explanation of the internal structure of leptons.

In section 5 the magnitude of the electromagnetic energy and the elastic energy of the electron are compared. It appears that the elastic energy based on Hooke's law may make a substantial contribution to the self-energy of the electron.

Summing up, in this work Parson's *ring model* and the toroidal model of leptons [3, 4] are combined. The latter model postulates the same basic relations (3.1) for all leptons. The simplified basic equations of (1.1) leads to Parson's *ring model* for all charged leptons and neutrino 1, but this approach is not applicable to neutrino 2 and 3. Utilizing different combinations of orthogonal harmonic oscillators, the calculated spring constants of all charged leptons and neutrino 1 look alike. For neutrinos 2 and 3, however, relatively small spring constants are obtained compared to that of neutrino 1. A summary of all expressions of lepton energies, spring constants and magnetic moments are given in table 1.

Furthermore, the lepton torus model predicts the magnetic dipole moments for all charged leptons, first order anomalous corrections included. In addition, theoretical expressions for the magnetic dipole moments are obtained for all neutrinos (first order correction for neutrino 1 also included), although the predicted magnetic dipole moments for the neutrinos have not yet been observed.

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