On Noncommutative Spacetimes and corrections to the Kerr-Newman Black Hole Entropy due to Quantization of Area, Mass and Entropy

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Abstract

Recently we have argued [1] that the noncommutativity of the spacetime coordinates is the answer to the question : Why is area, mass, entropy quantized ? Furthermore, it casts light into a deep interplay among black hole entropy, discrete calculus, number theory, theory of partitions, random matrix theory, fuzzy spheres, ... We extend our previous construction of Schwarzschild black holes and derive the corrections to the Kerr-Newman temperature and black hole entropy, to all orders, from the discrete mass transitions taken place among different mass states. The mass spectrum for Kerr, Kerr-Newman, and Reissner-Nordstrom black holes is explicitly obtained which reduces to the Schwarzschild case when the angular momentum and charge is set to zero. One of the most salient features in the expansion of the *modified* temperature $\mathcal{T} = T + c_1 \frac{T}{N} + c_2 \frac{T}{N^2} + \ldots$ is that it spells a correspondence between the loop expansion in QFT in powers of $\hbar$, after setting $\hbar \leftrightarrow (1/N)$. N is the principal quantum number labeling the spectrum of mass states and which is given by $N = l_3(l_3+2) - l_2(l_2+1) + l_1^2$, with $l_3 \geq l_2 \geq |l_1|$ being the quantum numbers associated with the hyper-spherical harmonics of the three-sphere S^3. These results can be extended to higher dimensions. To finalize, we should add that the deviation from a full thermal spectrum and the corrections to the Hawking temperature might be relevant to the solution of the Black Hole Information paradox.

Keywords : Noncommutative Geometry; Gravity, Black Hole, Entropy; Strings; Matrix Models; Partitions.

1 Introduction : Noncommutative Spacetimes, Quantization of Area, Mass and Entropy

Very recently we have explored how the Noncommutativity of the spacetime coordinates results in the Quantization of Area, Mass and Entropy of black holes [1]. It allowed to derive the Schwarzschild black hole entropy $\frac{A}{4G}$, the logarithmic corrections, and further corrections, from the discrete mass transitions taken place among different mass states in $D = 4$. The higher dimensional generalization of the results in $D = 4$ followed. The discretization of the entropy-mass relation $S = S(M)$ lead to an entropy quantization of the form $S = S(M_n) = n$, and such that one may always assign n "bits" to the discrete entropy, and in doing so, make contact with quantum information. The physical applications of mass quantization, like the counting of states contributing to the black hole entropy, black hole evaporation, and the direct connection to the black holes-string correspondence [2] via the asymptotic behavior of the number of partitions of integers, followed. We found that the recent large N Matrix model (fuzzy sphere) of [3] leads to very similar results for the black hole entropy as the physical model described in [1].

The idea of a Quantum Spacetime where the spacetime coordinates do not commute was proposed early on by Heisenberg and Ivanenko as a way to eliminate infinities from Quantum Field Theory. Snyder published the first concrete example [4] of a noncommutative algebra involving the spacetime coordinates, and it was generalized shortly after by Yang [5], to include noncommuting momentum variables as well. We learnt from General Relativity that the Poincare algebra cannot be implemented on a curved spacetime, but only on its flat tangent space (Minkowski spacetime). The momentum operators don't commute on a curved spacetime. And vice versa, by Born's principle of reciprocity [6], [7] the coordinate operators do not commute on a curved *momentum* space. This prompted the formulation of Quantum Mechanics and Quantum Field Theory in Noncommutative spacetimes (also called Noncommutative QFT), and which might cast some light in the formulation of Quantum Gravity by encoding both key aspects of a curved and a noncommuting spacetime (a curved noncommuting spacetime).

Given a flat $6D$ spacetime with coordinates $Y^A = \{Y^1, Y^2, Y^3, Y^4, Y^5, Y^6\}$, and a metric $\eta_{AB} = diag(-1, +1, +1, \ldots, +1)$, the Yang algebra [5] can be derived in terms of the $so(5,1)$ Lorentz algebra generators described by the angular momentum/boost operators

$$J^{AB} = -(Y^A\ \Pi^B\ -\ Y^B\ \Pi^A)\ =\ i\ Y^A \frac{\partial}{\partial Y_B}\ -i\ Y^B \frac{\partial}{\partial Y_A} \tag{1.1}$$

where $\Pi^A = -i(\partial/\partial Y_A)$ is the canonical conjugate momentum variable to Y^A. Their commutators are

$$[Y^A, Y^B]\ =\ 0,\ [\Pi^A, \Pi^B]\ =\ 0,\ \ [Y^A, \Pi^B]\ =\ i\,\eta^{AB},\ \ A, B = 1, 2, 3, 4, 5, 6 \tag{1.2}$$

The coordinates Y^A commute. The momenta Π^A also commute, and Y^A, Π^B obey the Weyl-Heisenberg algebra in $6D$.

Adopting the units $\hbar = c = 1$, the correspondence among the noncommuting $4D$ spacetime coordinates X^μ, the noncommuting momenta P^μ, and the Lorentz $so(5,1)$ algebra generators leading to the Yang algebra [5] is given by

$$X^\mu \leftrightarrow L_P \; J^{\mu 5} \; = \; - L_P \; (Y^\mu \; \Pi^5 \; - \; Y^5 \; \Pi^\mu)$$

$$P^\mu \leftrightarrow \frac{1}{\mathcal{L}} \; J^{\mu 6} \; = \; - \frac{1}{\mathcal{L}} \; (Y^\mu \; \Pi^6 \; - \; Y^6 \; \Pi^\mu), \quad \mu, \nu = 1, 2, 3, 4 \tag{1.3}$$

and which requires the introduction of an ultra-violet cutoff scale L_P given by the Planck scale, and an infra-red cutoff scale $\mathcal{L}$ that can be set equal to the Hubble scale R_H (which determines the cosmological constant). It is very important to emphasize that despite the introduction of two length scales $L_P, \mathcal{L}$ the Lorentz symmetry is not lost. This is one of the most salient features of the Snyder [4] and Yang [5] algebras.

One must include also the remaining $so(5,1)$ generators

$$\mathcal{N} \; \equiv \; J^{56} = -(Y^5 \; \Pi^6 - Y^6 \; \Pi^5), \;\; J^{\mu\nu} \; = \; -(Y^\mu \; \Pi^\nu - Y^\nu \; \Pi^\mu), \;\; \mu, \nu = 1, 2, 3, 4 \tag{1.4}$$

One can then verify that the Yang algebra is recovered after imposing the above correspondence (1.3)

$$[X^\mu, \; X^\nu] \; = \; - i \; L_P^2 \; J^{\mu\nu}, \;\; [P^\mu, \; P^\nu] \; = \; - i \; (\frac{1}{\mathcal{L}})^2 \; J^{\mu\nu}, \;\; \eta^{55} \; = \; \eta^{66} = 1 \tag{1.5}$$

$$[X^\mu, \; J^{\nu\rho}] \; = \; i \; (\eta^{\mu\rho} \; X^\nu \; - \; \eta^{\mu\nu} \; X^\rho) \tag{1.6}$$

$$[P^\mu, \; J^{\nu\rho}] \; = \; i \; (\eta^{\mu\rho} \; P^\nu \; - \; \eta^{\mu\nu} \; P^\rho \;) \tag{1.7}$$

$$[X^\mu, \; P^\nu] \; = \; - i \; \eta^{\mu\nu} \; \frac{L_P}{\mathcal{L}} \; \mathcal{N}, \;\; [J^{\mu\nu}, \mathcal{N}] = 0 \tag{1.8}$$

$$[X^\mu, \; \mathcal{N}] \; = \; i \; L_P \mathcal{L} \; P^\mu, \;\; [P^\mu, \; \mathcal{N}] \; = \; - i \; \frac{1}{L_P \mathcal{L}} \; X^\mu \tag{1.9}$$

and where the $[J^{\mu\nu}, J^{\rho\sigma}]$ commutators are the same as in the $so(3,1)$ Lorentz algebra in $4D$. They are of the form

$$[\; J^{\mu_1\mu_2}, \; J^{\nu_1\nu_2} \;] \; = \; - i \; \eta^{\mu_1\nu_1} \; J^{\mu_2\nu_2} \; + i \; \eta^{\mu_1\nu_2} \; J^{\mu_2\nu_1} \; +$$

$$i \; \eta^{\mu_2\nu_1} \; J^{\mu_1\nu_2} \; - i \; \eta^{\mu_2\nu_2} \; J^{\mu_1\nu_1}, \;\; \hbar = c = 1 \tag{1.10}$$

The generators are assigned to be Hermitian so there are i factors in the right-hand side of eq-(2.10) since the commutator of two Hermitian operators is anti-Hermitian. The $4D$ spacetime metric is $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$.

In [9] we discussed two approaches in the evaluation of the areal spectrum in $3D$ and associated with noncommutative coordinates that we labeled as operators as $\mathbf{x_i}; i = 1, 2, 3$.

One approach was to write the operator $L_P^{-2}\sum_{i=1}^{i=3}\mathbf{x_i}\mathbf{x^i}$ (in Planck units) as the difference $\sum_{i,j=1}^{i,j=4}\mathbf{J}_{ij}^2 - \sum_{i,j=1}^{i,j=3}\mathbf{J}_{ij}^2$ of the total orbital angular momentum squared in $D = 4$ and $D = 3$. So the eigenvalues can be obtained from the difference between the quadratic Casimirs of $SO(4)$ and $SO(3)$ given by $C_2[SO(4)] - C_2[SO(3)] = l_3(l_3+2) - l_2(l_2+1)$, where l_3 is the orbital angular momentum quantum number of the three-sphere S^3, and l_2 is the orbital angular momentum quantum number of the two-sphere S^2. In the very special case when $l_3 = l_2$ the difference $C_2[SO(4)] - C_2[SO(3)]$ is given by l_2 and such that $\sum_{i=1}^{i=3}\mathbf{x_i}\mathbf{x^i} = l_2 L_P^2$ turns out to be *linear* in the angular momentum quantum number of the two-sphere $l_2 = l$.

The eigenfunctions of the angular momentum operators $\mathbf{J}_{S^2}^2$ associated with S^2 are the spherical harmonics $Y_{lm}(\theta,\varphi)$ and which can be rewritten as $Y_{l_2 l_1}(\theta_2,\theta_1)$

$$Y_{l_2 l_1}(\theta_2,\theta_1) \ \equiv \ Y_{lm}(\theta,\varphi) \ = \ (-1)^m \sqrt{\frac{2l+1}{4\pi}} \ \sqrt{\frac{(l-m)!}{(l+m)!}} \ P_l^m(cos\theta)\, e^{im\varphi} \quad (1.11)$$

with $l_1 = m, l_2 = l; \theta_2 = \theta, \theta_1 = \varphi$ and where $P_{lm}(cos\theta)$ are the associated Legendre ploynomials.

The eigenfunctions of the angular momentum operators $\mathbf{J}_{S^3}^2$ associated with S^3 are given in terms of three angles $\theta_1 = \varphi, \theta_2 = \theta, \theta_3 = \xi$ and three quantum numbers l_1, l_2, l_3, obeying $l_3 \geq l_2 \geq |l_1|$, as follows [10]

$$Y_{l_1 l_2 l_3}(\theta,\varphi,\xi) \ = \ Y_{l_1 l_2}(\theta,\varphi) \ \sqrt{\frac{2l_3+2}{2} \ \frac{(l_3+l_2+1)!}{(l_3-l_2)!}} \ \sqrt{sin\xi} \ P_{l_3+\frac{1}{2}}^{-(l_2+\frac{1}{2})}(cos\xi) \quad (1.12)$$

where $P_{l_3+\frac{1}{2}}^{-(l_2+\frac{1}{2})}(cos\xi)$ is the associate Legendre function of the first kind that can be written in terms of the hypergeometric function ${}_2F_1$ as

$$P_{l_3+\frac{1}{2}}^{-(l_2+\frac{1}{2})}(cos\xi) \ \equiv \ \frac{1}{\Gamma(1+l_2+\frac{1}{2})} \ (\frac{1-cos\xi}{1+cos\xi})^{\frac{1}{2}(l_2+\frac{1}{2})} \ \times$$

$${}_2F_1\left(-(l_3+\frac{1}{2}), \ (l_3+\frac{1}{2})+1; \ 1+(l_2+\frac{1}{2}); \ \frac{1-cos\xi}{2}\right) \quad (1.13)$$

Note that because $Y_{l_1 l_2 l_3}(\theta,\varphi,\xi)$ factorizes $Y_{l_1 l_2}(\theta,\varphi)F_{l_3 l_2}(\xi)$, it can be seen also as an eigenfunction of $\mathbf{J}_{S^2}^2$ (the angular momentum operator associated with S^2) because $\mathbf{J}_{S^2}^2 Y_{l_1 l_2 l_3}(\theta,\varphi,\xi) = l_2(l_2+1)Y_{l_1 l_2 l_3}(\theta,\varphi,\xi)$ due to the factorization property and the trivial fact that $\mathbf{J}_{S^2}^2$ does not act on the extra angle ξ. Therefore one arrives at

$$(\sum_{i=1}^{i=3} \mathbf{x_i}\mathbf{x^i}) \ Y_{l_1 l_2 l_3} \ = \ L_P^2 \ (\mathbf{J}_{S^3}^2 - \mathbf{J}_{S^2}^2) \ Y_{l_1 l_2 l_3} \ = \ L_P^2 \ [l_3(l_3+2) - l_2(l_2+1)] Y_{l_1 l_2 l_3} \quad (1.14)$$

giving $L_P^2 l_2 Y_{l_1 l_2 l_3}$ for the right hand side in the special case when $l_3 = l_2$. Since $4\pi r^2$ is the area of a sphere, when the coordinates are noncommutative, we can label $\mathbf{r}^2$ as the square of the radial operator, and the area spectrum of the quantum sphere is $4\pi L_P^2[l_3(l_3+2) - l_2(l_2+1)]$. The areal spectrum becomes *linear* in the angular momentum when $l_3 = l_2 = l$

Let us explore the physical implications behind the eigenvalues and eigenfunctions of the area-operators described in terms of the angular momentum operators of (hyper) spheres $\mathbf{S}^3, \mathbf{S}^2$. The main starting point is eq-(1.14). If one sets $l_3 = l_2 = n$ in eq-(1.14) it yields $\frac{\mathbf{r}^2}{L_P^2} Y_{l_1 l_2 l_3} = n Y_{l_1 l_2 l_3}$, Given $G = L_P^2$ in $D = 4$, the area quantization of the spherical horizon of radius $r_h = 2GM$ can be recast also as a mass quantization condition as follows

$$r_h^2 \ = \ (2GM_n)^2 \ = \ n \ L_P^2 \ \Rightarrow \ \frac{4M_n^2}{m_P^2} \ = \ n \ \Rightarrow \ \frac{2M_n}{m_P} \ = \ \sqrt{n}, \quad m_P = (L_P)^{-1} \tag{1.15}$$

with n an integer $0, 1, 2, \ldots$. If there is a transition between two neighboring discrete mass states : $M_n \rightarrow M_{n-1}$, a thermal photon of energy $\omega_{n,n-1} = M_n - M_{n-1} = \Delta M_n$ is emitted (radiated), so that when $\Delta n = 1$ one learns from eq-(2.1), to a first order approximation, that

$$\frac{2\omega_{n,n-1}}{m_P} \ = \ \frac{2\Delta M_n}{m_P} \ \sim \ \frac{\Delta n}{2\sqrt{n}} \ = \ \frac{m_P}{4M_n}, \ (\Delta n = 1) \tag{1.16}$$

leading to

$$\omega_{n,n-1} \ = \ \Delta M_n \ \sim \ \frac{m_P^2}{8M_n} \ = \ \frac{1}{8GM_n} \tag{1.17}$$

It is important to emphasize that if the transition occurs between states that are not neighbors, $\Delta n \neq 1$, one may inclined to claim erroneously that the frequencies of the photons emitted appear to be integer multiples of $\omega_{n,n-1}$. This is an artifact of the first order approximation in eq-(1.17). A more rigorous result reveals that the frequencies are *not* integer-multiples of the frequency $\omega_{n,n-1}$ of eq-(1.7), because the mass states $M_n \sim m_P\sqrt{n}$ are *not* equally spaced, like it occurs in the energy levels of a harmonic oscillator.

In the black body radiation spectrum, Wien's displacement law sates that the wavelength at which the intensity per unit wavelength of the radiation has a local maximum or peak, is only a function of the temperature and given by $\lambda_{peak} = \frac{b}{T}$, where the constant $b \simeq 2.897 \times 10^{-3}$ m-K is Wien's displacement constant [24]. Since frequency is inversely proportional to the wavelength, the peak frequency turns out to be directly proportional to the black body temperature.

Hence, if one *postulates* that the frequency is the same as the temperature, $\omega_{n,n-1} = T_n$, one finds that $T_n \sim \frac{1}{8GM_n}$ is inversely proportional to the mass M_n. The latter expression corresponds to a temperature whose functional form is $T(M) = \frac{1}{8GM}$ and agrees with the Hawking temperature $T_H = \frac{1}{8\pi GM}$ up to a factor of π. The entropy corresponding to a temperature $T = T(M) = \frac{1}{8GM}$ is defined as

$$S \;=\; \int \frac{dM}{T(M)} \;=\; \int dM\;(8GM) \;=\; 4GM^2 \;=\; \frac{4M^2}{m_P^2} \tag{1.18}$$

and this quadratic behavior in the mass matches the entropy of a black hole $\frac{4\pi(2GM)^2}{4G} = \frac{Area}{4G}$, up to a factor of π. One may note that a simple rescaling $L_P \rightarrow \frac{L_P}{\sqrt{\pi}}$ in the first term of eq-(1.15) suffices to obtain the *exact* expression for the Black Hole entropy. In other words, one has $\frac{A_n}{4G} = n\pi$ to be more precise. Given the Bekenstein-Hawking black hole entropy $S = \frac{4\pi M^2}{m_P^2}$, its *discretized* form becomes $S_n = \frac{4\pi M_n^2}{m_P^2} = n\pi$, and such that it is quantized in n-bits, in the same way that one-quarter of the horizon's area is quantized in integer multiples of Planck-area cells (up to a multiple of π). In the remaining of this work we shall explain how to introduce the π factors properly.

Logarithmic corrections to the black hole entropy are obtained when one does *not* approximate the expression ΔM_n as displayed in eq-(1.17) but instead one evaluates *exactly* the mass increment ΔM_n by performing the binomial expansion in powers of $\frac{1}{n}$, with $n = \frac{4M_n^2}{m_P^2}$, as follows

$$\Delta M_n \;=\; \frac{m_P}{2}\;(\sqrt{n} \;-\; \sqrt{n-1}) \;=\; \frac{m_P}{2}\;\sqrt{n}\;\left(1 \;-\; \sqrt{1-\frac{1}{n}}\right) \;\sim$$

$$\frac{m_P}{2}\;\sqrt{n}\;\left(\frac{1}{2n} \;+\; \frac{1}{8n^2} \;+\; \ldots\right) \tag{1.19}$$

Upon substituting $n = \frac{4M_n^2}{m_P^2}$ in eq-(1.19), which stems from the area/mass quantization, gives then for the *two* leading terms in the binomial expansion the following

$$\Delta M_n = \omega_{n,n-1} \;=\; T_n \;=\; \frac{1}{8GM_n} \;+\; \frac{1}{128G^2M_n^3} \tag{1.20}$$

and such discrete expression (1.20) corresponds to a temperature-mass relation of the form

$$T = T(M) \;=\; \frac{1}{8GM} \;+\; \frac{1}{128G^2M^3} \tag{1.21}$$

and one then obtains in this manner the first order corrections to the Hawking temperature (up to π factors). Hence, the logarithmic corrections to the black hole entropy are obtained from the integral

$$S \;=\; \int \frac{dM}{T(M)} \;=\; \int dM\;\left(\frac{1}{8GM} \;+\; \frac{1}{128G^2M^3}\right)^{-1} \;=$$

$$\frac{A}{4G} \;-\; \frac{1}{4}\;ln(\frac{A}{G}+1) \qquad (1.22a)$$

after inserting the expression for the horizon area $A = 4\pi(2GM)^2$ in terms of the mass M and inserting the factors of π judiciously. The discrete version of eq-(1.22a) is

$$S_n = \frac{A_n}{4G} - \frac{1}{4}\, ln(\frac{A_n}{G} + 1), \quad \frac{A_n}{4G} = n\pi = \frac{4\pi M_n^2}{m_P^2} \qquad (1.22b)$$

Higher order corrections to the Hawking temperature and black hole entropy follow by including the higher order terms in the binomial expansion.

A similar procedure to obtain the logarithmic corrections to the black hole entropy, after relating the frequency of the radiated photon to the temperature in discrete mass transitions, can be found in [13] and references therein. The mass spectrum of black holes has a long history, see [14], [15], [16], [12], [17] among others. More recently, the quantum deformation of the Wheeler–DeWitt equation of a Schwarzchild black hole was studied by [13]. The quantum deformed black hole was based on a quantized model constructed from the quantum Heisenberg–Weyl $U_q(h_4)$ group. It was found that the event horizon area and the mass were quantized, degenerate, and bounded due to the nature of the quantum group when the deformation parameter was a root of unity.

In the next section we extend the above construction of Schwarzschild black holes and derive the corrections to the Kerr-Newman temperature and black hole entropy, to all orders, from the discrete mass transitions taken place among different mass states. The mass spectrum for Kerr, Kerr-Newman, and Reissner-Nordstrom black holes is explicitly obtained which reduces to the Schwarzschild case when the angular momentum and charge is set to zero. One of the most salient features in the expansion of the *modified* temperature $\mathcal{T} = T + c_1 \frac{T}{N} + c_2 \frac{T}{N^2} + \ldots$ is that it spells a correspondence between the loop expansion in QFT in powers of $\hbar$, after setting $\hbar \leftrightarrow (1/N)$. N is the principal quantum number labeling the spectrum of mass states and which is given by $N = l_3(l_3 + 2) - l_2(l_2 + 1) + l_1^2$, with $l_3 \geq l_2 \geq |l_1|$ being the quantum numbers associated with the hyper-spherical harmonics of the three-sphere S^3. These results can be extended to higher dimensions. To finalize, we should add that the deviation from a full thermal spectrum and the corrections to the Hawking temperature might be relevant to the solution of the Black Hole Information paradox.

Throughout this work we shall employ the units $\hbar = c = k_B = 1$.

2 Mass Spectrum of the Kerr, Kerr-Newman and Reissner-Nordstrom Black Holes

Having presented a review in the introduction of the area, mass quantization of the Schwarzschild black hole resulting from the noncommutativity of the spacetime coordinates, we shall proceed with the Kerr, Kerr-Newman and Reissner-Nordstrom Black Holes. Let us begin with the rotating massive Kerr black hole whose fundamental parameters are the mass M, and angular momentum J. The angular rotation frequency Ω_H of the black hole at the horizon is [18]

$$\Omega_H \ = \ \frac{J}{M} \ \frac{1}{r_+^2 + a^2}, \quad a = \frac{J}{M} \tag{2.1}$$

where $a \equiv J/M$ is the angular momentum per unit mass. The outer and inner horizon radius are

$$r_\pm \ = \ (GM) \ \pm \ \sqrt{(GM)^2 \ - \ (J/M)^2} \tag{2.2}$$

The area of the horizon is

$$A \ = \ \int d\theta \int d\phi \sqrt{g_{\theta\theta}(r_+, \theta, \phi) \ g_{\phi\phi}(r_+, \theta, \phi)} \ = \ 4\pi \ (r_+^2 \ + \ a^2) \tag{2.3}$$

and the Kerr black hole temperature is given by

$$T_{Kerr} \ = \ \frac{1}{2\pi} \ \frac{r_+ \ - \ GM}{r_+^2 \ + \ a^2} \tag{2.4}$$

The Smarr formula [19] in $D = 4$

$$M \ = \ 2 \, TS \ + \ 2 \, \Omega J \tag{2.5}$$

yields the Kerr black hole entropy

$$S \ = \ \frac{\beta}{2} \ (M \ - \ 2\Omega J) \ = \ \frac{4\pi \ (r_+^2 + a^2)}{4G}, \quad a \equiv \frac{J}{M} \tag{2.6}$$

after inserting the expression for $\beta = \frac{1}{T}$ with T given by (2.4), and the angular rotation frequency Ω of the black hole at the horizon given by eq-(2.1). The result in (2.6) is due to the equalities

$$(GM) \, (r_+^2 - a^2) \ = \ 2 \, (GM) \, r_+^2 \ -2 \, (GM)^2 \, r_+ \ = \ (r_+ - GM) \, (r_+^2 + a^2) \tag{2.7}$$

resulting from

$$r_+ \ - \ GM \ = \ \sqrt{(GM)^2 \ - \ (J/M)^2} \ = \ \sqrt{(GM)^2 \ - \ a^2} \tag{2.8}$$

and

$$r_+^2 \ + \ a^2 \ = \ 2(GM)^2 \ + \ 2GM \ \sqrt{(GM)^2 \ - \ (J/M)^2} \tag{2.9}$$

Given the entropy (2.6) the area quantization condition is chosen to be

$$\pi \ (r_+^2 \ + \ a^2)_N \ = \ \pi \ 2(GM_N)^2 \ + \ \pi \ 2GM_N \ \sqrt{(GM_N)^2 \ - \ (l_1/M_N)^2} \ =$$

$$\pi \ L_P^2 \ [l_3(l_3 + 2) \ - \ l_2(l_2 + 1) \ + \ l_1^2] \ = \ N\pi \ L_P^2, \quad N = 0, 1, 2, 3, \ldots \tag{2.10}$$

The integer N (the principal quantum number) is defined as

$$N \equiv l_3(l_3+2) - l_2(l_2+1) + l_1^2, \quad l_3 \geq l_2 \geq |l_1| \qquad (2.11)$$

where l_1 is the azimuthal quantum number corresponding to the Cartan generator $J_3 = J_z = J_{12}$ of the $SO(3)$ rotation group in $3D$. The angular momentum $\vec{J}$ of the Kerr black hole points in the z-axis direction [1] and its value is quantized $J_3 = l_1$ in positive/negative integer units of $\hbar$. Given l_2 the values of the $2l_2+1$ azimuthal quantum numbers are $l_1 = \{l_2, l_2-1, l_2-2, \ldots, 0, -1, -2, \ldots, -l_2\}$.

Choosing a given value for N leads to many different choices for the triplet $\{l_3, l_2, l_1\}$. The larger N is the larger the number of choices for $\{l_3, l_2, l_1\}$. Given eq-(2.10) one can then solve for M_N in terms of N and l_1. After some algebra, one finds the following Kerr black hole mass spectrum

$$M_N = \frac{m_P}{2} \sqrt{\frac{N^2 + 4l_1^2}{N}}, \quad N = 0, 1, 2, \ldots \qquad (2.12)$$

which leads to the quantization of the entropy

$$S_N = \frac{A_N}{4G} = \pi \, (r_+^2 + a^2)_N = \pi \, N \qquad (2.13)$$

The first law of Kerr black hole thermodynamics is

$$dM = T \, dS + \Omega \, dJ \Rightarrow (\frac{\partial S}{\partial M})_J = \frac{1}{T} \Rightarrow (\frac{\partial M}{\partial S})_J = T \qquad (2.14)$$

where the variations are performed keeping J fixed. Hence, one can derive the the discrete (quantized) version of the Kerr black hole's temperature by varying M_N with respect to $S_N = N\pi$, keeping l_1 fixed, as follows

$$T_N = (\frac{\partial M_N}{\partial S_N})_{l_1} = (\frac{\partial M_N}{\partial (N\pi)})_{l_1} = \frac{1}{\pi} (\frac{\partial M_N}{\partial N})_{l_1} = \frac{m_P}{2\pi} \frac{1 - 4(l_1^2/N^2)}{2\sqrt{(N^2+4l_1^2)/N}} =$$

$$\frac{m_P^2}{8\pi} \frac{1 - 4(l_1^2/N^2)}{M_N} = \frac{m_P^2}{8\pi} \frac{N^2 - 4l_1^2}{M_N \, N^2} \qquad (2.15)$$

Having found eq-(2.15) the next step is to invoke the quantization condition (2.10) in order to relate N, l_1 to the quantization values of r_+^2 and $a^2 = \frac{J^2}{M^2}$, which are denoted by r_{+N}^2 and $a_N^2 = \frac{l_1^2}{M_N^2}$, respectively. In this fashion, the mass and angular momentum quantization quantization leads to the following identifications

$$N \, L_P^2 = r_{+N}^2 + a_N^2, \quad N^2 \, L_P^4 = (r_{+N}^2 + a_N^2)^2, \quad l_1^2 = a_N^2 \, M_N^2 \qquad (2.16)$$

[1] We use the notation l_1 instead of m used by many authors because one may confuse m with mass

such that the discrete (quantized) temperature (2.15) can be rewritten as

$$T_N = \frac{1}{8\pi} \frac{(r_{+N}^2 + a_N^2)^2 - 4\, a_N^2\, (GM_N)^2}{GM_N\, (r_{+N}^2 + a_N^2)^2} \tag{2.17}$$

after using $G = L_P^2 = m_P^{-2}$. One can verify explicitly that the right hand side of eq-(2.17) is precisely the *same* as

$$T_{Kerr} = \frac{1}{2\pi} \frac{r_{+N} - GM_N}{r_{+N}^2 + a_N^2} \tag{2.18}$$

which is the discrete (quantized) version of the temperature of a Kerr black hole. After *dropping* the subscripts r_{+N}, M_N, a_N, for simplicity, one can show by inspection that the $difference$ between eq-(2.17) and eq-(2.18) is *zero*

$$\frac{1}{8\pi} \frac{(r_+^2 + a^2)^2 - 4a^2\, (GM)^2}{GM\, (r_+^2 + a^2)^2} - \frac{4}{8\pi} \frac{(r_+^2 + a^2)\, (GM)\, (r_+ - GM)}{GM\, (r_+^2 + a^2)^2} = 0 \tag{2.19}$$

To verify the validity of eq-(2.19) it suffices to use the definitions in eq-(2.2) and show by inspection that there is a precise cancellation of all the terms in (2.19). Therefore, one has *checked* that eq-(2.17) = eq-(2.18) and it leads to the discrete version of the Kerr black hole temperature, as expected, by *construction*, via the quantization conditions (2.10,2.16).

One can repeat this whole process for the Kerr-Newman black hole. The quantization condition of the Kerr-Newman black hole is now given by

$$(r_+^2 + a^2)_N = 2(GM_N)^2 - GQ_N^2 + 2GM_N \sqrt{(GM_N)^2 - GQ_N^2 - (l_1/M_N)^2} =$$

$$L_P^2\, [l_3(l_3+2) - l_2(l_2+1) + l_1^2] = N\, L_P^2, \quad N = 0, 1, 2, 3, \ldots \tag{2.20}$$

After some straightforward algebra one can solve for M_N in terms of N, Q_N, l_1 and obtain the spectrum of the Kerr-Newman black hole

$$M_N = \frac{m_P}{2} \sqrt{\frac{N^2 + 4\, l_1^2}{N - Q_N^2}} \tag{2.21}$$

If one wishes to avoid singularities, one finds that the quantized charge must be bounded as follow $Q_N^2 < N$. The discrete temperature turns out to be

$$T_N = \frac{m_P^2}{8\pi} \frac{N^2 - 2N\, Q_N^2 - 4l_1^2}{M_N\, (N - Q_N^2)^2} \tag{2.22}$$

Following the same arguments as above one can show that eq-(2.22) is the same as the discrete version of the Kerr-Newman black hole temperature and which has the same functional form as eq-(2.18) but now the outer horizon is given by

$$r_{+N} \ = \ GM_N \ + \ \sqrt{(GM_N)^2 \ - GQ_N^2 \ - \ (l_1/M_N)^2}, \ \ a_N^2 \ = \ \frac{l_1^2}{M_N^2} \qquad (2.23)$$

All the above equations reduce to the Schwarzschild black hole case when $l_1 = Q_N = 0$. The Reissner-Nordstrom black hole results follow from all the Kerr-Newman results simply by setting $l_1 = 0$.

We learnt from the selection rules in quantum atomic transitions that $\Delta j = \pm 1$ and which are consistent with the fact that a photon carries spin 1. Conservation of angular momentum requires that the quantum atomic state (after the photon emission) has decreased its angular momentum by one unit (of $\hbar$). And vice versa, it must increase its angular momentum by one unit after a photon absorption. Based on this selection rule, using discrete calculus, and given the Kerr black hole mass spectrum $M_N = \frac{m_P}{2}\sqrt{(N^2 + 4l_1^2)/N}$, one has

$$\Delta M_N \ = \ \frac{\partial M_N}{\partial N} \ \Delta N \ + \ \frac{\partial M_N}{\partial l_1} \ \Delta l_1 \ + \ \frac{1}{2!}\frac{\partial^2 M_N}{\partial N^2} \ (\Delta N)^2 \ + \ \frac{1}{2!}\frac{\partial^2 M_N}{\partial l_1^2} \ (\Delta l_1)^2 \ +$$

$$\frac{\partial M}{\partial N} \ \frac{\partial M_N}{\partial l_1} \ \Delta N \ \Delta l_1 \ + \ \frac{1}{3!} \ \frac{\partial^3 M_N}{\partial N^3} \ (\Delta N)^3 \ + \ \frac{\partial^3 M_N}{\partial l_1^3} \ (\Delta l_1)^3 \ + \ \ldots \qquad (2.24)$$

involving *both* a variation in N and l_1. The mass-energy content of a Kerr black hole involves an internal energy (linked to the temperature), and a rotational energy (linked to the angular momentum). In order to extract the temperature, we shall keep l_1 fixed, $\Delta l_1 = 0$ in (2.24) leading to

$$(\Delta M_N)_{l_1} \ = \ (\frac{\partial M_N}{\partial N})_{l_1} \ \Delta N + \frac{1}{2!}(\frac{\partial^2 M_N}{\partial N^2})_{l_1} \ (\Delta N)^2 + \frac{1}{3!}(\frac{\partial^3 M_N}{\partial N^3})_{l_1} \ (\Delta N)^3 + \ \ldots \qquad (2.25)$$

so that one may find the higher corrections to the discrete temperature T_N given in eq-(2.18) by simply setting $\Delta N = 1$. The first term in eq-(2.25) yields T_N (2.18), while the remaining terms furnish the higher order corrections. When $\Delta N = 1$, eq-(2.25) can be rewritten in terms of T_N and its derivatives as

$$(\Delta M_N)_{l_1} \ = \ T_N \ + \ \frac{1}{2!}(\frac{\partial T_N}{\partial N})_{l_1} \ + \ \frac{1}{3!}(\frac{\partial^2 T_N}{\partial N^2})_{l_1} \ + \ \ldots \qquad (2.26)$$

Since the discrete (quantized) entropy is $S_N = N\pi$, the actual variations should be with respect to $N\pi$ in order to reproduce the correct π factors. The *continuum* version of eq-(2.26) is obtained by a simple replacement $M_N \to M$, $T_N \to T$, and $S_N \to S$. Hence, the *modified* temperature for the Kerr black hole is postulated to be given by the expression

$$\mathcal{T} \ = \ T \ + \ \frac{1}{2!}(\frac{\partial T}{\partial S})_{l_1} \ + \ \frac{1}{3!}(\frac{\partial^2 T}{\partial S^2})_{l_1} \ + \ \ldots \qquad (2.27)$$

And the first law of thermodynamics leads to the *modified* entropy

$$\mathcal{S} \; = \; \int \beta \; (dM \; - \; \Omega_H \; dJ) \; = \; \int \frac{dM}{\mathcal{T}} \; - \; \int \frac{\Omega_H}{\mathcal{T}} \; dJ \tag{2.28}$$

In the case of the Schwarzschild black hole one can show that the first order corrections to the temperature is what leads to the *logarithmic* corrections to the black hole entropy. Given $T = T(M) = \frac{1}{8\pi GM}$; $S = 4\pi GM^2 \Rightarrow S \sim T^{-2}$, $T \sim S^{-1/2}$, and this way one can express the functional dependence of $T(S)$ or $S(T)$. The modified entropy is

$$\mathcal{S} \; = \; \int \frac{dM}{\mathcal{T}} \; = \; \int \frac{dM}{T \; + \; \frac{1}{2!}\frac{dT}{dS} \; + \; \frac{1}{3!}\frac{d^2T}{dS^2} \; + \; \ldots} \tag{2.29}$$

The integral (2.29) is of the form

$$\int \frac{dM}{M' \; + \; \frac{1}{2!}M'' \; + \; \ldots} \; = \; \int \frac{dM}{M' \; \big(1 \; + \; \frac{1}{2!}(M''/M') \; + \; \ldots\big)} \; \sim$$

$$\int \frac{dM}{M'} \; (1 \; - \; \frac{1}{2}(M''/M') \; - \; \ldots) \tag{2.30}$$

with $M' = T = \frac{dM}{dS}$; $M'' = T' = \frac{d^2M}{dS^2}, \ldots$. The last term of (2.30) was obtained by recurring to the expansion $(1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots$. The first integral $\int \frac{dM}{M'} = \int dS = S$ yields the Bekenstein-Hawking entropy $\frac{A}{4G}$, while the second integral

$$-\frac{1}{2}\int \; dM \; \frac{M''}{(M')^2} \; = \; \frac{1}{2}\int dM \; \frac{d}{dS}(\frac{1}{M'}) \; = \; \frac{1}{2}\int \; \frac{dM}{dS} \; d(\frac{1}{M'}) \; =$$

$$\frac{1}{2}\int \; \frac{1}{(dS/dM)} \; d(\frac{dS}{dM}) \; = \; \frac{1}{2} \; ln(\frac{dS}{dM}) \; = \; \frac{1}{2} \; ln(T^{-1}) \; =$$

$$\frac{1}{4} \; ln(\frac{A}{4G}) \; + \; \frac{1}{2} \; ln(4\sqrt{\pi}) \tag{2.31}$$

is what furnishes the logarithmic corrections to the Schwarzschild black hole entropy. The sign change of the log terms in eq-(2.31) compared to those in eq-(1.22a) results because we chose above $\Delta N = 1$ (absorption of photon) instead of $\Delta N = -1$ (emission of photon). Choosing $\Delta N = -1$ will then affect the *relative* sign in the first two terms of the expansion in eq-(2.25). This is the reason why there is a sign change in eq-(2.31) compared to that in eq-(1.22a). Also, one should add that because the last term of (2.30) was obtained by recurring to the expansion $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots$, one ends up with $ln(A/4G)$ instead of $ln(1 + A/4G)$ as in eq-(1.22a).

If one includes the higher order terms in (2.30) one will generate the additional terms of the form $(A/4G)^{-1} + (A/4G)^{-2} + (A/4G)^{-3} + \ldots$ as expected. The salient feature of defining the modified temperature by eq-(2.25) is that the logarithmic corrections occur for all Schwarzschild-Tangherlini black holes in dimensions $D \geq 4$. In the case of the Kerr black hole matters are more complicated because one is dealing with *partial* derivatives instead of total derivatives due to the fact that two variables are involved : the mass and the angular momentum.

Next we shall exploit the expressions in eqs-(2.26,2.27) in order to obtain the first order (and higher orders) corrections to the discrete Kerr black temperature. The expression for $T_N = (\frac{\partial M}{\partial N})_{l_1}$ was already obtained in eq-(2.17), and its *continuum* limit was shown to agree precisely with the Kerr black hole temperature T_{Kerr} in eq-(2.18). The leading correction to T_N (after taking care of the π factors involved in the value of the entropy $S_N = N\pi$) is

$$\frac{1}{2!} \; (\frac{\partial T}{\partial (N\pi)})_{l_1} \;=\; \frac{1}{16\pi^2} \; \frac{8 \; a_N^2 \; N \; M_N^3 - \; (N^2 \; - \; 4a_N^2 \; M_N^2) \; N^2 \; T_N}{G \; N^4 \; M_N^2} \tag{2.32}$$

Given the mass spectrum (2.12), one finds that for very large $N >> a$, the leading corrections are of the order $T_1 \sim (T_N/GM_N^2) \sim T_N/N$ compared to T_N. This is a sign of consistency.

The correspondence given in eq-(2.16) allows to express N, N^2 appearing in the above eq-(2.32) in terms of M_N and l_1 and such that one can read-off the functional form of $\mathcal{T}_N = \mathcal{T}(M_N, l_1)$, and in turn, by taking the continuum limit $\mathcal{T}_N \rightarrow \mathcal{T}, T_N \rightarrow T; M_N \rightarrow M, \, l_1 \rightarrow aM$, one will be able to find the sought-after expression for the modified temperature $\mathcal{T} = \mathcal{T}(M,J) = T(M,J) + \ldots$. After tedious algebra, from eq-(2.32), and the correspondence given in eq-(2.16), one finds that the continuum limit of the first leading correction to the Kerr black hole temperature is

$$T_1 \;=\; \frac{1}{16\pi^2} \; \frac{8 \; a^2 \; (GM)^3 \; - \; [\; (r_+^2 \; + \; a^2)^2 \; - \; 4a^2 \; (GM)^2 \;] \; T \; (r_+^2 \; + \; a^2)}{G \; M^2 \; (r_+^2 \; + \; a^2)^3} \tag{2.33}$$

One can check that by setting $a = 0$ in (2.33) it furnishes the leading correction to the Schwarzschild black hole temperature

$$T_1^{(Schwarz)} \;=\; - \; \frac{1}{128 \; \pi^3 \; G^2 \; M^3} \tag{2.34}$$

which agrees (up to π factors) with the temperature correction to the Hawking temperature appearing in eq-(1.22a). Once again, the sign change is due to the fact that eq-(2.34) was derived by setting $\Delta N = 1$ (photon absorption, $\Delta M_N > 0$), whereas eq-(1.22a) was based in setting $\Delta N = -1$ (photon emission, $\Delta M_N < 0$).

After a laborious algebra one finds that the expression in (2.33) agrees with

$$(\frac{\partial T}{\partial S})_J \;=\; (\frac{\partial M}{\partial S})_J \; (\frac{\partial T}{\partial M})_J \;=\; T \; (\frac{\partial T}{\partial M})_J \tag{2.35}$$

where

$$(\frac{\partial T}{\partial M})_J \ = \ \frac{1}{16\pi^2} \ \frac{16\pi \ a^2 \ (GM)^3 \ (r_+ - GM)^{-1} \ - \ [\ (r_+^2 \ + \ a^2)^2 \ - \ 4a^2 \ (GM)^2 \ ]}{G \ M^2 \ (r_+^2 \ + \ a^2)^2} \tag{2.36}$$

with $T = T_{Kerr} = T(M,J)$ is given by eq-(2.4). Therefore, to sum up, one has $T_1 = (\frac{\partial T}{\partial S})_J$. The *higher* order sub-leading corrections to the temperature are obtained by taking further derivatives

$$(\frac{\partial^2 T}{\partial S^2})_J \ = \ (\frac{\partial T}{\partial S})_J \ (\frac{\partial T}{\partial M})_J \ + \ T \ (\frac{\partial^2 T}{\partial S \partial M})_J \ = \ T \ (\frac{\partial T}{\partial M})^2_J \ + \ T^2 \ (\frac{\partial^2 T}{\partial M^2})_J; \ \ \ldots \tag{2.37}$$

One finds that for very large $N >> a$, the sub-leading corrections are of the form $T_N/N^2; T_N/N^3; \ldots$ compared to T_N. This expansion of the corrections to the black hole temperature in powers of $(1/N)$ resembles the loop expansion in QFT in powers of $\hbar$. A correspondence between $\hbar$ and N of the form $\hbar \leftrightarrow (1/N)$ is not new. A known example is the large N limit of Self Dual $SU(\infty)$ Yang Mills when we showed in [25] that the equations for self dual $SU(\infty)$ Yang Mills lead to the Plebanski heavenly equations for Self Dual Gravity. A Moyal deformation quantization of $SU(2)$ Self Dual Yang-Mills also furnishes the Plebanski heavenly equations for Self Dual Gravity. The large $N = \infty$ limit of the Moyal brackets reduce to the classical Poisson brackets. This is consistent with the fact that if $\hbar \leftrightarrow (1/N)$, then $N \to \infty$ amounts to $\hbar \to 0$. Another example of this classical/quantum correspondence in the large N limit is the AdS/CFT, gravity/$SU(N)$ Yang-Mills correspondence [26].

Finally, by inserting the sum of the expressions (2.4) and (2.33) $\mathcal{T} = T + T_1$, into the two integrals (2.28) one can derive the first order corrections to the Kerr black hole entropy. A similar procedure works out for the Kerr-Newman and Reissner-Nordstrom black hole entropies obtained by including the black hole's electric potential Φ_H at the horizon

$$\Phi_H \ = \ Q \ \frac{r_+}{r_+^2 + a^2} \tag{2.38}$$

An important remark is in order concerning the first law of black hole thermodynamics and the Smarr formula. The most general Smarr formulae for massive, charged, rotating black holes in higher dimensions, and with a cosmological constant, were discussed in full detail by [19]. If one uses the Smarr formula (2.5) by replacing T for $T + T_1 \ldots$ in the Schwarzschild black hole it becomes clear that one will *not* be able to generate the logarithmic corrections to the entropy and which were recovered by recurring to the first law $dM = \mathcal{T} dS = (T + T_1) dS$, as displayed explicitly by the integrals in eqs-(2.230, 2.31). Therefore one must use the first law of thermodynamics which leads to the modified Kerr-Newman entropy given by the integrals

$$\mathcal{S} \ = \ \int \frac{dM}{\mathcal{T}} \ - \ \int \frac{\Omega_H}{\mathcal{T}} \ dJ \ - \ \int \ \frac{\Phi_H}{\mathcal{T}} \ dQ \tag{2.39}$$

with $\mathcal{T} = T + T_1 + T_2 + \ldots$, where $T = T(M, J, Q)$ is the Kerr-Newman black hole temperature. The results obtained in this work can be generalized to higher dimensions after recurring to hyper spherical harmonics in order to derive the hyper-areas spectrum [1].

3 Concluding Remarks

We end this work with some important final remarks. The deviation from a full thermal spectrum and the corrections to the Hawking temperature might be relevant to the solution of the Black Hole Information paradox since pure states may no longer evolve to mixed states. Dvali [27] has provided a model-independent argument indicating that for a black hole of entropy N the non-thermal deviations from Hawking radiation, per each emission time, are of order $(1/N)$, as opposed to $exp(-N)$, and argued that his fact abolishes the standard a priori basis for the information paradox. The features of the non-corrected thermal (non-thermal) spectrum and the quantum corrected thermal (non-thermal) spectrum were analyzed by [28]. Consequently, these differences provide a possible way towards experimentally analyzing whether the radiation spectrum of black hole is thermal or non-thermal with or without high order quantum corrections.

To our knowledge, our approach to evaluate the area and mass spectrum of black holes, and the higher order corrections to the black holes entropies and temperatures, based on the noncommutativity of spacetime coordinates, is new, or not widely known, despite that the literature on the logarithmic corrections to black hole entropies is vast. We refer to the work of Sen [20], and the large number of references therein, where he derived the logarithmic corrections to the Kerr black hole entropy by evaluating in full rigour the Euclidean gravitational path integral. Tanaka [8] in the past has studied quantum black holes within the context of an underlying noncommutative quantized space-time, and has explored the holographic relations in Yang's quantized space-time algebra. Valtancoli [11] is another author who has studied the spectrum of spheres in a noncommutative Snyder geometry. For different approaches to the quantization of the black holes area spectrum see [21]. The authors [29] have shown that the standard quantum gravitational logarithmic correction to Bekenstein-Hawking entropy is equivalent to a running gravitational "constant" dependent on the horizon area $G_{eff}(A)$ that could also lead to a coupling between black hole masses and cosmological expansion.

Recently, the Landauer principle from information theory in the context of area quantization of the Schwarzschild black hole has been explored by [22]. It is also based on a quantum-mechanical perspective where Hawking evaporation can be interpreted in terms of transitions between the discrete states of the area (or mass) spectrum. A very extensive and rigorous analysis of the entropy and spectrum of near-extremal black holes involving semiclassical brane solutions to non-perturbative problems can be found in [23].

Most importantly, in our opinion, is that the noncommutativity of the spacetime coordinates is the answer to the question : Why is area, mass, entropy quantized ? In this work we have not touched the generalized uncertainty principle (GUP) relations because we have focused mainly on the spectral questions. The GUP is bound to play an important role. Furthermore, as discussed in [1], these results cast light into a deep interplay among black hole entropy, discrete calculus, number theory, theory of partitions, random matrix theory, fuzzy spheres, ... For recent work on the interplay between number theory and black holes see [30] where the authors used their methods to revisit the UV/IR connection that relates black hole microstate counting to modular forms. They also provided a microscopic interpretation of the logarithmic corrections to the entropy of BPS black holes. We hope that our results in this work will further strengthen the links between number theory and black holes.

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References

[1] Carlos Castro Perelman, "A glimpse into Quantum Gravity : A Noncommutative Spacetime results in Quantization of Area, Mass and Entropy", submitted to the IJGMMP, 2024.

[2] G. Horowitz and J. Polchinski, "A correspondence principle for black holes and strings", Phys. Rev. **D 55** (1997) 6189.

[3] Chu, "A Matrix Model proposal for QG and the QM of Black Holes", arXiv : 2406.01466

[4] H. Snyder, "Quantized space-time", Physical Review, **67** (1) (1947) 38.

[5] C. Yang, "On Quantized Spacetime", Phys. Rev. **72** (1947) 874.

[6] M. Born, "Elementary Particles and the Principle of Reciprocity", Nature **1949**, 163, 207.

[7] C. Castro Perelman, "On Born's Reciprocal Relativity, Algebraic Extensions of the Yang and Quaplectic Algebra, and Noncommutative Curved Phase Spaces", Universe 2023, **9** (3) , 144.

C. Castro Perelman, "Born's Reciprocal Relativity Theory, Curved Phase Space, Finsler Geometry and the Cosmological Constant " Annals of Physics **416**, May 2020, 168143.

[8] S. Tanaka, "A Short Essay on Quantum Black holes and underlying noncommutative quantized spacetime", Class. Quantum. Grav. **34** (2017) 015007. Found. Phys. **39**, 510 (2009).

S. Tanaka, "Kinematical reduction of spatial degrees of freedom and holographic relation in Yang's quantized space-time algebra, Found. Phys. **39**, 510 (2009)

S. Tanaka, "Space-time quantization and matrix model", Nuovo Cimento **B 114**, 49 (1999),

[9] C. Castro Perelman, "The Clifford-Yang Algebra, Noncommutative Clifford Phase Spaces and the Deformed Quantum Oscillator" IJGMMP **20**, No. 5 (2023) 2350078.

C. Castro Perelman, "A Noncommutative Spacetime Realization of Quantum Black Holes, Regge Trajectories and Holography" Journal of Geometry and Physics, Volume **173**, March 2022, 1044352022

[10] J.A Avery and J.S Avery, *Hyperspherical Harmonics and Their Physical Applications* (World Scientific, 2018).

K. Granzow, "N-dimensional Total Orbital Angular-Momentum Operator" J. Math. Phys. **4** no. 7 (1963) 897.

K. Granzow, "N-dimensional Total Orbital Angular-Momentum Operator II. Explicit Representation" J. Math. Phys. **5** no. 10 (1964) 1474.

A. Higuchi, "Symmetric tensor spherical harmonics on the N-sphere and their application to the de Sitter group $SO(N, 1)$", Journal of Mathematical Physics. **28** (7) (1987) 1553.

G. Compere, and D. A. Nichols, "Classical and Quantized General-Relativistic Angular Momentum", arXiv : 2103.17103.

[11] P. Valtancoli, "On the area of the sphere in a Snyder geometry" arXiv : 2102.04424.

P. Valtancoli, "Harmonic oscillators in a Snyder geometry" arXiv : 1306.0116.

P. Valtancoli, "Dirac oscillator and minimal length", J. Math. Phys. **58** (2017) 6, 063504.

[12] J.Makela, P. Repo, M. Luomajoki and J. Piilonen, "Quantum-mechanical model of the Kerr-Newman black hole". Phys. Rev. **D 64** (2001) 024018.

[13] S. Jalalzadeh, "Quantum Black hole–White hole entangled states", arXiv : 2203.09968.

[14] J.D. Bekenstein, "Black holes and entropy", Phys. Rev. **D 7** (1973) 2333.

S. W. Hawking, "Particle creation by black holes," Commun. Math. Phys. **43** (3) (1975) 199–220.

[15] V. F. Mukhanov, "Are black holes quantized?", JETP Lett. **44** (1986) 63–66.

J. D. Bekenstein, V. F. Mukhanov, "Spectroscopy of the quantum black hole", Phys. Lett. **B 360** (1995) 7–12.

J. D. Bekenstein, "The quantum mass spectrum of the Kerr black hole", Lett. Nuovo Cim. **11** (1974) 467.

[16] J. Louko and J. Makela, "Area spectrum of the Schwarzschild black hole", Phys. Rev. **D 54** (1996) 4982–4996.

[17] S. Das, P. Ramadevi and U. A. Yajnik, "Black hole area quantization", Mod.Phys. Lett. **A 17** (2002) 993–1000.

[18] R. M. Wald, *General Relativity* (University of Chicago Press, 1984).

[19] Kamal Hajian, H. Oszahin, and B.Tekin, "First law of black hole thermodynamics and Smarr formula with a cosmological constant", 2103.10983.

[20] A. Sen, "Logarithmic corrections to rotating extremal black hole entropy in four and five dimensions" Gen Relativ. Gravitation, **44**, 1947–1991 (2012).

[21] M. R. Setare, and E. C. Vagenas, "Area Spectrum of Kerr and extremal Kerr Black Holes from Quasinormal Modes", Mod. Phys. Lett. **A 20** (2005) 1923-1932.

[22] B. Bagchi, A. Ghosh, and S. Sen, "Landauer's principle and black hole area quantization", arXiv:2408.02077.

M. Cortes and A. Liddle, "Hawking evaporation and the Landauer Principle", arXiv:2407.08777

[23] Sergio Hernández-Cuenca, "Entropy and Spectrum of Near-Extremal Black Holes: semiclassical brane solutions to non-perturbative problems", arXiv:2407.20321.

[24] Black-body radiation, https://en.wikipedia.org/wiki/Black-body_radiation

[25] C. Castro (Perelman), $SU(\infty)$ (super) Gauge Theories and Self Dual (super) Gravity " Journal of Mathematical Physics **34** no.2 (1993) 681.

[26] J. M. Maldacena, " The large N limit of superconformal field theories and supergravity" Adv. Theor. Math. Phys. **2** (1998) 231-252.

[27] G. Dvali, "Non-Thermal Corrections to Hawking Radiation Versus the Information Paradox", arXiv:1509.04645

[28] Z. Tian, and J. Jing, "Towards experimentally studying some puzzles of Hawking radiation" Gen. Relativ. Gravit. (2014) **46** , 1779.

[29] H. Lu, S. Di Gennaro, Y C Ong, "Generalized Entropy Implies Varying-G: Horizon Area Dependent Field Equations and Black Hole-Cosmology Coupling" arXiv:2407.00484.

[30] L. Apolo, S. Bintanja, A. Castro, D. Liska, "The light we can see: Extracting black holes from weak Jacobi forms", arXiv:2407.06260.