SOLUTION OF WALLIS'S INTEGRAL USING COMPLEX FUNCTIONS

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Introduction

This article shows the value of the Wallis integral when n is an even number, n≥2 using the integration of a complex function. Proof of the Wallis product is generally derived using partial integrals, but here derivation using complex integrals is introduced. The value of the Wallis integral is given by the following equation [1].

$$W_n = \int_0^{\pi/2} cos^n\,\theta d\theta = \int_0^{\pi/2} sin^n\,\theta d\theta$$

$$W_n = \frac{(n-1)!!}{n!!} \qquad \text{(n: odd, n≥1)}$$

$$W_n = \frac{(n-1)!!}{n!!}\frac{\pi}{2} \qquad \text{(n: even, n≥2)}$$

Calculations

cosθ and sinθ can be displayed as follows using the complex number z using the following relation: Using Euler's formula, cosθ and sinθ can be expressed by the following equations[2].
where, $z = e^{i\theta}(0 \le \theta \le 2\pi)$

(1) $$cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right),\ sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

(2) $$\int_0^{2\pi} \cos^n\theta\, d\theta$$

defined for n ≥ 0 as an integer, and equation (2) can be expressed as follows.

(3) $$\int_0^{2\pi} cos^n\,\theta d\theta = \frac{1}{2^n i}\int_C \left(z + \frac{1}{z}\right)^n \frac{dz}{z} = \frac{1}{2^n i}\int_0^{2\pi} \left(z + \frac{1}{z}\right)^n \frac{dz}{z}$$

Integration path, $C: |\,z\,| = 1$

(4) $$\frac{1}{z}\left(z + \frac{1}{z}\right)^n = \frac{1}{z}\left(z^n + nz^{n-1}\frac{1}{z} + n(n-1)z^{n-2}\frac{1}{z^2} + \cdots + \frac{1}{z^n}\right)$$

In the expansion equation of (4), the term 1/z appears in the n/2nd term.

(5) $$\int_C z^n \, dz$$

Equation (5) is 0 when n ≠ -1, and 2πi holds when n = -1.
Using the relation in (4), (5) the result of the calculation of equation (3) is

(6) $$2\pi \left(\frac{n!}{\left(\frac{n}{2}!\right)^2} \right) \frac{1}{2^n}$$

where, n is an even number, n≥2

(7) $$\int_0^{2\pi} cos^n \, \theta d\theta = 2\pi \left(\frac{n!}{\left(\frac{n}{2}!\right)^2} \right) \frac{1}{2^n}$$

This equation holds when n is an even number, n≥2. The result of (6) is also true for sinθ.

Investigation

(7) Equation is an even function when n is an even number. Therefore, this value takes a positive value and is a periodic function, so it is 1/4 times of equation (7) in the range of interval 0 to π/2.

Therefore, the following equation holds.

$$\int_0^{\pi/2} cos^n \, \theta d\theta = \int_0^{\pi/2} sin^n \, \theta d\theta = \frac{\pi}{2} \left(\frac{n!}{\left(\frac{n}{2}!\right)^2} \right) \frac{1}{2^n}$$

Conclusions

The formula for the Wallis integral holds the following formula:

It was shown that it can be derived from the complex function approach as fairly simple calculations.

$$\int_0^{\pi/2} cos^n \, \theta d\theta = \int_0^{\pi/2} sin^n \, \theta d\theta = \frac{(n-1)!!}{n!!} \frac{\pi}{2} = \frac{\pi}{2} \left(\frac{n!}{\left(\frac{n}{2}!\right)^2} \right) \frac{1}{2^n}$$

where n is an even number, n≥2

References

[1] Shunichi Tachibana, Kiyomasa Narita, Chie Nara, Exercise: Calculus, Kyoritsu Publishing (1992).
[2] Keishi Baba,Yutaka Takasugi, Complex functions Campus seminar, Masema Publishing (2006).