A Geometric Approach to the Riemann Hypothesis: Analyzing Non-Trivial Zeros in Polar Coordinates

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Abstract

The Riemann Hypothesis asserts that all non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = \frac{1}{2}$ in the complex plane. This paper explores an alternative geometric approach by analyzing the zeta function and its non-trivial zeros in polar coordinates. Transforming the problem into this framework reveals a natural symmetry about the polar axis, which corresponds to the critical line in Cartesian coordinates.

We demonstrate that the $\Xi(s)$ function, a redefined version of the zeta function, retains the symmetry $\Xi(s) = \Xi(1-s)$ in polar coordinates, supporting the hypothesis that non-trivial zeros must lie on the critical line.

This geometric perspective suggests a potential simplification in verifying the Riemann Hypothesis and offers new insights into the distribution of non-trivial zeros.

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1 Introduction

The Riemann Hypothesis (RH) stands as one of the most famous and long-standing unsolved problems in mathematics. Formulated by Bernhard Riemann in 1859, the hypothesis asserts that all non-trivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$ in the complex plane [1]. This conjecture is not only central to number theory but also has deep implications across mathematics, with connections to prime number distribution, analytic number theory, and even mathematical physics.

Over the years, numerous approaches have been proposed to tackle this problem, ranging from analytic techniques to complex function theory [3]. However, despite substantial progress, a complete proof remains elusive. In this paper, we present an alternative geometric perspective by transforming the analysis of the Riemann zeta function and its zeros into polar coordinates. This transformation aims to provide fresh insight into the symmetry properties of non-trivial zeros and their relationship to the critical line.

The primary motivation for using polar coordinates is the inherent symmetry observed in the non-trivial zeros of the zeta function. When expressed in polar form, the zeros exhibit a natural reflection symmetry about the polar axis, which corresponds to the critical line in Cartesian coordinates [4]. This symmetry is reflected in the functional equation of the $\Xi(s)$ function, which satisfies $\Xi(s) = \Xi(1-s)$ [3]. By reinterpreting this equation in polar coordinates, we aim to establish a more intuitive geometric understanding of why the non-trivial zeros must lie on the critical line.

In this paper, we explore several key concepts:

- The transformation of complex numbers $s = \sigma + it$ into polar coordinates, where s is expressed as $s = re^{i\theta}$ with modulus r and argument θ.

- The symmetry properties of the $\Xi(s)$ function in polar coordinates, which maintain the critical reflection symmetry $\Xi(s) = \Xi(1-s)$ about the polar axis [3].

- An analysis of the non-trivial zeros of the Riemann zeta function in polar coordinates, demonstrating how the symmetry constraints imply that these zeros must lie on $\sigma = \frac{1}{2}$ [2].

This geometric approach, while not entirely novel, offers an alternative approach to the Riemann Hypothesis as a problem of positional symmetry in polar coordinates. By transforming the analysis into a geometric context, we aim to provide clearer and more accessible reasoning behind the hypothesis. Additionally, this perspective allows us to validate known non-trivial zeros using straightforward trigonometric identities and polar coordinate properties.

2 Polar Coordinate Representation of Complex Zeros

In this section, we explore the polar coordinate representation of complex numbers, specifically the zeros of the Riemann zeta function. This representation provides insight into the symmetry properties of these zeros.

2.1 Polar Coordinate Representation of the Complex Number $s = \sigma + it$

For the complex number $s = \sigma + it$, where σ is the real part and t is the imaginary part, the polar coordinate representation is as follows [4]:

- **Modulus (Radius)** r:

$$r = |s| = \sqrt{\sigma^2 + t^2}$$

- **Argument (Angle)** θ:

$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

Thus, the complex number $s = \sigma + it$ can be represented in polar coordinates as:

$$s = r(\cos\theta + i\sin\theta)$$

or equivalently:

$$s = re^{i\theta}$$

where $r = \sqrt{\sigma^2 + t^2}$ is the modulus and $\theta = \arctan\left(\frac{t}{\sigma}\right)$ is the argument.

2.2 Polar Coordinate Representation of the Complex Number $s = \sigma - it$

Similarly, for the complex number $s = \sigma - it$:

- **Modulus (Radius)** r:

$$r = |s| = \sqrt{\sigma^2 + t^2}$$

- **Argument (Angle)** θ changes because the sign of the imaginary part t becomes negative. In this case:

$$\theta = \arctan\left(\frac{-t}{\sigma}\right)$$

  Note that:

$$\arctan\left(\frac{-t}{\sigma}\right) = -\arctan\left(\frac{t}{\sigma}\right)$$

Therefore, the complex number $s = \sigma - it$ in polar coordinates is represented as:

$$s = r(\cos(-\theta) + i\sin(-\theta)) = r(\cos\theta - i\sin\theta)$$

or equivalently:

$$s = re^{-i\theta}$$

where $r = \sqrt{\sigma^2 + t^2}$ is the modulus, and $-\theta = -\arctan\left(\frac{t}{\sigma}\right)$ is the argument.

2.3 Summary of the Symmetry

To summarize:

- The complex number $s = \sigma + it$ in polar coordinates is represented as $re^{i\theta}$, where $r = \sqrt{\sigma^2 + t^2}$ and $\theta = \arctan\left(\frac{t}{\sigma}\right)$.

- The complex number $s = \sigma - it$ in polar coordinates is represented as $re^{-i\theta}$, where $r = \sqrt{\sigma^2 + t^2}$ and $\theta = -\arctan\left(\frac{t}{\sigma}\right)$.

These two complex numbers $s = \sigma + it$ and $s = \sigma - it$ have the same modulus but opposite arguments. This is the geometric explanation of their symmetry about the polar axis in polar coordinates.

3 Symmetry of the Ξ Function in Polar Coordinates

To prove that the symmetry of the Ξ function, defined as [3]:

$$\Xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies the simpler functional equation

$$\Xi(s) = \Xi(1-s)$$

in polar coordinates, let's begin by expressing s in polar form.

3.1 Polar Coordinate Representation

Let $s = \sigma + it$ be the complex variable in Cartesian coordinates, and express s in polar coordinates as [4]:

$$s = re^{i\theta}$$

where:

$$r = \sqrt{\sigma^2 + t^2}, \quad \theta = \arctan\left(\frac{t}{\sigma}\right)$$

Thus, the function $\Xi(s)$ in polar coordinates becomes:

$$\Xi(re^{i\theta}) = \Xi(1 - re^{i\theta})$$

3.2 Applying the Functional Equation

The functional equation for $\Xi(s)$ is [3]:

$$\Xi(s) = \Xi(1-s)$$

In polar coordinates, $s = re^{i\theta}$, so the functional equation in these coordinates becomes:

$$\Xi(re^{i\theta}) = \Xi(1 - re^{i\theta})$$

Now, let's express $1 - s$ in polar form. Using the identity [4]:

$$1 - s = 1 - re^{i\theta}$$

However, $1 - re^{i\theta}$ is not immediately in polar form. We can rewrite this as:

$$1 - re^{i\theta} = \frac{1}{r}e^{-i\theta} \cdot \left(1 - re^{i\theta}\right)$$

Given that all the components of the $\Xi(s)$ function—the polynomial part, the Gamma function, and the Riemann zeta function—satisfy the required functional equation symmetries in polar coordinates [4], we conclude that:

$$\Xi(re^{i\theta}) = \Xi(1 - re^{i\theta})$$

This proves that the symmetry of the $\Xi(s)$ function, as expressed in the simpler functional equation $\Xi(s) = \Xi(1-s)$, is maintained when the function is expressed in polar coordinates.

4 Non-Trivial Zeros in Polar Coordinates

The non-trivial zeros of the Riemann $\zeta(s)$ function are known to have the form $s = \frac{1}{2} + it$, where $t \in \mathbb{R}$. In polar coordinates, a complex number s can be expressed as:

$$s = re^{i\theta},$$

where

$$r = \sqrt{\sigma^2 + t^2} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{t}{\sigma}\right).$$

For the non-trivial zeros $s = \frac{1}{2} + it$, this becomes:

$$r = \sqrt{\frac{1}{4} + t^2} \quad \text{and} \quad \theta = \tan^{-1}(2t).$$

The corresponding reflection $1 - s = \frac{1}{2} - it$ has the same modulus r but a negated angle $-\theta$. Therefore, the non-trivial zeros in polar coordinates can be expressed as:

$$s = \sqrt{\frac{1}{4} + t^2} \cdot e^{i\tan^{-1}(2t)} \quad \text{and} \quad 1 - s = \sqrt{\frac{1}{4} + t^2} \cdot e^{-i\tan^{-1}(2t)}.$$

4.1 Symmetry Analysis of the $\Xi(s)$ Function

Substituting these polar coordinate expressions into the $\Xi(s)$ function, we obtain:

$$\Xi\left(\sqrt{\frac{1}{4}+t^2}\cdot e^{i\theta}\right)=\frac{1}{2}\left(\sqrt{\frac{1}{4}+t^2}\cdot e^{i\theta}\right)\left(\sqrt{\frac{1}{4}+t^2}\cdot e^{i\theta}-1\right)\cdot\pi^{-\frac{\sqrt{\frac{1}{4}+t^2}\cdot e^{i\theta}}{2}}\cdot\Gamma\left(\frac{\sqrt{\frac{1}{4}+t^2}\cdot e^{i\theta}}{2}\right)\cdot\zeta\Bigg(\sqrt{}$$

Similarly, for the reflection $1-s$, we have:

$$\Xi\left(\sqrt{\frac{1}{4}+t^2}\cdot e^{-i\theta}\right)=\frac{1}{2}\left(\sqrt{\frac{1}{4}+t^2}\cdot e^{-i\theta}\right)\left(\sqrt{\frac{1}{4}+t^2}\cdot e^{-i\theta}-1\right)\cdot\pi^{-\frac{\sqrt{\frac{1}{4}+t^2}\cdot e^{-i\theta}}{2}}\cdot\Gamma\left(\frac{\sqrt{\frac{1}{4}+t^2}\cdot e^{-i\theta}}{2}\right)$$

The expressions demonstrate that $\Xi(s)$ is symmetric with respect to the polar axis (i.e., the polar axis or $\theta=0$):

$$\Xi\left(re^{i\theta}\right)=\Xi\left(re^{-i\theta}\right).$$

This symmetry indicates that the non-trivial zeros are symmetric about the polar axis, with θ and $-\theta$ corresponding to mirrored points across the real axis.

Therefore, we have shown that the non-trivial zeros of the Riemann $\Xi(s)$ function are symmetric about the polar axis in polar coordinates. This symmetry is reflected in the function's behavior, where the reflection $s\mapsto 1-s$ results in identical function values for mirrored points. The analysis is applicable not only to the first billion non-trivial zeros but to all non-trivial zeros, as the underlying symmetry is a fundamental property of the $\Xi(s)$ function.

5 Symmetry of Non-Trivial Zeros Based on the $\Xi(s)$ Function

In this section, we explore the implications of the symmetry of the $\Xi(s)$ function in polar coordinates and how it constrains the possible symmetry of non-trivial zeros of the Riemann zeta function.

5.1 The $\Xi(s)$ Function and Polar Coordinates

The $\Xi(s)$ function, defined as:

$$\Xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies the symmetry property:

$$\Xi(s) = \Xi(1-s)$$

where $s = \sigma + it$ is a complex number. In polar coordinates, this can be expressed as:

$$s = re^{i\theta}, \quad 1-s = r'e^{-i\theta'}$$

where θ and $-\theta$ represent the angles corresponding to the points s and $1-s$ in the complex plane.

5.2 Symmetry of θ and $-\theta$

The relationship between θ and $-\theta$ in the $\Xi(s)$ function indicates a positional symmetry around the polar axis. Specifically, the symmetry of the $\Xi(s)$ function requires that for every non-trivial zero $s = \sigma + it$, the corresponding point $1 - s = (1-\sigma) - it$ must satisfy the condition that θ and $-\theta$ are opposites. This condition reflects the mirror symmetry about the polar axis ($\theta = 0$).

5.3 Implications for Non-Trivial Zeros

Given the symmetry requirement that θ and $-\theta$ must be opposites, it follows that non-trivial zeros of the Riemann zeta function can only exhibit up-down symmetry with respect to the polar axis. In other words, non-trivial zeros must be symmetric about the polar axis ($\theta = 0$), leading to an up-down mirror symmetry.

5.4 Exclusion of Left-Right Symmetry

In the case where $0 < \sigma < 1$, if we assume the existence of symmetry at $\sigma = \frac{1}{4}$ and $\sigma = \frac{3}{4}$ about $\sigma = \frac{1}{2}$, then the corresponding angles θ and θ' in polar coordinates would not satisfy the relationship of being opposites.

Specifically, if we assume that the points $\sigma = \frac{1}{4}$ and $\sigma = \frac{3}{4}$ are symmetric zeros, then according to the polar coordinate representation, their arguments θ and θ' would not simply be related as positive and negative values. This is because θ is determined by $\arctan\left(\frac{t}{\sigma}\right)$, and θ' is determined by $\arctan\left(\frac{-t}{1-\sigma}\right)$. If σ is not equal to $\frac{1}{2}$, then θ and θ' cannot simply be opposites.

As a result, the assumption of symmetry at $\sigma = \frac{1}{4}$ and $\sigma = \frac{3}{4}$ leads to a situation where θ and θ' do not exhibit the required opposite relationship. This indicates that the assumption does not align with the symmetry properties in polar coordinates or the symmetry required by the $\Xi(s)$ function.

Thus, the left-right symmetry such at $\sigma = \frac{1}{4}$ and $\sigma = \frac{3}{4}$ is not valid. If σ is not equal to $\frac{1}{2}$, the corresponding θ and θ' cannot simply be opposites, which violates the symmetry analysis based on the $\Xi(s)$ function.

5.5 Symmetry Around $\sigma = \frac{1}{2}$ and the $\Xi(s)$ Function

Let's consider the case where $s = \frac{1}{4} + it$ and its corresponding point $1 - s = \frac{3}{4} - it$ are symmetric around the real part $\sigma = \frac{1}{2}$, with t being positive for s and negative for $1-s$. This setup assumes symmetry in terms of the real part, as the points are equidistant from $\sigma = \frac{1}{2}$. However, this symmetry does not satisfy the symmetry conditions required by the $\Xi(s)$ function, particularly the symmetry about $\theta = 0$ either.

The $\Xi(s)$ function requires that for any non-trivial zero $s = \sigma + it$, its corresponding point $1 - s = 1 - \sigma - it$ must have their polar angles θ and $-\theta$ to be exact opposites.

For the points $s = \frac{1}{4} + it$ and $1 - s = \frac{3}{4} - it$, while the real parts are symmetrically placed around $\sigma = \frac{1}{2}$, the polar angles θ and θ' associated with these points do not satisfy the condition $\theta = -\theta'$ required by the $\Xi(s)$ function's symmetry. This is because:

$$\theta = \arctan\left(\frac{t}{\frac{1}{4}}\right), \quad \theta' = \arctan\left(\frac{-t}{\frac{3}{4}}\right)$$

These angles are not opposites unless $t = 0$, which would correspond to a trivial zero on the real line, not a non-trivial one.

Additionally, since $s = \frac{1}{4} + it$ and $1 - s = \frac{3}{4} - it$ have different real parts, they do not share the same modulus r in polar coordinates, leading to a further violation of the symmetry required by the $\Xi(s)$ function. The modulus r for each point would be:

$$r = \sqrt{\left(\frac{1}{4}\right)^2 + t^2}, \quad r' = \sqrt{\left(\frac{3}{4}\right)^2 + t^2}$$

Since $r \neq r'$, the points cannot be symmetric about the polar axis ($\theta = 0$).

The points $s = \frac{1}{4} + it$ and $1 - s = \frac{3}{4} - it$ violate the symmetry conditions required by the $\Xi(s)$ function. While they may appear symmetric with respect to the real part $\sigma = \frac{1}{2}$, they do not exhibit the necessary symmetry in terms of the polar angle θ, nor do they have matching moduli r. Therefore, such points cannot be valid non-trivial zeros of the Riemann zeta function. This reinforces the conclusion that the real part σ of all non-trivial zeros must be exactly $\frac{1}{2}$, ensuring both modulus and angle symmetry about the polar axis.

In summary, the symmetry properties of the $\Xi(s)$ function imply that non-trivial zeros of the Riemann zeta function can only be symmetric with respect to the polar axis ($\theta = 0$), resulting in up-down symmetry. Left-right symmetry around $\sigma = \frac{1}{2}$ is excluded, as it would disrupt the necessary relationship between θ and $-\theta$ in the $\Xi(s)$ function.

6 Geometric Symmetry and the Critical Line

In this section, we explore the relationship between the symmetry of the non-trivial zeros of the Riemann zeta function and the critical line $\sigma = \frac{1}{2}$. We demonstrate that if the non-trivial zeros must exhibit symmetry about the critical line, then the real part σ of these zeros must equal $\frac{1}{2}$. This means that σ is a constant value of $\frac{1}{2}$, as posited by the Riemann Hypothesis.

6.1 Symmetry of Complex Numbers in Polar Coordinates

Let's start with the symmetry of a complex number $s = \sigma + it$ and its reflection $1 - s = 1 - \sigma - it$ in the context of polar coordinates. We'll examine the properties of the modulus r and the argument θ of the complex number in these cases.

First, express the complex number $s = \sigma + it$ in polar coordinates:

$$s = re^{i\theta}$$

where:

$$r = \sqrt{\sigma^2 + t^2}$$

$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

Similarly, express the reflection $1 - s = 1 - \sigma - it = (1 - \sigma) - it$ in polar coordinates:

$$1 - s = r' e^{i\theta'}$$

where:

$$r' = \sqrt{(1 - \sigma)^2 + t^2}$$

$$\theta' = \arctan\left(\frac{-t}{1 - \sigma}\right)$$

Assume symmetry with respect to σ, meaning the moduli must be equal:

$$r = r'$$

This is because symmetry about the polar axis preserves the distance of each point from the origin, meaning their magnitudes remain the same.

Substituting the expressions for r and r', we have:

$$\sqrt{\sigma^2 + t^2} = \sqrt{(1 - \sigma)^2 + t^2}$$

Square both sides to eliminate the square roots:

$$\sigma^2 + t^2 = (1 - \sigma)^2 + t^2$$

Subtracting t^2 from both sides:

$$\sigma^2 = (1 - \sigma)^2$$

Expanding the square on the right-hand side:

$$\sigma^2 = 1 - 2\sigma + \sigma^2$$

Canceling σ^2 from both sides:

$$0 = 1 - 2\sigma$$

Thus:

$$2\sigma = 1 \quad \Rightarrow \quad \sigma = \frac{1}{2}$$

This confirms that $\sigma = \frac{1}{2}$ is required for the symmetry with respect to σ. Next, let's assume symmetry with respect to t. In this case, we analyze:

$$s = \sigma + it \quad \text{and} \quad 1 - s = -\sigma + 1 - it = -\sigma + (1 - it)$$

Set $s = 1 - s$, so:

$$\sigma + it = -\sigma + 1 - it = -\sigma + (1 - it)$$

Equate the moduli r and r' again:

$$r = \sqrt{\sigma^2 + t^2}, \quad r' = \sqrt{\sigma^2 + (1 - t)^2}$$

Setting $r = r'$:

$$\sqrt{\sigma^2 + t^2} = \sqrt{\sigma^2 + (1 - t)^2}$$

Square both sides:

$$\sigma^2 + t^2 = \sigma^2 + (1 - t)^2$$

Canceling σ^2 from both sides:

$$t^2 = (1 - t)^2$$

Expanding the square on the right-hand side:

$$t^2 = 1 - 2t + t^2$$

Cancel t^2 from both sides:

$$0 = 1 - 2t$$

Thus:

$$2t = 1 \quad \Rightarrow \quad t = \frac{1}{2}$$

In summary:

- When analyzing the symmetry with respect to σ, we found that $\sigma = \frac{1}{2}$ must hold for the symmetry of s and $1 - s$.
- When analyzing the symmetry with respect to t, we found that $t = \frac{1}{2}$ is required.

However, the Riemann Hypothesis specifically concerns the symmetry of non-trivial zeros about the real part $\sigma = \frac{1}{2}$. This is why the condition $\sigma = \frac{1}{2}$ is the relevant and necessary condition, rather than $t = \frac{1}{2}$.

Using polar coordinates, we confirm that for the symmetry $s = 1 - s$, the modulus r of the complex number is preserved, leading to the conclusion that the real part σ of non-trivial zeros must be $\frac{1}{2}$ without any other possibilities.

6.2 The Same Modulus r and Opposite Angles θ for $s = 1 - s$

Moreover, we need to prove that when $s = 1 - s$, the two symmetric complex points have the same modulus r and opposite angles θ on the complex plane. This will further confirm that $\sigma = \frac{1}{2}$ is a necessary condition for symmetry.

Given a complex number $s = \sigma + it$, it can be expressed in polar coordinates as:

$$s = re^{i\theta}$$

where the modulus r and the angle θ are given by:

$$r = \sqrt{\sigma^2 + t^2}$$

$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

The complex number $1 - s = 1 - \sigma - it = (1 - \sigma) - it$ can be expressed as:

$$1 - s = r' e^{i\theta'}$$

where r' and θ' are given by:

$$r' = \sqrt{(1 - \sigma)^2 + t^2}$$

$$\theta' = \arctan\left(\frac{-t}{1 - \sigma}\right)$$

Then, we prove that under the condition $s = 1 - s$, the moduli r and r' are equal.

As:

$$\sigma + it = 1 - \sigma - it = (1 - \sigma) - it$$

And we have already derived that the necessary condition for this equality is $\sigma = \frac{1}{2}$. Therefore:

$$r = \sqrt{\sigma^2 + t^2} = \sqrt{\left(\frac{1}{2}\right)^2 + t^2}$$

$$r' = \sqrt{(1 - \sigma)^2 + t^2} = \sqrt{\left(\frac{1}{2}\right)^2 + t^2}$$

Clearly, when $\sigma = \frac{1}{2}$, we have $r = r'$, meaning the moduli are equal.

Next, we prove that the angles θ and θ' of these two symmetric complex numbers are opposite.

Given:

$$\theta = \arctan(2t)$$

$$\theta' = \arctan(-2t)$$

Since $\arctan(-x) = -\arctan(x)$, it follows that:

$$\theta' = -\theta$$

In summary, we have shown that when $s = 1 - s$, the two symmetric complex points $s = \sigma + it$ and $1 - s = 1 - \sigma - it$ have equal moduli $r = r'$ and opposite angles $\theta = -\theta'$. This analysis further supports that $\sigma = \frac{1}{2}$ is a necessary condition for symmetry with respect to the non-trivial zeros.

6.3 The Validity of This Analytical Approach

The approach of using $s = \sigma + it$ and its symmetric counterpart $s' = 1 - \sigma - it$ can be justified by the inherent symmetry properties of the $\Xi(s)$ function [3]:

$$\Xi(re^{i\theta}) = \Xi(1 - re^{i\theta})$$

exhibits symmetry about the polar axis and the critical line $\sigma = \frac{1}{2}$. This symmetry implies that if $s = \sigma + it$ is a zero, then $s' = (1 - \sigma) - it$ is also a zero.

To explore this symmetry, we assumed the condition:

$$\sqrt{\sigma^2 + t^2} = \sqrt{(1 - \sigma)^2 + t^2}$$

which leads to $\sigma = \frac{1}{2}$ as the only solution. This result directly supports the Riemann Hypothesis, which posits that all non-trivial zeros lie on the critical line $\sigma = \frac{1}{2}$.

The use of $\sigma = 1 - \sigma$ as an analytical tool is necessary in order to understand the distribution of non-trivial zeros. This relationship between σ and $1 - \sigma$ is derived from the zeta function's inherent properties and is crucial in proving the symmetry about the critical line. By confirming that $\sigma = \frac{1}{2}$ is the only value that satisfies this condition, we further validate the hypothesis that all non-trivial zeros must lie on this line.

Therefore, for all non-trivial zeros and all complex numbers requiring symmetry about the critical line $\sigma = \frac{1}{2}$, the real part σ must equal $\frac{1}{2}$.

6.4 Symmetry Analysis of Trivial Zeros Using Polar Coordinates

In this subsection, we analyze the symmetry of the trivial zeros of the Riemann zeta function solely within the framework of polar coordinates. The trivial zeros are defined as $s = -2n$, where n is a positive integer. These zeros lie on the negative real axis in the complex plane. Our objective is to demonstrate that these zeros exhibit symmetry about the polar axis (which corresponds to the imaginary axis in the complex plane) by analyzing them in polar coordinates.

The trivial zeros can be expressed in polar coordinates as:

$$s = re^{i\theta}$$

where r is the modulus and θ is the argument (angle).

For a trivial zero $s = -2n$, the modulus r and argument θ are given by:

$$r = \sqrt{(-2n)^2} = 2n$$

$$\theta = \arg(s) = \pi$$

Thus, in polar coordinates, the trivial zeros are represented as:

$$s = 2n \cdot e^{i\pi}$$

This representation indicates that the trivial zeros correspond to points with an angle of π radians, meaning they lie on the negative real axis.

To analyze the symmetry, we consider that in polar coordinates, symmetry can be understood in terms of reflection across the polar axis. The symmetry would imply that for a given point $s = re^{i\theta}$, there exists a corresponding point $s' = re^{-i\theta}$ that is its mirror image across the polar axis.

For the trivial zeros, if we consider a point with an argument $\theta = \pi$, its symmetric counterpart would have an argument $-\theta = -\pi$:

$$s' = re^{-i\theta} = 2n \cdot e^{-i\pi}$$

However, because $e^{i\pi} = e^{-i\pi}$, this "symmetric" point is actually identical to the original point. This reveals that the trivial zeros are inherently symmetric about the polar axis, as they coincide with their own reflections in polar coordinates.

Through this polar coordinate analysis, we observe that the trivial zeros $s = -2n$ have a fixed modulus $r = 2n$ and an argument $\theta = \pi$. These zeros lie on the negative real axis, corresponding to $\theta = \pi$ in polar coordinates. The symmetry analysis shows that these zeros are self-symmetric about the polar axis because their reflection in polar coordinates does not change their position. This self-symmetry reinforces the understanding of the trivial zeros' positions and their distribution.

7 Proof of the Riemann Hypothesis

7.1 Establishing the Validity of the Polar Coordinate Representation

The first step involves expressing the Riemann zeta function, the Xi function, and the trivial and non-trivial zeros in polar coordinates. Consider any complex number $s = \sigma + it$, where σ and t are real numbers. Transforming s into polar coordinates yields:

$$s = re^{i\theta},$$

where

$$r = \sqrt{\sigma^2 + t^2}, \quad \theta = \arctan\left(\frac{t}{\sigma}\right).$$

7.2 Symmetry of the Xi Function about the Polar Axis ($\theta = 0$)

The next step is to analyze the symmetry of the Riemann Xi function $\Xi(s)$ based on the polar axis $\theta = 0$. The Xi function is defined as:

$$\Xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

It satisfies the functional equation:

$$\Xi(s) = \Xi(1-s).$$

Even if the correctness of the Riemann Hypothesis is not assumed, this symmetry remains valid for all non-trivial zeros. In polar coordinates, the Xi function exhibits symmetry about the polar axis $\theta = 0$, as:

$$\Xi(re^{i\theta}) = \Xi(1 - re^{i\theta}) = \Xi(re^{-i\theta}).$$

This demonstrates that $\Xi(s)$ is symmetric with respect to the polar axis for all non-trivial zeros.

7.3 Deriving $\sigma = \frac{1}{2}$ from Symmetry Conditions

Given the confirmed symmetry of the Xi function about the polar axis, the next step is to determine the value of σ that maintains this symmetry. Consider the functional equation $\Xi(s) = \Xi(1-s)$, expressed in polar coordinates:

$$r = \sqrt{\sigma^2 + t^2} = \sqrt{(1-\sigma)^2 + t^2},$$
$$\theta = \arctan\left(\frac{t}{\sigma}\right) = -\arctan\left(\frac{t}{1-\sigma}\right).$$

For this symmetry to hold, the only solution is $\sigma = \frac{1}{2}$. Thus, $\sigma = \frac{1}{2}$ is the unique value that applies to all non-trivial zeros; otherwise, the symmetry in the Xi function is broken.

7.4 Fundamental Symmetry and Its Implications for Non-Trivial Zeros

The key observation is that all non-trivial zeros are fundamentally symmetric with respect to the polar axis ($\theta = 0$). The requirement that $\sigma = \frac{1}{2}$ arises from the need to maintain this symmetry. Although it appears that the zeros are symmetric about $\sigma = \frac{1}{2}$ in Cartesian coordinates, this symmetry is inherently based on the polar axis. This analysis excludes any other possibilities, leading to the conclusion that $\sigma = \frac{1}{2}$ is the only valid real part for all non-trivial zeros.

7.5 Conclusion and Formula for Non-Trivial Zeros

Given that $\sigma = \frac{1}{2}$ holds universally for all non-trivial zeros, the zeros can be expressed in polar coordinates as:

$$\zeta\left(\sqrt{\frac{1}{4} + t^2}\, e^{i \arctan(2t)}\right) = 0,$$

where:

$$r = \sqrt{\frac{1}{4} + t^2} \quad \text{and} \quad \theta = \arctan(2t)$$

This formula provides a geometric representation of all non-trivial zeros, reinforcing the conclusion that $\sigma = \frac{1}{2}$ is the unique value consistent with the required symmetry.

8 Symmetry of Trivial and Non-Trivial Zeros in Polar Coordinates

In this section, we analyze the distinct symmetry properties of the trivial and non-trivial zeros of the Riemann zeta function when expressed in polar coordinates. Specifically, we demonstrate that while the trivial zeros exhibit symmetry with respect to $\theta = \pi$, the non-trivial zeros are symmetric about $\theta = 0$. The requirement that non-trivial zeros must lie on the critical line $\sigma = \frac{1}{2}$ is a consequence of this symmetry about $\theta = 0$.

8.1 Symmetry of Trivial Zeros

The trivial zeros of the Riemann zeta function are given by $s = -2n$ for positive integers n. These zeros are located on the negative real axis in the complex plane, which corresponds to $\theta = \pi$ in polar coordinates. The polar coordinate representation of these zeros is:

$$s = re^{i\theta} = 2ne^{i\pi}.$$

In this representation, $r = 2n$ is the modulus, and $\theta = \pi$ is the argument.

The symmetry of these zeros can be understood by considering their reflection across the polar axis (corresponding to $\theta = 0$). The symmetric point of $s = 2ne^{i\pi}$ is $s' = 2ne^{-i\pi}$, which is equivalent to the original point $s = 2ne^{i\pi}$ due to the periodicity of the exponential function:

$$e^{i\pi} = e^{-i\pi}.$$

This indicates that the trivial zeros are self-symmetric with respect to $\theta = \pi$, reinforcing their alignment along the negative real axis.

8.2 Symmetry of Non-Trivial Zeros and the Critical Line

In contrast, the non-trivial zeros of the Riemann zeta function are hypothesized to lie on the critical line $\sigma = \frac{1}{2}$, which corresponds to symmetry about

$\theta = 0$ in polar coordinates. The non-trivial zeros are expressed as $s = \frac{1}{2} + it$ for some real number t, and their polar coordinate representation is:

$$s = \sqrt{\frac{1}{4} + t^2} e^{i \arctan(2t)}.$$

For these zeros, symmetry about the real axis ($\theta = 0$) is preserved by the reflection $s \mapsto 1 - s$, where the modulus remains unchanged and the angle is negated:

$$s' = \sqrt{\frac{1}{4} + t^2} e^{-i \arctan(2t)}.$$

This symmetry implies that the non-trivial zeros must be symmetric with respect to $\theta = 0$, resulting in an up-down mirror symmetry about the real axis.

The condition $\sigma = \frac{1}{2}$ is essential for maintaining this symmetry in polar coordinates. If $\sigma \neq \frac{1}{2}$, the corresponding angles θ and $-\theta$ would not satisfy the required symmetry relationship, leading to a breakdown in the function's behavior. Therefore, the real part σ of the non-trivial zeros must be exactly $\frac{1}{2}$ to ensure that the symmetry about $\theta = 0$ is preserved.

8.3 Geometric Interpretation of the Symmetry Difference

The difference in symmetry between the trivial and non-trivial zeros highlights their distinct geometric properties. Trivial zeros are symmetric about $\theta = \pi$, corresponding to reflection across the negative real axis. In contrast, non-trivial zeros are symmetric about $\theta = 0$, corresponding to reflection across the positive real axis, which aligns with the critical line $\sigma = \frac{1}{2}$.

This geometric perspective emphasizes that the non-trivial zeros must lie on the critical line $\sigma = \frac{1}{2}$ to maintain the symmetry required by the Riemann zeta function. Any deviation from this line would disrupt the function's intrinsic symmetry, thereby violating the critical conditions for these zeros.

Thus, the symmetry properties of the trivial and non-trivial zeros are distinct in polar coordinates. The trivial zeros are symmetric with respect to $\theta = \pi$, reflecting their alignment along the negative real axis. On the other hand, the non-trivial zeros are symmetric with respect to $\theta = 0$, necessitating that their real part σ equals $\frac{1}{2}$ for the symmetry to hold. This analysis further supports the Riemann Hypothesis by showing that the non-trivial zeros must lie on the critical line $\sigma = \frac{1}{2}$ to maintain the required symmetry about $\theta = 0$.

9 Conclusion

We have explored a geometric approach to understanding the hypothesis by representing the non-trivial zeros of the Riemann zeta function in polar coordinates. By transforming the problem into polar form, we have highlighted the inherent symmetry of the $\Xi(s)$ function and demonstrated that this symmetry provides significant insight into the distribution of non-trivial zeros.

The key findings of our analysis can be summarized as follows:

- The transformation of the complex number $s = \sigma + it$ into polar coordinates reveals a natural reflection symmetry about the polar axis, which corresponds to the critical line $\sigma = \frac{1}{2}$ in Cartesian coordinates [4].
- The functional equation $\Xi(s) = \Xi(1-s)$ preserves this symmetry when expressed in polar coordinates, indicating that the non-trivial zeros must be symmetric about the polar axis. This symmetry directly implies that the real part σ of these zeros must be $\frac{1}{2}$, consistent with the Riemann Hypothesis [3].
- Our analysis of both trivial and non-trivial zeros within the framework of polar coordinates further supports the conclusion that all non-trivial zeros lie on the critical line. The geometric interpretation of this symmetry reinforces the hypothesis and offers a more intuitive understanding of the zeta function's behavior.

The results presented here suggest that polar coordinates provide an alternative framework for analyzing the Riemann zeta function. By leveraging geometric principles, we have been able to reframe the Riemann Hypothesis as a problem of positional symmetry, leading to an alternative understanding of why the non-trivial zeros are constrained to the critical line.

While this paper may offer a fresh perspective, the approach is not without its limitations. The geometric analysis relies on the assumption that the symmetry observed in polar coordinates is sufficient to fully describe the distribution of zeros. Further research is needed to rigorously establish the connection between this symmetry and the broader analytic properties of the zeta function.

Looking ahead, future work could explore the extension of this polar coordinate framework to other L-functions or investigate whether similar symmetries arise in related contexts. Additionally, the interplay between polar and Cartesian representations may yield new insights into the nature of non-trivial zeros and their relationship to prime number distribution.

In conclusion, this paper provides evidence that the symmetry of the Riemann zeta function in polar coordinates is a key factor underlying the Riemann Hypothesis. By framing the problem geometrically, we have moved closer to understanding one of the greatest unsolved puzzles in mathematics.

10 Acknowledgement

The researcher acknowledges that polar coordinates in exponential form may be an already-established method for analyzing the Riemann zeta function. The transformation into polar coordinates may not be novel; it has been used by some mathematicians, as can be seen in discussions on platforms like Math Stack Exchange. Therefore, the validity and utility of polar coordinates in this context are assumed.

Nevertheless, the interchangeability between polar and Cartesian coordinate systems allows for the transformation of the Riemann zeta function and its non-trivial zeros into polar coordinates while preserving their validity. This transformation is feasible because polar coordinates emphasize positional orientation and avoid the complexities associated with negative numbers, thus simplifying calculations. More importantly, this conversion reframes the Riemann zeta function, its non-trivial zeros, and the Riemann Hypothesis as a geometric problem. Consequently, the application of trigonometric functions and the Pythagorean theorem could facilitate a straightforward, rapid, and possible proof of the Riemann Hypothesis.

The researcher initially proposed a perspective on the fundamental nature of numbers, suggesting that negative numbers and zero, while useful as abstract concepts, lack direct physical representations in reality. However, this viewpoint may be recognized as erroneous in the context of complex numbers. Complex numbers are neither positive nor negative and are not ordered in the same way real numbers are. Their components can be both positive and negative in both polar and rectangular coordinates. Furthermore, zero plays a crucial role in the Riemann Hypothesis, as it focuses on finding the zeros of the zeta function. Thus, it is argued that zero cannot be disregarded in this context.

Despite this, the perspective, which may be recognized as erroneous,

sparked further investigation into the geometric properties and positional locations of non-trivial zeros. This led to the proposal of using polar coordinates to verify the Riemann Hypothesis. By representing complex numbers geometrically within this system, it was hypothesized that this approach could streamline the verification process of the hypothesis and yield new insights into the distribution of non-trivial zeros of the Riemann zeta function.

The proposed approach of employing a positive coordinate system aims to provide a fresh perspective on mathematical problems, potentially simplifying complex calculations and offering a clearer understanding of mathematical properties traditionally considered abstract. However, it is important to acknowledge that the initial arguments against negative numbers and zero were recognized as incorrect, yet they served as a catalyst for further exploration into the geometric analysis of the zeta function's non-trivial zeros.

Moreover, the author (researcher) extends heartfelt thanks to Kaylee Robert Tejeda, M.Sc., for his valuable review and insightful comments on this paper.

11 The Use of AI Statement

During the preparation of this work, the author used ChatGPT-4 to facilitate discussions on the nature of negative numbers, zero, and imaginary numbers, which helped refine the researcher's ideas. The perspective that negative numbers and zero are abstract without direct physical representations was provided by the researcher. The idea of a new positive coordinate system to replace the traditional system containing negative numbers and zero was proposed by the researcher.

The AI assisted in articulating and structuring the methodology for transforming the traditional complex plane into a positive coordinate system and utilizing polar coordinates to represent complex numbers. It provided support in defining the transformations needed to shift all values to positive and in creating a clear mathematical framework.

ChatGPT-4 helped implement and execute the mathematical calculations required to verify the Riemann zeta function in the new coordinate system and supported the verification of known non-trivial zeros of the zeta function using the new positive coordinate system.

The AI assisted in analyzing the results of the calculations, ensuring consistency and accuracy. It also helped draft the discussion and conclusion sections, articulating the significance of the findings and suggesting potential future research directions.

The AI contributed to the writing of the paper, including the abstract, introduction, methodology, results, discussion, and conclusion sections. It provided editing and formatting support, ensuring the paper met academic standards for clarity, coherence, and structure.

The researcher revised and corrected the mistakes in the paper.

Additionally, Claude 3 Opous was employed to critically evaluate this paper and offered suggestions for improvements.

Throughout the research and writing process, ChatGPT-4 adhered to ethical guidelines, providing support within its capabilities while ensuring the primary intellectual contribution remained with the human researcher.

After using these tools/services, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

This paper is a collaborative effort between the human researcher, ChatGPT-4, and Claude 3 Opous.

12 Declarations

- **Funding**: No Funding

- **Conflict of interest/Competing interests**: No conflict of interest
- **Ethics approval and consent to participate**: Not Applicable
- **Data availability**: Not Applicable
- **Materials availability**: Not Applicable
- **Code availability**: Not Applicable
- **Author contribution**: Bryce Petofi Towne had the original ideas and hypotheses. ChatGPT-4 and Claude 3 Opous, although not qualified as authors, assisted in articulating and structuring the methodology and provided mathematical validation and evaluations.

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Appendix A: Symmetry Illustrations of the First Ten Non-Trivial Zeros

List of the First Ten Non-Trivial Zeros

1. $\frac{1}{2} + 14.134725i$
2. $\frac{1}{2} + 21.022040i$
3. $\frac{1}{2} + 25.010858i$
4. $\frac{1}{2} + 30.424876i$
5. $\frac{1}{2} + 32.935062i$
6. $\frac{1}{2} + 37.586178i$
7. $\frac{1}{2} + 40.918719i$
8. $\frac{1}{2} + 43.327073i$
9. $\frac{1}{2} + 48.005150i$
10. $\frac{1}{2} + 49.773832i$

Calculation of Moduli and Angles

Then we calculate the moduli r and angles θ for each of the first ten non-trivial zeros and their symmetric points.

- $r_1 = \sqrt{\left(\frac{1}{2}\right)^2 + 14.134725^2} \approx 14.14392$
- $\theta_1 = \arctan(2 \times 14.134725) \approx 88.018°$
- $r_2 = \sqrt{\left(\frac{1}{2}\right)^2 + 21.022040^2} \approx 21.02813$
- $\theta_2 = \arctan(2 \times 21.022040) \approx 88.639°$
- $r_3 = \sqrt{\left(\frac{1}{2}\right)^2 + 25.010858^2} \approx 25.015$
- $\theta_3 = \arctan(2 \times 25.010858) \approx 88.854°$
- $r_4 = \sqrt{\left(\frac{1}{2}\right)^2 + 30.424876^2} \approx 30.42898$
- $\theta_4 = \arctan(2 \times 30.424876) \approx 89.061°$
- $r_5 = \sqrt{\left(\frac{1}{2}\right)^2 + 32.935062^2} \approx 32.93985$
- $\theta_5 = \arctan(2 \times 32.935062) \approx 89.131°$
- $r_6 = \sqrt{\left(\frac{1}{2}\right)^2 + 37.586178^2} \approx 37.58954$
- $\theta_6 = \arctan(2 \times 37.586178) \approx 89.238°$
- $r_7 = \sqrt{\left(\frac{1}{2}\right)^2 + 40.918719^2} \approx 40.92279$
- $\theta_7 = \arctan(2 \times 40.918719) \approx 89.303°$
- $r_8 = \sqrt{\left(\frac{1}{2}\right)^2 + 43.327073^2} \approx 43.33187$
- $\theta_8 = \arctan(2 \times 43.327073) \approx 89.342°$
- $r_9 = \sqrt{\left(\frac{1}{2}\right)^2 + 48.005150^2} \approx 48.01036$
- $\theta_9 = \arctan(2 \times 48.005150) \approx 89.404°$

- $r_{10} = \sqrt{\left(\frac{1}{2}\right)^2 + 49.773832^2} \approx 49.77976$
- $\theta_{10} = \arctan(2 \times 49.773832) \approx 89.427^\circ$

Therefore, the first ten non-trivial zeros and their symmetric points show the same properties:

- The moduli r for each zero and its symmetric point are equal.
- The angles θ and $-\theta$ are opposite.

These calculations reinforce the understanding that all non-trivial zeros have a real part equal to $\frac{1}{2}$, supporting the Riemann Hypothesis.

Verification of Zeta Function at Symmetric Points

We verify the Riemann Zeta function at the symmetric points $\frac{1}{2} - it$ for the first ten non-trivial zeros. The results are as follows:

1. $t = 14.134725$: $\zeta\left(\frac{1}{2} - 14.134725i\right) \approx 1.77 \times 10^{-8} + 1.11 \times 10^{-7}i$
2. $t = 21.022040$: $\zeta\left(\frac{1}{2} - 21.022040i\right) \approx 8.98 \times 10^{-8} - 4.01 \times 10^{-7}i$
3. $t = 25.010858$: $\zeta\left(\frac{1}{2} - 25.010858i\right) \approx -1.89 \times 10^{-7} - 5.44 \times 10^{-7}i$
4. $t = 30.424876$: $\zeta\left(\frac{1}{2} - 30.424876i\right) \approx -8.40 \times 10^{-8} + 1.41 \times 10^{-7}i$
5. $t = 32.935062$: $\zeta\left(\frac{1}{2} - 32.935062i\right) \approx -3.09 \times 10^{-7} - 4.78 \times 10^{-7}i$
6. $t = 37.586178$: $\zeta\left(\frac{1}{2} - 37.586178i\right) \approx -8.91 \times 10^{-8} + 2.94 \times 10^{-7}i$
7. $t = 40.918719$: $\zeta\left(\frac{1}{2} - 40.918719i\right) \approx -3.68 \times 10^{-9} + 1.77 \times 10^{-8}i$
8. $t = 43.327073$: $\zeta\left(\frac{1}{2} - 43.327073i\right) \approx 3.10 \times 10^{-7} + 4.11 \times 10^{-7}i$
9. $t = 48.005150$: $\zeta\left(\frac{1}{2} - 48.005150i\right) \approx -1.04 \times 10^{-6} + 9.11 \times 10^{-7}i$
10. $t = 49.773832$: $\zeta\left(\frac{1}{2} - 49.773832i\right) \approx 3.11 \times 10^{-7} + 6.02 \times 10^{-7}i$

The Riemann Zeta function values at the symmetric points $\frac{1}{2} - it$ for the first ten non-trivial zeros are extremely close to zero, with minor deviations due to numerical precision. This confirms that these symmetric points are indeed zeros of the Riemann Zeta function, further supporting the symmetry and the Riemann Hypothesis.

A Appendix B: Mathematical Transformations and Properties

A.1 Transformation to Polar Coordinates

The Riemann zeta function $\zeta(s)$ is defined for complex numbers $s = \sigma + it$ with $\sigma > 1$ as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^{\sigma+it}} = \sum_{n=1}^{\infty} \frac{1}{n^s}[3]$$

This series converges absolutely for $\sigma > 1$ and can be analytically continued to other values of s (except $s = 1$).

The analytic continuation of the zeta function extends its domain to the entire complex plane, excluding $s = 1$. This continuation is essential for defining $\zeta(s)$ beyond the region where the original series converges.

Two key formulas used in analytic continuation are:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s)[3]$$

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1}\, dx$$

These integral representations converge for all s in the complex plane except $s = 1$, preserving the analytic nature of $\zeta(s)$ in polar coordinates as well. The functional equation of the Riemann zeta function implies a symmetry about the critical line $\sigma = \frac{1}{2}$ [3]:

$$\zeta(re^{i\theta}) = 2^{re^{i\theta}} \pi^{re^{i\theta}-1} \sin\left(\frac{\pi re^{i\theta}}{2}\right) \Gamma(1-re^{i\theta})\zeta(1-re^{i\theta})$$

The Ξ function, defined as:

$$\Xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

satisfies the simpler functional equation:

$$\Xi(s) = \Xi(1-s)$$

and in polar coordinates:

$$\Xi(re^{i\theta}) = \Xi(1-re^{i\theta})$$

To express s in polar coordinates, we write:

$$s = \sigma + it = re^{i\theta}$$

where:

$$r = \sqrt{\sigma^2 + t^2}$$
$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

Given a complex number $s = \sigma + it$, we transform it into polar coordinates as follows [4]:

$$s = r(\cos\theta + i\sin\theta)$$

where:

$$r = \sqrt{\sigma^2 + t^2}$$
$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

The magnitude r of the complex number s in the traditional system is [4]:

$$|s| = \sqrt{\sigma^2 + t^2}$$

In the polar coordinate system, the magnitude r is defined as [4]:

$$r = \sqrt{\sigma^2 + t^2}$$

Since the magnitude is preserved, we have:

$$|s| = r$$

The phase θ in the traditional system is:

$$\phi = \arctan\left(\frac{t}{\sigma}\right)$$

In the polar coordinate system, the phase θ is [4]:

$$\theta = \arctan\left(\frac{t}{\sigma}\right)$$

Since the phase is preserved, we have:

$$\phi = \theta$$

To show that $s = \sigma + it$ is preserved in polar coordinates, we start with [4]:

$$s = r(\cos\theta + i\sin\theta)$$

Substitute r and θ:

$$s = \sqrt{\sigma^2 + t^2}\left(\cos\left(\arctan\left(\frac{t}{\sigma}\right)\right) + i\sin\left(\arctan\left(\frac{t}{\sigma}\right)\right)\right)$$

Using the trigonometric identities [4]:

$$\cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}, \quad \sin(\arctan(x)) = \frac{x}{\sqrt{1+x^2}}$$

Let $x = \frac{t}{\sigma}$, then [4]:

$$\cos\left(\arctan\left(\frac{t}{\sigma}\right)\right) = \frac{\sigma}{\sqrt{\sigma^2 + t^2}}$$

$$\sin\left(\arctan\left(\frac{t}{\sigma}\right)\right) = \frac{t}{\sqrt{\sigma^2 + t^2}}$$

Substituting these back [4]:

$$s = \sqrt{\sigma^2 + t^2}\left(\frac{\sigma}{\sqrt{\sigma^2 + t^2}} + i\frac{t}{\sqrt{\sigma^2 + t^2}}\right)$$

Simplifying:

$$s = \sigma + it$$

This confirms that the transformation preserves the representation $s = \sigma + it$ and this also applies to $s = \sigma - it$.

A.2 Verification of Properties

To verify that the zeta function's properties are consistent in polar coordinates, we provide detailed steps:

1. **Series Representation**:

$$\zeta(re^{i\theta}) = \sum_{n=1}^{\infty} \frac{1}{n^{re^{i\theta}}} [3]$$

2. **Continuity and Differentiability**: The transformation from Cartesian to polar coordinates is smooth, and $\zeta(re^{i\theta})$ inherits the continuity and differentiability of $\zeta(s)$.

3. **Functional Equation in Polar Form**: The functional equation $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$ in polar coordinates becomes [3]:

$$\zeta(re^{i\theta}) = 2^{re^{i\theta}}\pi^{re^{i\theta}-1}\sin\left(\frac{\pi re^{i\theta}}{2}\right)\Gamma(1-re^{i\theta})\zeta(1-re^{i\theta})$$

   Given that the gamma function $\Gamma(s)$ and the sine function $\sin(s)$ are well-defined and analytic in the complex plane [4], the symmetry and analytic continuation properties hold in the polar form.

4. **Symmetry**: Using the Ξ function [3], which satisfies $\Xi(s) = \Xi(1-s)$, we confirm that the symmetry about the critical line $\sigma = \frac{1}{2}$ is maintained:

$$\Xi(re^{i\theta}) = \Xi(1-re^{i\theta})$$

B Appendix C: Verification and Analysis of Non-Trivial Zeros

B.1 Verification of Formula for Non-Trivial Zeros

To verify the formula $\zeta\left(\sqrt{\frac{1}{4}+t^2}\, e^{i\arctan(2t)}\right) = 0$ for non-trivial zeros of the Riemann zeta function, we selected 30 known non-trivial zeros and computed the zeta function values using the given formula.

The results are summarized in the following table:

B.2 Analysis

The values of $\zeta\left(\sqrt{\frac{1}{4}+t^2}\, e^{i\arctan(2t)}\right)$ for the selected non-trivial zeros are extremely close to zero, with both real and imaginary parts being on the order of 10^{-13} or smaller. This strongly suggests that the given formula holds true for these zeros.

These results indicate that the formula $\zeta\left(\sqrt{\frac{1}{4}+t^2}\, e^{i\arctan(2t)}\right) = 0$ accurately represents the non-trivial zeros of the Riemann zeta function for the

t (Imaginary part of zero)	$\Re(\zeta)$	$\Im(\zeta)$
14.1347251417347	-1.61×10^{-16}	4.93×10^{-15}
21.0220396387716	1.41×10^{-14}	4.77×10^{-14}
25.0108575801457	-4.07×10^{-15}	1.50×10^{-14}
30.4248761258595	-2.80×10^{-15}	-1.03×10^{-14}
32.9350615877392	-5.02×10^{-15}	1.17×10^{-14}
37.5861781588256	-3.73×10^{-14}	-1.26×10^{-13}
40.9187190121475	7.62×10^{-15}	3.32×10^{-15}
43.327073280914	1.11×10^{-12}	-1.46×10^{-12}
48.0051508811672	3.72×10^{-14}	2.82×10^{-14}
49.7738324776723	-3.64×10^{-15}	6.71×10^{-15}
52.9703214777145	-6.67×10^{-15}	9.52×10^{-14}
56.4462476970634	1.33×10^{-14}	4.67×10^{-15}
59.3470440026026	1.57×10^{-13}	3.08×10^{-13}
60.8317785246098	1.17×10^{-14}	-3.83×10^{-15}
65.1125440480819	4.63×10^{-13}	4.89×10^{-13}
67.0798105294942	-5.28×10^{-15}	3.99×10^{-14}
69.5464017111739	6.91×10^{-14}	-1.62×10^{-13}
72.0671576744819	1.79×10^{-14}	-1.92×10^{-15}
75.7046906990839	-5.06×10^{-14}	-3.79×10^{-14}
77.1448400688748	8.12×10^{-15}	-6.14×10^{-15}
79.3373750202493	1.27×10^{-13}	-1.37×10^{-13}
82.910380854086	-5.32×10^{-14}	-5.85×10^{-14}
84.7354929805171	-5.97×10^{-15}	9.47×10^{-14}
87.4252746131252	-1.32×10^{-14}	-5.74×10^{-14}
88.8091112076345	-2.34×10^{-14}	5.16×10^{-14}
92.4918992705583	-3.63×10^{-13}	-3.92×10^{-13}
94.6513440405198	-6.37×10^{-14}	-1.07×10^{-13}
95.8706342282453	4.79×10^{-14}	-2.13×10^{-14}
98.8311942181937	-1.89×10^{-14}	8.13×10^{-14}
101.317851005731	-3.04×10^{-13}	-1.19×10^{-12}

Table 1: Verification of the formula for known non-trivial zeros of the Riemann zeta function

tested cases, providing further support to the hypothesis that all non-trivial zeros lie on the critical line $\sigma = \frac{1}{2}$.